École Polytechnique Fédérale de Lausanne

College of Management of Technology

MGT-626 Concepts in Operations, Economics and Strategy (Prof. Weber)

Problem Set 3

Spring 2014

Issued: Wednesday, February 26, 2014	**Due:** Friday, March 7, 2014

Problem 3.1 (Cournot Oligopoly) In a market where all $N \geq 2$ firms are offering homogeneous products, the evolution of the "sticky" market price $p(t)$ as a function of time $t \geq 0$ is described by the IVP

$$\dot{p}(t) = f(p(t), u^1(t), \ldots, u^N(t)), \quad p(0) = p_0,$$

where the initial price $p_0 > 0$ and the continuously differentiable excess demand function $f : \mathbb{R}^{1+N} \to \mathbb{R}$ are given, with

$$f(p, u^1, \ldots, u^N) = \alpha \left(a - p - \sum_{i=1}^{N} u^i \right).$$

The constant $\alpha > 0$ is a given adjustment rate, and $a > 0$ represents the known market potential. Each firm $i \in \{1, \ldots, N\}$ produces the output $u^i(t) \in \mathcal{U} = [0, \bar{u}]$, given the large capacity limit $\bar{u} > 0$, resulting in the production cost

$$C(u^i) = cu^i + \frac{(u^i)^2}{2},$$

where $c \in [0, a)$ is a known constant. Given the other firms' strategy profile $u^{-i}(t) = \mu^{-i}(t, p(t))$, $t \geq 0$, each firm i maximizes its infinite-horizon discounted profit,

$$J^i(u^i) = \int_0^\infty e^{-rt} \left(p(t)u^i(t) - C(u^i(t)) \right) dt,$$

where $r > 0$ is a common discount rate.

(i) Formulate the differential game $\Gamma(p_0)$ in normal form.

(ii) Determine the unique Nash equilibrium u_0 (together with an appropriate initial price p_0) for a 'static' version of this Cournot game, where each firm i can choose only a constant production quantity $u_0^i \in \mathcal{U}$ and where the price p_0 is adjusted only once at time $t = 0$ and remains constant from then on.

(iii) Find a symmetric open-loop Nash equilibrium $u^*(t)$, $t \geq 0$, of $\Gamma(p_0)$, and compute the corresponding equilibrium turnpike $(\bar{p}^*, \bar{u}^*)$, namely, the long-run equilibrium state-control tuple. Compare it to your solution in (i), and explain the intuition behind your findings.

(iv) Find a symmetric Markov-perfect (closed-loop) Nash equilibrium $\mu^*(t, p)$, $(t, p) \in \mathbb{R}_+^2$, of $\Gamma(p_0)$, and compare it to your results in (iii).

(v) How do your answers in parts (ii)–(iv) change as the market becomes competitive, as $N \to \infty$?

Problem 3.2 (Duopoly Pricing Game) Two firms in a common market are competing on price. At time $t \geq 0$, firm $i \in \{1,2\}$ has a user base $x_i(t) \in [0,1]$. Given the firms' pricing strategy profile $p(t)=(p^i(t), p^j(t))$,[1] $t \geq 0$, this user base evolves according to the ODE

$$\dot{x}_i = x_i(1 - x_i - x_j)[\alpha(x_i - x_j) - (p^i(t) - p^j(t))], \quad x_i(0) = x_{i0},$$

where $j \in \{1,2\} \setminus \{i\}$, and the initial user base $x_{i0} \in (0, 1 - x_{j0})$ is given. Intuitively, firm i's installed base increases if its price is smaller than $\alpha(x_i - x_j) + p^j$, where the constant $\alpha \geq 0$ determines the importance of the difference in installed bases as 'brand premium.' For simplicity, assume that the firms are selling information goods at zero marginal cost. Hence, firm i's profit is

$$J^i(p^i|p^j) = \int_0^\infty e^{-rt} p^i(t)\, [\dot{x}_i(t)]_+ \, dt,$$

where $r > 0$ is a given common discount rate.

(i) Formulate the differential game $\Gamma(x_0)$ in normal form.

(ii) Show that any admissible state trajectory $x(t)$ of the game $\Gamma(x_0)$ moves along the curve $\mathcal{C}(x_0) = \{(x_1, x_2) \in [0,1]^2 : x_1 x_2 = x_{10} x_{20}\}$.

(iii) Show that an open-loop state-control trajectory $(x^*(t), p^*(t))$ in the game $\Gamma(x_0)$ is characterized as follows.

1. If $x_{10} = x_{20}$, then $p^*(t) \equiv 0$ and $x^*(t) \equiv x_0$.
2. If $x_{i0}^* > x_{j0}^*$, then $p^{j*}(t) = 0$, and $(x_i^*(t), -x_j^*(t))$ increases along the curve $\mathcal{C}(x_0)$, converging to the stationary point

$$\bar{x} = (\bar{x}_i, \bar{x}_j) = \left(\frac{1 - \sqrt{1 - 4x_{i0}x_{j0}}}{2}, \frac{1 + \sqrt{1 - 4x_{i0}x_{j0}}}{2} \right).$$

(iv) Plot a typical Nash-equilibrium state-control trajectory.

(v) Provide an intuitive interpretation of the Nash-equilibrium strategy profile of $\Gamma(x_0)$ determined earlier, and discuss the corresponding managerial conclusions. What features of the game are not realistic?

Problem 3.3 (Screening) Assume that you are the product manager for a company that produces digital cameras. You have identified two consumer types $\theta \in \Theta = \{\theta_L, \theta_H\}$, "amateurs" ($\theta_L$) and "professionals" ($\theta_H$), where $\theta_H > \theta_L > 0$. Based on the results of a detailed survey of the two groups you find that the choice behavior of a type-θ consumer can be represented approximately by the utility function

$$U(t, x, \theta) = \theta\left(1 - (1 - x)^2\right)/2 - t,$$

[1] It is without loss of generality to assume that $p(t) \in [0, P]^2$, where $P > 0$ can be interpreted as a (sufficiently large) maximum willingness to pay.

where t is the price of a camera and x is an internal (scalar) 'quality index' that you have developed, which orders the different camera models of the company. A consumer's utility is zero if he does not buy. Assume that the production cost of a digital camera of quality x is $C(x) = cx$, where $c \in (0, 1)$. The survey also showed that the proportion of professionals among all consumers is equal to $\mu \in (0, 1)$.

(i) If your company has production capacity for only a single camera type, which can only be sold at a single price, determine the profit-maximizing price and quality, (t^m, x^m), of that product. Under what conditions can you avoid 'shutdown,' so that both amateurs and professionals would end up buying this product? What are the company's profits, Π^m?

(ii) If you could perfectly distinguish amateurs from professionals, what would be the profit-maximizing menu of products, $\{(\hat{t}_i, \hat{x}_i)\}_{i \in \{L,H\}}$, to offer? Determine the associated *first-best* level of profits, $\hat{\Pi}$.

(iii) If you cannot distinguish between consumer types, what is the optimal *second-best* menu of products, $\{(t_i^*, x_i^*)\}_{i \in \{L,H\}}$? Determine your company's second-best profits Π^* in that case.

(iv) Determine the optimal nonlinear menu of products,

$$\{(t^*(\theta), x^*(\theta))\}_{\theta \in \Theta},$$

for the case where $\Theta = [0, 1]$ and all types are equally likely.