École Polytechnique Fédérale de Lausanne

College of Management of Technology

MGT-626 Concepts in Operations, Economics and Strategy (Prof. Weber)

Problem Set 2

Spring 2014

Issued: Friday, February 21, 2014	**Due:** Wednesday, February 26, 2014

Problem 2.1 (Controllability and 'Golden Rule') Consider a model for determining a firm's dynamic policy about what amount $u_1(t)$ to spend on advertising and what price $u_2(t)$ to charge for its homogeneous product at any time $t \geq 0$. The advertising effect x_1 tracks the advertising expenditure, and the firm's installed base x_2 increases when demand $D(x, u_2)$ is larger than the number of products βx_2 that fail due to obsolescence. The evolution of the state variable $x = (x_1, x_2)$ is described by a system of ODEs,

$$\begin{aligned} \dot{x}_1 &= -\alpha_1 x_1 + u_1, \\ \dot{x}_2 &= D(x, u_2) - \beta x_2, \end{aligned}$$

where[1]

$$D(x, u_2) = [1 - x_2 - \gamma u_2]_+ (\alpha_2 x_1 + \alpha_3 x_2)$$

denotes demand, and $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ are given positive constants. The control at time $t \geq 0$ is $u(t) = (u_1(t), u_2(t)) \in \mathcal{U} = [0, \bar{u}_1] \times [0, 1/\gamma]$. Assume that the initial state $x(0) = (x_{10}, x_{20}) \in (0, \bar{u}_1/\alpha_1) \times (0, 1)$ is known. The constant $\bar{u}_1 > 0$ is a given upper limit on advertising expenditure.

(i) Show that, without any loss of generality, one can restrict attention to the case where $\gamma = 1$.

(ii) Sketch phase diagrams for the two cases where $\beta < \alpha_3$ and $\beta \geq \alpha_3$. (Hint: The interesting controls to consider are those that drive the system at (or close to) either maximum or minimum velocity (in terms of the right-hand side of the system equation) in the different directions.)

(iii) Determine a nontrivial compact set $\mathcal{C} \subset (0, 1) \times (0, \bar{u})$ of 'controllable states,' which are such that they can be reached from any other state in that set in finite time. Be sure to show how one could steer the system from x to $\hat{x}$ for any $\hat{x}, x \in \mathcal{C}$. Explain what happens when $x(0) \notin \mathcal{C}$.

(iv) Now consider the problem of maximizing the firm's discounted infinite-horizon profit,

$$J(u) = \int_0^\infty e^{-rt} \left(u_2(t) D(x(t), u_2(t)) - c u_1(t) \right) dt,$$

where $r > 0$ is a given discount rate and $c > 0$ is the unit cost of advertising, with respect to bounded measurable controls $u = (u_1, u_2)$ defined a.e. on $\mathbb{R}_+$, with values in the compact control set $\mathcal{U}$. Can you determine a state $\hat{x} = (\hat{x}_1, \hat{x}_2)$ such that if $x(0) = \hat{x}$, it would be optimal for the firm to stay at that state forever? (Hint: Compare the system's turnpike with the golden rule.)

[1] For any $z \in \mathbb{R}$, the 'nonnegative part of z' is denoted by $[z]_+ = \max\{0, z\}$.

(v) Check if the equilibrium state $\hat{x}$ of part (iv) is contained in the set $\mathcal{C}$ of part (iii). Based on this, explain *intuitively* how to find and implement an optimal policy. Try to verify your policy numerically for an example with $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma, r, c) = (1, .05, .1, .6, 100, .1, .05)$.

Problem 2.2 (Exploitation of an Exhaustible Resource) Let $x(0) = x_0 > 0$ be the initial stock of an exhaustible (also known as 'nonrenewable' or 'depletable') resource. The utility (to society) of consuming the resource at the nonnegative rate $c(t)$ at time $t \in [0, T]$ (for a given time horizon $T > 0$) is $U(c(t))$, where $U : \mathbb{R}_+ \to \mathbb{R}$ is a utility function that is twice continuously differentiable, increasing, and strictly concave on $\mathbb{R}_{++}$. For any bounded, measurable consumption path $c : [0, T] \to [0, \bar{c}]$, bounded by the maximum extraction rate $\bar{c} > 0$, the *social welfare* is

$$W(c) = \int_0^T e^{-rt} U(c(t))\, dt,$$

where $r > 0$ is the social discount rate. The stock of the resource evolves according to

$$\dot{x} = -c,$$

provided that the feasibility constraint

$$c(t) \in [0, \mathbf{1}_{\{x(t) \geq 0\}} \bar{c}\,]$$

is satisfied a.e. on $[0, T]$, where $\mathbf{1}$ is the indicator function.

(i) Formulate the social planner's dynamic welfare maximization problem as an optimal control problem.[2]

(ii) Using the PMP, provide necessary optimality conditions that need to hold on an optimal state-control path (x^*, c^*). If $\psi(t)$ is the (absolutely continuous) adjoint variable in the PMP, denote by $\nu(t) = e^{rt}\psi(t)$, for all $t \in [0, T]$, the current-value adjoint variable.

(iii) Let $\eta = -c\, U_{cc}(c)/U_c(c) > 0$ be the relative risk aversion (or the 'elasticity of the marginal utility of consumption'). Using the conditions in (ii), prove the *Hotelling rule*, that $\dot{\nu} = r\nu$ on $[0, T]$. Explain its economic significance using intuitive arguments. Show also that

$$\frac{\dot{c}}{c} = -\frac{r}{\eta},$$

that is, the relative growth rate of consumption on an optimal resource extraction path is proportional to the ratio of the discount rate and the relative risk aversion.

(iv) Find a welfare-maximizing policy $c^*(t)$, $t \in [0, T]$, when $U(c) = \ln(c)$. Compute the corresponding optimal state trajectory $x^*(t)$, $t \in [0, T]$.

(v) Is it possible to write the optimal policy in (iv) in terms of a feedback law μ, that is, in the form $c^*(t) = \mu(t, x^*(t))$, $t \in [0, T]$?

[2]Instead of letting the control constraint depend on the state, it may be convenient to introduce a constraint on the state endpoint $x(T)$ because $x(T) \leq x(t)$ for all $t \in [0, T]$.

(vi) Redo parts (i)–(v) when $T \to \infty$. (For each one it is enough to note and discuss the key changes.)

Problem 2.3 (Exploitation of a Renewable Resource) Let $x(t)$ represent the size of an animal population that produces a useful by-product $y(t)$ (e.g., cows produce milk, bees produce honey) at time $t \in [0, T]$, where $T > 0$ is a given time horizon. The production of the by-product is governed by the production function $y = F(x)$, where $F : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, increasing, strictly concave, and such that $F(0) = 0$. A fraction of $u(t) \in [0, 1]$ of the by-product is extracted at time t, and the remaining fraction $1 - u(t)$ is left with the animals, so that their population evolves according to

$$\dot{x} = \alpha(x - \bar{x}) + (1 - u(t))F(x),$$

where $\bar{x} \geq 0$ is a given critical mass for the population to be able to grow, and $\alpha \geq r$ is a given growth rate. Each unit of the by-product that is extracted can be sold at a profit of 1. A firm is trying to maximize its profit,

$$J(u) = \int_0^T e^{-rt} u(t) F(x(t))\, dt,$$

where $r > 0$ is a given discount rate, subject to the 'sustainability constraint'

$$x(0) = x(T) = x_0,$$

where $x_0 \leq \bar{x}$, with $\alpha x_0 + F(x_0) > \alpha \bar{x}$, is the given initial size of the animal population.

(i) Formulate the firm's profit-maximization problem as an optimal control problem.

(ii) Use the PMP to provide necessary optimality conditions. Provide a phase diagram of the Hamiltonian system of ODEs.

(iii) Characterize the optimal policy $u^*(t)$, $t \in [0, T]$, and show that in general it is discontinuous.

(iv) Describe the optimal policy in words. How is this policy influenced by r and α?

(v) For $\alpha = 1$, $r = .1$, $x_0 = 10$, $\bar{x} = 12$, $T = 2$, and $F(x) = \sqrt{x}$, provide an approximate numerical solution of the optimal control problem in (i), that is, plot the optimal state trajectory $x^*(t)$ and the optimal control trajectory $u^*(t)$ for $t \in [0, T]$.