École Polytechnique Fédérale de Lausanne

College of Management of Technology

MGT-626 Concepts in Operations, Economics and Strategy (Prof. Weber)

Problem Set 1

Spring 2014

Issued: Monday, February 17, 2014 **Due:** Friday, February 21, 2014

Problem 1.1 (Growth Models) Given the domain $\mathcal{D} = \mathbb{R}^2_{++}$ and initial data $(t_0, x_0) \in \mathcal{D}$ such that $(t_0, x_0) \gg 0$,[1] solve the initial-value problem

$$\dot{x} = f(t, x), \quad x(t_0) = x_0,$$

where $f(t, x) : \mathcal{D} \to \mathbb{R}$ is one of the continuous functions given in parts (i)–(iii). For this, assume that $\bar{x} > x_0$ is a finite carrying capacity, and that $\alpha, \beta, \gamma > 0$ are given parameters.

(i) $f(t,x) = \alpha\gamma \left(1 - \left(\frac{x}{\bar{x}}\right)^{1/\gamma}\right) x$ (Generalized Logistic Growth).

(ii) $f(t,x) = \alpha x \ln\left(\frac{\bar{x}}{x}\right)$ (Gompertz Growth); show that Gompertz growth is obtained from generalized logistic growth in (i) as $\gamma \to \infty$.

(iii) $f(t,x) = \left(1 - \frac{x}{\bar{x}}\right)(\alpha x + \beta)\rho(t)$ (Bass Diffusion), where $\rho(t) = 1 + \delta u(t)$. The continuous function $u(t)$ with values in $[0, \bar{u}]$ describes the relative growth of advertising expenditure for some $\bar{u} > 0$, and the constant $\delta \geq 0$ determines the sensitivity of the Bass product diffusion process to advertising.[2]

(iv) Let $\delta = 0$. Find some product diffusion data for your favorite innovation, and estimate the values of the parameters α, β, γ for the models under (i)–(iii). Plot the three different growth curves together with your data. (There is no need to develop econometrically rigorous estimation procedures; the point of this problem is to become familiar with some software that can produce graphical output and to experiment with the different growth models.)

Problem 1.2 (Population Dynamics with Competitive Exclusion) Consider the evolution of two interacting populations, $\xi_1(\tau)$ and $\xi_2(\tau)$. At time $\tau \geq 0$, the evolution of the populations, with given positive initial sizes of ξ_{10} and ξ_{20}, is described by the following initial-value problem:

$$\begin{aligned}
\dot{\xi}_1 &= a_1 \xi_1 \left(1 - \frac{\xi_1}{\bar{\xi}_1} - b_{12}\frac{\xi_2}{\bar{\xi}_1}\right), \quad \xi_1(0) = \xi_{10}, \\
\dot{\xi}_2 &= a_2 \xi_2 \left(1 - \frac{\xi_2}{\bar{\xi}_2} - b_{21}\frac{\xi_1}{\bar{\xi}_2}\right), \quad \xi_2(0) = \xi_{20},
\end{aligned}$$

where the constants $a_i, \bar{\xi}_i > \xi_{i0}$, $i \in \{1, 2\}$, are positive and $b_{12}, b_{21} \in \mathbb{R}$.

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[1] Given two vectors $a, b \in \mathbb{R}^n$, we write that $a = (a_1, \ldots, a_n) \gg (b_1, \ldots, b_n) = b$ if and only if $a_i > b_i$ for all $i \in \{1, \ldots, n\}$.

[2] This diffusion model was developed by Frank M. Bass in 1969 for $\delta = 0$. The original model fits diffusion data surprisingly well, as shown by Bass et al. (1994). It was included in the *Management Science* special issue on "Ten Most Influential Titles of Management Science's First Fifty Years."

(i) Using a linear variable transformation, de-dimensionalize the model, such that the evolution of the two populations is described by functions $x_1(t)$ and $x_2(t)$ that satisfy

$$\begin{aligned} \dot{x}_1 &= x_1 \left(1 - x_1 - \beta_{12} x_2\right), \quad x_1(0) = x_{10}, &(1)\\ \dot{x}_2 &= \alpha x_2 \left(1 - x_2 - \beta_{21} x_1\right), \quad x_2(0) = x_{20}, &(2)\end{aligned}$$

where $\alpha, \beta_{12}, \beta_{21}, x_{10}, x_{20}$ are appropriate positive constants. Interpret the meaning of these constants. The description (1)–(2) is used for the following parts.

(ii) Determine any steady states of the system (1)–(2). Show that there is at most one positive steady state (that is, strictly inside the positive quadrant of the (x_1, x_2)-plane). Under what conditions on the parameters does it exist?

(iii) Determine the stability properties of each steady state determined in part (ii).

(iv) Under the assumption that there is a positive steady state, draw a qualitative diagram of the state trajectories in phase space (given any admissible initial conditions).

(v) How does the phase diagram in part (iv) support the conclusion (from evolutionary biology) that *when two populations compete for the same limited resources, one population usually becomes extinct*? Discuss your findings using a practical example in economics.

Problem 1.3 (Predator-Prey Dynamics with Limit Cycle) Consider two interacting populations of prey, $x_1(t)$, and predators, $x_2(t)$, which for $t \geq 0$ evolve according to

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1) - \frac{\delta x_1 x_2}{x_1 + \beta}, \quad x_1(0) = x_{10}, &(3)\\ \dot{x}_2 &= \alpha x_2 \left(1 - \frac{x_2}{x_1}\right), \quad x_2(0) = x_{20}, &(4)\end{aligned}$$

where α, β, δ are positive constants, and $x_0 = (x_{10}, x_{20}) \gg 0$ is a given initial state.

(i) Find all steady states of the system (3)–(4). Show that there is a unique positive steady state x^*.

(ii) What is the smallest value δ for which the system can exhibit (as a function of α, β) unstable behavior around the positive steady state?

(iii) Show that if the positive steady state is unstable, there exists a (stable?) limit cycle. For that case, draw a qualitative diagram of solution trajectories in the phase space.

(iv) Discuss qualitative differences of the diagram obtained in part (iii) from the Lotka-Volterra predator-prey model. Which model do you expect to more robustly fit data? Explain.