MGT-621 Microeconomics

Screening

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1 Introduction

A decision maker may face a situation in which decision-relevant information is held privately by other economic agents. For instance, suppose the decision maker is a refrigerator salesman. In a discussion with a potential buyer he is thinking about the right price to announce. Naturally, the client's private value for a refrigerator of this type is a piece of *hidden information* that the salesman would love to know before announcing his price. It would prevent him from announcing a price that is too high, in which case there would be no trade, or else a price that is too low, in which case the buyer is left with surplus that the seller would rather pocket himself. In particular, the decision maker would like to charge a person with a higher value for the refrigerator more than a person with a lower value, provided he can at least cover his actual cost of furnishing the item. — Thus the key question we would like to ask is, what incentive could an economic agent possibly have to reveal a piece of hidden information, if an advantage could be obtained by announcing something untruthful? More specifically, can the decision maker devise a *mechanism* that an agent might find attractive enough to participate in (instead of ignoring the decision maker and doing something else) and that at the same time induces revelation of private information such as the agent's willingness to pay? We would expect that in order for agents to voluntarily disclose private information to a decision maker, whom we will

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henceforth also refer to as the *principal*, they must be offered a nonnegative *information rent*, which they would be unable to obtain without their private information: an appropriate *screening* mechanism should make it advantageous for agents (compared to their *status quo* or an "outside option") to disclose private information, even when this very information is likely to be used "against" themselves. As shown below, the *revelation principle* (cf. Proposition 1) guarantees that without loss of generality the principal can limit his search of appropriate mechanisms to those in which all agents find it optimal to announce their private information truthfully (so-called *direct* mechanisms). An agent's piece of private information is commonly referred to as her *type*. The refrigerator salesman, before announcing a price for the product, thus wishes to know the type of the potential buyer allowing him to infer her willingness to pay. As an example, if there are two or more agents competing for the item, then a truth-revealing mechanism can be implemented using a second-price auction. In the next section, we will provide a solution for the refrigerator salesman's mechanism design problem when the buyer's private information is binary (i.e., when there are only two possible types).

Overview. In this summary we review the basics of static mechanism design in settings where a principal faces a single agent of uncertain type. The aim of the resulting "screening contract" is for the principal to obtain the agent's type information in order to avert adverse selection, maximizing his payoffs. We discuss nonlinear pricing as an important application.

2 A Model with Two Types

Consider the refrigerator salesman from the last section and assume that he could face a buyer of type θ_L or θ_H, whereby $\theta_L < \theta_H$. The buyer's type $\theta \in \{\theta_L, \theta_H\} = \Theta$ is related to her willingness to pay in the following way: if the salesman announces a price p for a refrigerator of quality $q \in \mathcal{Q} \subset \mathbb{R}$ (with $\mathcal{Q}$ compact), the (type-dependent) buyer's utility is equal to zero if she does not buy (while exercising her outside option), and it is

$$u(q, \theta) - p \geq 0, \qquad (1)$$

if she does buy. The function $u : \mathcal{Q} \times \Theta \to \mathbb{R}$, assumed to be strictly increasing in (q, θ) and convex in q, represents the buyer's preferences.[1] In order to design an appropriate screening mechanism that distinguishes between the two types, the salesman needs a contracting device (i.e., an *instrument*), such as a product characteristic that he is able to vary. The sales contract could then specify the product characteristic, say, the product's quality q, for which the buyer would need to pay a price $p(q)$.[2] The idea for the design of a screening mechanism is that the salesman proposes a *menu of contracts* containing variations of the instrument, from which the buyer is expected to select her most desirable one. Since there are only two possible types, the salesman needs at most two different contracts. Let us index the contracts by the quality on offer, $q \in \{q_L, q_H\}$. The buyer – no matter what type – cannot be forced to sign the sales contract: her participation is voluntary. Thus, inequality (1) needs to be satisfied for any participating type $\theta \in \{\theta_L, \theta_H\}$. Furthermore, at price p_H the buyer of type θ_H should prefer quality q_H,

$$u(q_H, \theta_H) - p_H \geq u(q_L, \theta_H) - p_L, \tag{2}$$

and conversely at price p_L a buyer of type L should prefer q_L, so that

$$u(q_L, \theta_L) - p_L \geq u(q_H, \theta_L) - p_H. \tag{3}$$

Assume that the unit costs for a refrigerator of quality q is $c(q)$, where $c : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function. The contract design problem is to choose $\{(q_L, p_L), (q_H, p_H)\}$ (with $p_L = p(q_L)$ and $p_H = p(q_H)$) such as to maximize the salesman's expected profit,

$$\bar{\Pi}(p_L, p_H, q_L, q_H) = (1 - \mu)\,[p_L - c(q_L)] + \mu\,[p_H - c(q_H)]\,, \tag{4}$$

where $\mu = \text{Prob}(\tilde{\theta} = \theta_H) = 1 - \text{Prob}(\tilde{\theta} = \theta_L) \in (0, 1)$ denotes the salesman's prior belief about the probability of being confronted with type θ_H as opposed to θ_L.[3] The optimization problem,

$$\max_{\{(q_L, p_L), (q_H, p_H)\}} \bar{\Pi}(p_L, p_H, q_L, q_H), \tag{5}$$

[1] We assume here that the buyer's preferences are quasi-linear in wealth.

[2] It is important to note at this point that the instrument needs to be *observable* by the buyer and *verifiable* by a third party, so that a sales contract specifying a payment $p(q)$ for a quality q can be enforced by a benevolent court of law.

[3] Since the buyer's type is unknown to the decision maker, he treats $\tilde{\theta}$ as a *random variable* with realizations in the *type space* Θ.

is subject to the *individual rationality* (or *participation*) constraint (1) as well as the *incentive compatibility* constraints (2) and (3). The general solution to this *mechanism design problem* is complicated and depends on the form of u. Its solution is simplified, if u has increasing differences in (q, θ). In other words, let us assume that $u(q, \theta_H) - u(q, \theta_L)$ is increasing in q, or equivalently that

$$\hat{q} \geq q \quad \Rightarrow \quad u(\hat{q}, \theta_H) - u(q, \theta_H) \geq u(\hat{q}, \theta_L) - u(q, \theta_L). \tag{6}$$

Condition (6) implies that the marginal gain from additional quality is greater for type θ_H (the "high type") than for type θ_L (the "low type"). To further simplify the principal's constrained optimization problem, we first show that the low type's participation constraint is binding. Indeed, if this was not the case, then $u(q_L, \theta_L) - p_L > 0$ and thus,

$$u(q_H, \theta_H) - p_H \geq u(q_L, \theta_H) - p_L \geq u(q_L, \theta_L) - p_L > 0,$$

which would allow the principal to increase prices for both high and low type as neither participation constraint is binding. As an additional consequence of this proof the individual rationality constraint for the high type can be neglected while it is binding for the low type. This makes the principal's problem substantially easier. Another simplification is achieved by noting that the high type's incentive compatibility constraint (2) must be active. If this was not true, then

$$u(q_H, \theta_H) - p_H > u(q_L, \theta_H) - p_L \geq u(q_L, \theta_L) - p_L = 0,$$

whence it would be possible to increase p_H without breaking (1) for the high type: a contradiction. Moreover, it is then possible to neglect (3), since incentive compatibility for the low type is implied by the fact that (2) is binding and the sorting condition (6) holds, $p_H - p_L = u(q_H, \theta_H) - u(q_L, \theta_H) \geq u(q_H, \theta_L) - u(q_L, \theta_L)$. The last inequality with $\theta_H > \theta_L$ also implies that $q_H > q_L$. To summarize, we can therefore drop the high type's participation constraint (1) and the low type's incentive compatibility constraint (3) from the principal's program, which is now constrained by the high type's incentive compatibility constraint,

$$p_H - p_L = u(q_H, \theta_H) - u(q_L, \theta_H), \tag{7}$$

and the low type's participation constraint,

$$u(q_L, \theta_L) = p_L. \tag{8}$$

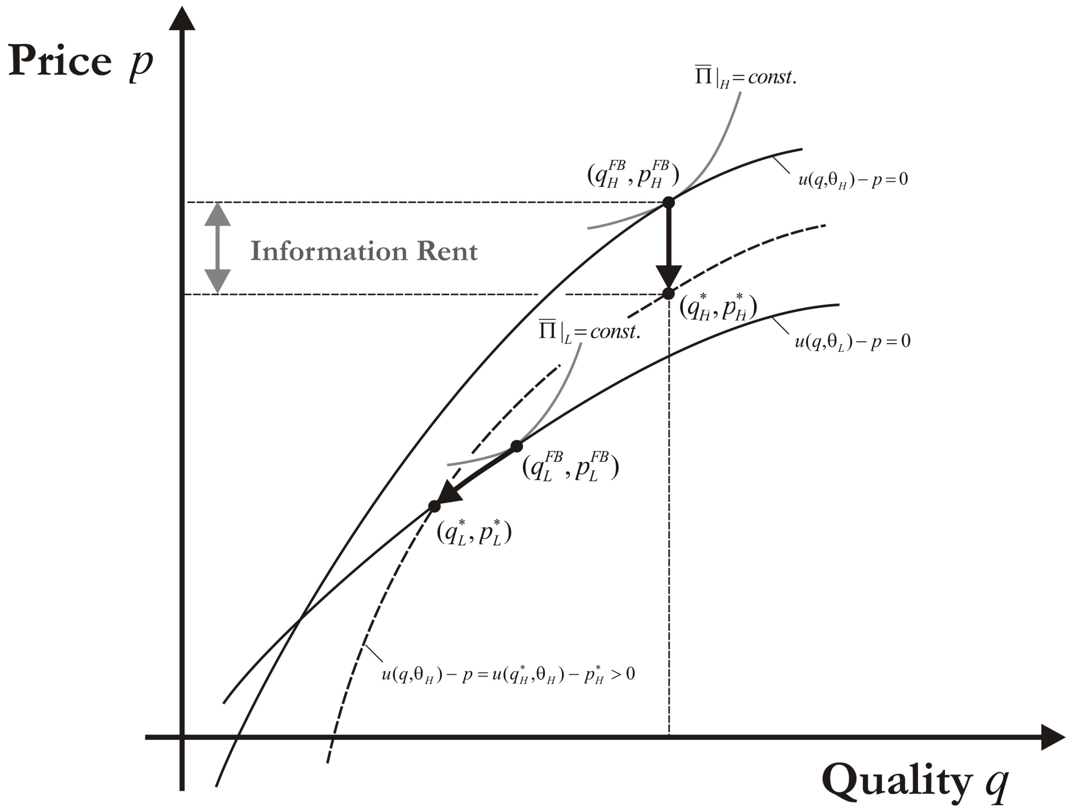

Figure 1: First-best and second-best solution of the model with two types.

Equations (7)–(8) allow us to substitute p_L and p_H into the salesman's expected profit (4). With this, the contract design problem (5) subject to (1)–(3) can be reformulated as an unconstrained optimization problem,

$$\max_{q_L, q_H \in \mathcal{Q}} \Big\{ (1-\mu)\, [u(q_L, \theta_L) - c(q_L)] \\ + \mu \Big[u(q_H, \theta_H) - c(q_H) - \Big(u(q_L, \theta_H) - u(q_L, \theta_L) \Big) \Big] \Big\}.$$

The problem therefore decomposes into the two independent maximization problems

$$q_H^* \in \arg\max_{q_H \in \mathcal{Q}} \Big\{ u(q_H, \theta_H) - c(q_H) \Big\}, \tag{9}$$

and

$$q_L^* \in \arg\max_{q_L \in \mathcal{Q}} \Big\{ u(q_L, \theta_L) - c(q_L) - \frac{\mu}{1-\mu} \Big(u(q_L, \theta_H) - u(q_L, \theta_L) \Big) \Big\}. \tag{10}$$

From (9)–(10) the salesman can determine p_H^* and p_L^* using (7)–(8). In order to confirm that indeed $q_H^* > q_L^*$ as initially assumed, let us first consider the *first-best* solution to the mechanism design problem, $\{(q_L^{\text{FB}}, p_L^{\text{FB}}), (q_H^{\text{FB}}, p_H^{\text{FB}})\}$, i.e., the solution under full information. Indeed, if the salesman knows the type of the buyer, then

$$q_j^{\text{FB}} \in \arg\max_{q_j \in \mathcal{Q}} \{ u(q_j, \theta_j) - c(q_j) \}, \quad j \in \{L, H\}, \tag{11}$$

and

$$p_j^{\text{FB}} = u(q_j, \theta_j), \quad j \in \{L, H\}. \tag{12}$$

Comparing (11) and (9) we find that $q_H^* = q_H^{\text{FB}}$. In other words, even in the presence of hidden information *the high type will be provided with the first-best quality level.* As a result of the supermodularity assumption (6) on u, the first-best solution $q^{\text{FB}}(\theta)$ is increasing in θ. Hence, $\theta_L < \theta_H$ implies that $q_L^{\text{FB}} = q^{\text{FB}}(\theta_L) < q^{\text{FB}}(\theta_H) = q_H^{\text{FB}}$. In addition, supermodularity of u implies that for the low type the *second-best* solution q_L^* in (10) cannot exceed the first-best solution q_L^{FB} in (11), since $(u(q_L, \theta_H) - u(q_L, \theta_L))$, a nonnegative function increasing in q_L, is subtracted from the first-best maximand in order to obtain the second-best solution (which therefore cannot be larger than the first-best solution). Hence, we have shown that

$$q_H^* = q_H^{\text{FB}} > q_L^{\text{FB}} \geq q_L^*. \tag{13}$$

We conclude that in a hidden-information environment the low type is furnished with an inefficient quality level compared to the first-best. Moreover, the low type is left with zero surplus (since $p_L^* = u(q_L^*, \theta_L)$), while the high type enjoys a positive information rent (from (7) and (12)),

$$p_H^* = p_H^{\text{FB}} - \underbrace{(u(q_L^*, \theta_H) - u(q_L^*, \theta_L))}_{\text{Information Rent}}. \tag{14}$$

The mere possibility that a low type exists thus exerts a *positive externality* on the high type, whereas the net surplus of the low type remains unchanged (and equal to zero) when moving from the principal's first-best to his second-best solution (cf. Figure 1). — If the principal's prior belief is such that he thinks the high type is very likely (i.e., μ is close enough to one), then he adopts a *shutdown solution*, in which he effectively stops supplying the good to the low type. Let $\underline{q} = \min \mathcal{Q}$ be the lowest quality level (provided at cost $c(\underline{q})$). Then (10) implies that for

$$\mu \geq \frac{u(\underline{q}, \theta_L) - c(\underline{q})}{u(\underline{q}, \theta_H) - c(\underline{q})} \equiv \mu_0 \tag{15}$$

the principal shuts down (charges zero price for no product or a costless minimum-quality product), i.e., he starts selling exclusively to the high type. In that case, the high type's information rent collapses to zero, as she is unable to derive a positive externality from the now valueless (for the principal) low type. The following example intends to clarify some of the general notions introduced in this section for a widely used parametrization of the two-type model.

Example 1 Assume that the consumer's private value for the product is proportional to both her type θ and the product's quality q,[4] so that $u(q, \theta) = \theta q$, whereby $\theta \in \{\theta_L, \theta_H\}$ with $\theta_H > \theta_L > 0$ and $q \in [0, \bar{q}]$ with some (large enough) maximum achievable quality level $\bar{q}$. The cost of a product of quality q is assumed to be quadratic, $c(q) = \gamma q^2/2$ for some positive constant $\gamma \geq \theta_H/\bar{q}$ (so that $\bar{q} \geq \theta_H/\gamma$). Note first that $u(q, \theta)$ exhibits increasing differences in (q, θ), since $u(q, \theta_H) - u(q, \theta_L) = q(\theta_H - \theta_L)$ is increasing in q so that condition (6) is indeed satisfied. Assuming a principal's prior $\mu \in (0, 1)$

[4] In this formulation, the type parameter θ can be interpreted as the marginal utility of quality, $\theta = u_q(q, \theta)$. The higher the type for a given quality level, the higher the marginal utility for extra quality. The underlying heuristic is that "power users" are often able to capitalize more on quality improvements than less sophisticated occasional users.

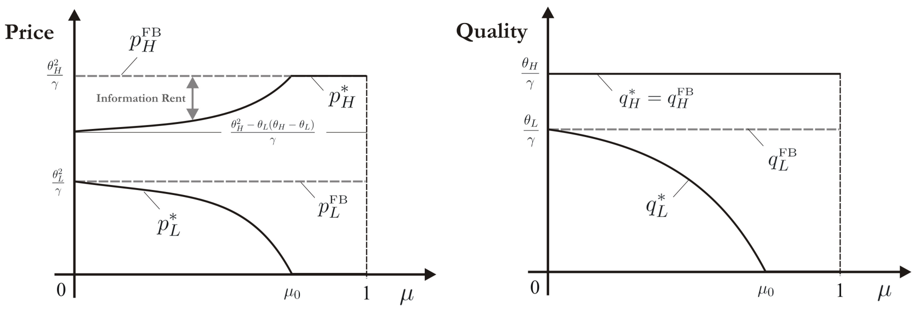

Figure 2: Comparison of first-best and second-best solutions in terms of expected profit ($\bar{\Pi}^{\mathrm{FB}}$ vs. $\bar{\Pi}^*$) as well as expected welfare ($\bar{W}^{\mathrm{FB}}$ vs. $\bar{W}^*$) for Example 1.

of the same form as before, we can therefore use the general results obtained earlier to get

$$q_H^* = q_H^{\mathrm{FB}} = \theta_H/\gamma,$$

and

$$q_L^* = \left[\theta_L - \frac{\mu}{1-\mu}(\theta_H - \theta_L)\right]_+ /\gamma < q_L^{\mathrm{FB}},$$

whereby $q_j^{\mathrm{FB}} = \theta_j/\gamma$ for $j \in \{L, H\}$. Let

$$\mu_0 = \frac{\theta_L}{\theta_H},$$

denote the threshold probability for the high type as in (15): for $\mu \geq \mu_0$, we have that $q_L^* = 0$. In other words, if the high type is more likely than μ_0, then the principal offers a single product of (efficient) high quality, while the low type is excluded from the market. The corresponding second-best prices are given by $p_L^* = u(q_L^*, \theta_L) = q_L^*\theta_L$ and $p_H^* = p_L^* + (u(q_H^*, \theta_H) - u(q_L^*, \theta_H)) = p_L^* + q_H^*(\theta_H - \theta_L)/(1-\mu)$ (cf. Figure 2), whence the principal's expected profit under this optimal screening mechanism becomes

$$\bar{\Pi}^* = \bar{\Pi}(p_L^*, p_H^*, q_L^*, q_H^*) = \begin{cases} \dfrac{\theta_L^2 + \mu\theta_H^2 - 2\mu\theta_L\theta_H}{2\gamma(1-\mu)}, & \text{if } \mu \leq \mu_0, \\ \dfrac{\mu\theta_H^2}{2\gamma}, & \text{otherwise.} \end{cases}$$

By contrast, the first-best profit is

$$\bar{\Pi}^{\mathrm{FB}} = \bar{\Pi}(p_L^{\mathrm{FB}}, p_H^{\mathrm{FB}}, q_L^{\mathrm{FB}}, q_H^{\mathrm{FB}}) = \frac{1}{2\gamma}\left(\theta_L^2 + \mu\left(\theta_H^2 - \theta_L^2\right)\right).$$

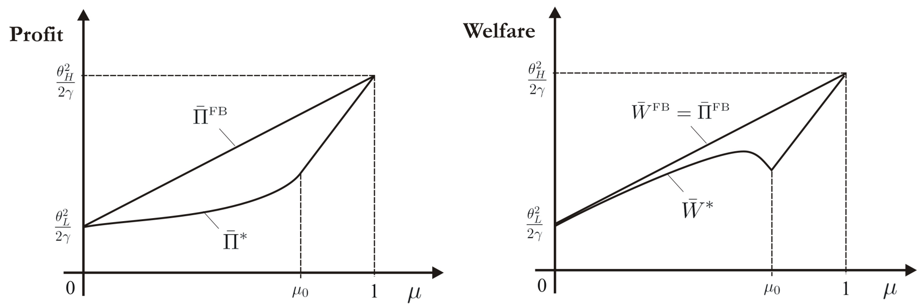

Figure 3: Comparison of first-best and second-best solutions in terms of expected profit ($\bar{\Pi}^{\mathrm{FB}}$ vs. $\bar{\Pi}^*$) as well as expected welfare ($\bar{W}^{\mathrm{FB}}$ vs. $\bar{W}^*$) for Example 1.

In the absence of type uncertainty, i.e., if $\mu \in \{0, 1\}$, we have that $\bar{\Pi}^* = \bar{\Pi}^{\mathrm{FB}}$. With uncertainty, i.e., for $\mu \in (0, 1)$, it is $\bar{\Pi}^* < \bar{\Pi}^{\mathrm{FB}}$. Figure 3 shows how $\bar{\Pi}^*$ and $\bar{\Pi}^{\mathrm{FB}}$ differ as a function of μ. – Let us now consider the social welfare (i.e., the sum of the buyer's and seller's surplus in expectation) as a function of μ. In the absence of hidden type information, the seller is able to appropriate all the surplus in an efficient manner, and thus, first-best expected welfare $\bar{W}^{\mathrm{FB}}$ equals first-best expected profit $\bar{\Pi}^{\mathrm{FB}}$. The second-best expected welfare, $\bar{W}^* = \bar{\Pi}^* + \mu(u(q_H^*, \theta_H) - p_H^*)$ is not necessarily monotonic in μ: as μ increases the seller is able to appropriate more information rent from the high type, while at the same time he is losing revenue from the low type for which he keeps decreasing quality (as a function of μ) until the shutdown point μ_0 is reached, at which all low types are excluded from the market. From then on, high types are charged the efficient price and second-best welfare linearly approaches the first best for $\mu \to 1^-$. □

3 Mechanism Design

Let us now consider the screening problem in a more abstract mechanism design setting. A principal faces one agent[5] of unknown type $\theta \in \Theta \subset \mathbb{R}$ and can offer her a contract (x, t), where $x \in \mathcal{X} \subset \mathbb{R}$ is a consumption input for the agent provided by the principal and $t \in \mathcal{T} \subset \mathbb{R}$ denotes a monetary transfer from the agent to the principal. We assume that for all

[5] We treat the more general mechanism design problem involving N agents later in the course, where we discuss the design of auctions.

$\theta \in \Theta$ an agent's preference preorder over allocations (x, t) can be obtained by evaluating $u(x, t, \theta)$, where $u : \mathcal{X} \times \mathcal{T} \times \Theta \to \mathbb{R}$ is a sufficiently smooth utility function, that is increasing in x, θ and decreasing in t.

Assumption 1 $u_t < 0 < u_x, u_\theta$.

The principal typically controls the agent's choice of x; however, as part of the mechanism the principal can *commit* to a certain set of rules for making the allocation decision (cf. Footnote 6). His prior beliefs about the distribution of agents on Θ are described in terms of a cdf $F : \Theta \to [0, 1]$. — Before the principal provides the agent's consumption input, the agent decides about her participation in the mechanism, and she can send a message $m \in \mathcal{M}$ to the principal, whereby the message space $\mathcal{M}$ is a (measurable) set specified by the principal. For instance, if the principal is a salesman as before, the set $\mathcal{M}$ might contain the different products on offer.[6] For simplicity let us assume that the message space contains a "null message" of the type "I would not like to participate." Allowing the agent not to participate means that the principal needs to consider the agent's individual rationality constraint when designing his mechanism.

Definition 1 *Given* $\{\Theta, F, u\}$ *a* communication mechanism $\Gamma = \langle \mathcal{M}, y \rangle$ *consists of a (measurable) message space* $\mathcal{M}$ *and a mapping* $y : \mathcal{M} \to \mathcal{X} \times \mathcal{T}$, *which assigns an allocation* $y(m) = (x, t)(m)$ *to any message* $m \in \mathcal{M}$.

Let the principal's preferences over allocations (x, t) for an agent of type θ be represented by a (sufficiently smooth) utility function $v : \mathcal{X} \times \mathcal{T} \times \Theta \to \mathbb{R}$. The problem of finding a mechanism that (in expectation) maximizes the principal's utility can be greatly simplified using the revelation principle, which is essentially due to Gibbard (1973), Green and Laffont (1979), and Myerson (1979), which we present here in a simplified one-agent version.

Proposition 1 (Revelation Principle) *If for a given mechanism* $\Gamma = \langle \mathcal{M}, y \rangle$ *an agent of type* $\theta \in \Theta$ *finds it optimal to send a message* $m^*(\theta)$, *then there exists a* direct revelation mechanism $\Gamma^d = \langle \Theta, y^d \rangle$, *such that* $y^d(\theta) = y(m^*(\theta))$ *and the agent finds it optimal to report her type truthfully under* Γ^d.

[6]To obtain more general results we can assume that $\mathcal{M}$ contains all possible (probabilistic) convex combinations over its elements, so that $m \in \mathcal{M}$ in fact represents a probability distribution over $\mathcal{M}$.

Proof. The proof is trivial. Since under mechanism $\Gamma = \langle \mathcal{M}, y \rangle$ the agent finds it optimal to report $m^*(\theta)$, it needs to be the case that

$$m^*(\theta) \in \arg\max_{m \in \mathcal{M}} u(y(m), \theta) \tag{16}$$

for $\theta \in \Theta$. If the principal sets $y^d(\theta) = y(m^*(\theta))$, then clearly it is optimal for the agent to send $m^d(\theta) = \theta$ in the new mechanism Γ^d, which is therefore a direct revelation mechanism. ■

The revelation principle implies that in his search for an optimal mechanism the principal can limit himself – without loss of any generality – to *direct* (i.e., truth-telling) mechanisms. Note that the fact that the principal is able to commit to a certain mechanism is essential for the revelation principle to work. Commitment to a mechanism allows the principal to promise the agent(s) that indeed a revelation mechanism is applied so that truthful messages are incentive compatible.[7]

Definition 2 *A direct mechanism $\langle \Theta, y \rangle$ is* implementable*, if the direct allocation $y : \Theta \to \mathcal{X} \times \mathcal{T}$ satisfies the agent's incentive compatibility (or truth-telling) constraint, i.e., if*

$$u(x(\theta), t(\theta), \theta) \geq u(x(\hat{\theta}), t(\hat{\theta}), \theta), \tag{17}$$

for all $\theta, \hat{\theta} \in \Theta$.

Note that the direct mechanisms in Proposition 1 are implementable. Relation (17) is just a restatement of (16). – In the analysis that follows we restrict our attention to differentiable mechanisms, i.e., mechanisms in which the allocation function y is differentiable (and all the relevant sets are convex and open).

Assumption 2 (Sorting Condition) *The marginal rate of substitution between the agent's consumption input (i.e., the "good") and money is monotonic in the agent's type, i.e.,*

$$\frac{\partial}{\partial \theta} \left(-\frac{u_x(x, t, \theta)}{u_t(x, t, \theta)} \right) > 0. \tag{18}$$

[7] In environments with *renegotiation*, where the principal is unable to commit to a revelation mechanism, the revelation principle fails to apply and it may become optimal for the principal to select an indirect mechanism.

For *quasi-linear* utility functions of the form $u(x, \theta) - t$, sorting condition (18) is nothing else than increasing differences in (x, θ), i.e., (18) reduces to (6) or – given differentiability – to the familiar supermodularity condition $u_{x\theta} > 0$. The following proposition provides a complete and useful characterization of an implementable direct mechanism.

Proposition 2 (Implementation Theorem) *The direct mechanism* $\langle \Theta, y \rangle$ *with* $y = (x, t) : \Theta \to \mathcal{X} \times \mathcal{T}$ *twice differentiable is implementable if and only if for all* $\theta \in \Theta$*:*

$$u_x(\theta, x(\theta), t(\theta))x'(\theta) + u_t(\theta, x(\theta), t(\theta))t'(\theta) = 0, \tag{19}$$

and

$$x'(\theta) \geq 0. \tag{20}$$

Proof. $\Rightarrow$: Consider an agent of type $\theta \in \Theta$ and a direct mechanism $\langle \Theta, y \rangle$. In choosing her message $m(\theta)$, the agent solves,

$$m(\theta) \in \arg\max_{\hat{\theta} \in \Theta} u(x(\hat{\theta}), t(\hat{\theta}), \theta),$$

for which the first-order necessary optimality condition can be written as

$$u_x(x(\hat{\theta}), t(\hat{\theta}), \theta)x'(\hat{\theta}) + u_t(x(\hat{\theta}), t(\hat{\theta}), \theta)t'(\hat{\theta}) = 0. \tag{21}$$

Hence, any direct mechanism must necessarily satisfy (21) for $\hat{\theta} = \theta$, i.e., equation (19) for all $\theta \in \Theta$. The necessary optimality condition (21) becomes sufficient if in addition the corresponding second-order condition,

$$u_{xx}(x')^2 + 2u_{xt}x't' + u_{tt}(t')^2 + u_x x'' + u_t t'' \leq 0, \tag{22}$$

is satisfied at a $\hat{\theta}$ that solves (21). At a truth-telling optimum, relation (22) needs to be satisfied for $\hat{\theta} = \theta$. Differentiating equation (19) (note that it holds for all $\theta \in \Theta$) with respect to θ we obtain

$$\left(u_{xx}x' + u_{xt}t' + u_{x\theta}\right) x' + u_x x'' + \left(u_{xt}x' + u_{tt}t' + u_{t\theta}\right) t' + u_t t'' = 0,$$

so that the second-order condition (22) becomes

$$u_{x\theta}x' + u_{t\theta}t' \geq 0,$$

or equivalently, using the fact that by (19) $t' = -u_x x'/u_t$,

$$u_t x' \frac{\partial}{\partial \theta} \left(\frac{u_x}{u_t} \right) \geq 0.$$

Since by Assumption 1 the agent's utility decreases in her transfer to the principal, i.e., $u_t < 0$, we have by Assumption 2 that necessarily $x' > 0$ on Θ. $\Leftarrow$: In order to demonstrate that (19) and (20) are sufficient for the direct mechanism $\langle \Theta, (x,t) \rangle$ to be implementable, we need to show that (17) in Definition 2 holds for all $\theta, \hat{\theta} \in \Theta$. If we set

$$U(\hat{\theta}, \theta) = u(x(\hat{\theta}), t(\hat{\theta}), \theta),$$

then the first-order and second-order optimality conditions can be written in the form $U_1(\theta,\theta) = 0$, and $U_{11}(\theta,\theta) \leq 0$. If $\hat{\theta} \leq \theta$, then

$$\begin{aligned} U(\theta,\theta) - U(\hat{\theta},\theta) &= \int_{\hat{\theta}}^{\theta} U_1(\vartheta,\theta) d\vartheta \qquad (23) \\ &= \int_{\hat{\theta}}^{\theta} u_t(x(\vartheta), t(\vartheta), \theta) \left(\frac{u_x(x(\vartheta), t(\vartheta), \theta)}{u_t(x(\vartheta), t(\vartheta), \theta)} x'(\vartheta) + t'(\vartheta) \right) d\vartheta. \end{aligned}$$

From (19) and (20), together with Assumption 2, we obtain that for $\vartheta \leq \theta$:

$$0 = \frac{u_x(x(\vartheta), t(\vartheta), \vartheta)}{u_t(x(\vartheta), t(\vartheta), \vartheta)} x'(\vartheta) + t'(\vartheta) \geq \frac{u_x(x(\vartheta), t(\vartheta), \theta)}{u_t(x(\vartheta), t(\vartheta), \theta)} x'(\vartheta) + t'(\vartheta).$$

By Assumption 1, $u_t < 0$, so that the RHS of (23) is nonnegative, i.e., $U(\theta,\theta) \geq U(\hat{\theta},\theta)$. If $\hat{\theta} > \theta$, then (23) still holds. By reversing the integration bounds we obtain that similarly Assumptions 1 and 2 lead us to the same conclusion, namely that the RHS of (23) is nonnegative. In other words, the direct mechanism $\langle \Theta, (x,t) \rangle$ is by (17) implementable, which completes the proof. ■

The implementation theorem implies a *representation* of all implementable direct mechanisms, which the principal can use to find an optimal screening mechanism:

1. Choose an arbitrary increasing schedule $x(\theta)$ for the agent's consumption good.

2. For the x in the last step, find a transfer schedule $t(\theta) - t_0$ by solving the differential equation (19). Note that this transfer schedule is only determined up to a constant t_0. The constant t_0 can be chosen so as to satisfy the agent's participation constraint.

3. Invert the schedule in step 1, i.e., find $\varphi(x) = \{\theta \in \Theta : x(\theta) = x\}$, which may be set-valued. The (unit) price schedule as a function of

the consumption choice, $p(x) \in t(\varphi(x))$ can then be written in the form[8]

$$p(x) = \begin{cases} t(\theta), & \text{for some } \theta \in \varphi(x) \neq \emptyset, \\ \infty, & \text{if } \varphi(x) = \emptyset. \end{cases}$$

Note that whenever $\varphi(x)$ is set-valued, different types obtain the same price for the same consumption choice. This is referred to as *bunching*, since different types are "bunched" together. Bunching occurs for neighboring types, whenever $x'(\theta) = 0$ on an interval of positive length.

In the following section we discuss a solution of the mechanism design problem for differentiable contracts and a continuum of types for the special case, when both the principal's and the agent's utility functions are quasi-linear in wealth.

4 A Model with a Continuum of Types

Consider again the refrigerator salesman from the first section, but this time assume that the potential buyer he meets may stem from a continuum of types $\Theta \subset \mathbb{R}$ (where Θ is convex and compact), $\Theta = [\underline{\theta}, \bar{\theta}]$. Let both the principal's and the agent's utility be quasi-linear in wealth. The principal's prior beliefs of the distribution of agent types is given by the cdf $F : \Theta \to [0, 1]$. The principal wishes to maximize his expected utility,

$$\int_{\Theta} (v(x, \theta) + t(\theta))\, dF(\theta), \tag{24}$$

subject to

$$x'(\theta) \geq 0, \tag{25}$$

$$u_x(x(\theta), \theta)x'(\theta) - t'(\theta) = 0, \tag{26}$$

for all $\theta \in \Theta$, and

$$\int_{\Theta} (u(x, \theta) - t(\theta))\, dF(\theta) \geq u_0. \tag{27}$$

From (26) we obtain

$$t(\theta) = t_0 + \int_{\underline{\theta}}^{\theta} u_x(x(\vartheta), \vartheta)x'(\vartheta)d\vartheta,$$

[8] This transformation is sometimes referred to as "taxation principle."

where $t(\underline{\theta}) = t_0$. We can thus rewrite the principal's objective function in the form

$$\int_{\Theta} \left(v(x(\theta), \theta) + t_0 + \int_{\underline{\theta}}^{\theta} u_x(x(\vartheta), \vartheta) x'(\vartheta) d\vartheta \right) dF(\theta).$$

The constant t_0 can be chosen such that the agent's individual rationality constraint is satisfied in expectation (i.e., (27) holds), so that

$$t_0 = \int_{\Theta} \left(u(x(\theta), \theta) - \int_{\underline{\theta}}^{\theta} u_x(x(\vartheta), \vartheta) x'(\vartheta) d\vartheta - u_0 \right) dF(\theta).$$

The principal's optimization problem thus simplifies to

$$\max_{x(\cdot)} \int_{\Theta} \left(u(x(\theta), \theta) + v(x(\theta), \theta) \right) dF(\theta), \tag{28}$$

subject to

$$x' \geq 0.$$

Let $\xi(\theta) = x'(\theta)$. Then the principal's contract design problem can be rewritten as an optimal control problem (OCP) (cf. Appendix). The Hamiltonian for this OCP is

$$H(x, \xi, \psi, \theta) = \left(u(x(\theta), \theta) + v(x(\theta), \theta) \right) f(\theta) + \psi(\theta) \xi(\theta). \tag{29}$$

According to Pontryagin's maximum principle (Proposition 4), necessary optimality conditions for the OCP are as follows:

$$\psi'(\theta) = -H_x(x^*(\theta), \xi^*(\theta), \psi(\theta), \theta) = -\left(u_x(x^*(\theta), \theta) + v_x(x^*(\theta), \theta) \right) f(\theta), \tag{30}$$

$$\frac{dx^*(\theta)}{d\theta} = \xi^*(\theta), \tag{31}$$

$$\psi(\underline{\theta}) = \psi(\bar{\theta}) = 0 \tag{32}$$

$$\xi^*(\theta) \in \arg\max_{\xi \geq 0} H(x^*(\theta), \xi, \psi(\theta), \theta). \tag{33}$$

Since there is no initial condition for $x(\theta)$ we can set $\psi(\underline{\theta}) = 0$ (which implies an initial condition on x, since the equations for the evolution of $x(\theta)$ and $\psi(\theta)$ are coupled). Thus,

$$0 = \psi(\bar{\theta}) - \psi(\underline{\theta}) = -\int_{\underline{\theta}}^{\bar{\theta}} \left(u_x(x^*(\theta), \theta) + v_x(x^*(\theta), \theta) \right) dF(\theta). \tag{34}$$

The Hamiltonian is linear in ξ and thus we have a *complementary slackness* condition, that

$$\left(\frac{dx^*(\theta)}{d\theta}\right)\left(\int_{\underline{\theta}}^{\theta}\left(u_x(x(\vartheta),\vartheta)+v_x(x(\vartheta),\vartheta)\right)dF(\vartheta)\right)=0,$$

for all $\theta \in [\underline{\theta},\bar{\theta}]$. If $\xi > 0$ over an interval, then $\psi = 0$ there, thus also $\psi' = u_x + v_x = 0$ there. If $\xi = 0$ over an interval $[\theta_0, \theta_1]$, then by continuity of ψ:

$$\psi(\theta_0) = \psi(\theta_1) = 0;$$

by continuity of x^*:

$$x^*(\theta_0) = x^*(\theta_1).$$

The last two equations yield the two unknowns θ_0, θ_1.

5 Application: Nonlinear Pricing

Let us consider the application of the model with a continuum of types to nonlinear pricing. Let the principal's and agent's utility be as in the last section. The standard approach to nonlinear pricing makes use of a slightly different version of the implementation theorem for quasi-linear utilities.

Proposition 3 (Implementation Theorem; alternate version) *Let* $\langle \Theta, (x,t)\rangle$ *(where* $\Theta = [\underline{\theta},\bar{\theta}]$*) be a differentiable direct mechanism and set* $U(\theta,\theta) = u(x,\theta) - t(\theta)$. *The mechanism is implementable if and only if*

$$U(\theta,\theta) = U(\underline{\theta},\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(x(\vartheta),\vartheta)d\vartheta \tag{35}$$

and $x'(\theta) \geq 0$ *for all* $\theta \in \Theta$.

Let $v(x,\theta) = -c(x)$. It is useful to rewrite the principal's objective function, $t(\theta) - c(x(\theta))$, in terms of social surplus,

$$s(x(\theta),\theta) = u(x(\theta),\theta) - c(x(\theta)).$$

In this manner, we obtain – using Proposition 3 – his mechanism design problem in a "relaxed" form (i.e., neglecting $x' \geq 0$ for a moment),

$$\max_{x(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left[s(x(\theta),\theta) - \int_{\underline{\theta}}^{\theta} u_\theta(x(\vartheta),\vartheta)d\vartheta\right] dF(\theta).$$

Note that via integration by parts (with F differentiable and $F' = f$), we get

$$\int_{\underline{\theta}}^{\bar{\theta}} \left(\int_{\underline{\theta}}^{\theta} u_\theta(x(\vartheta), \vartheta) d\vartheta \right) dF(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} u_\theta(x(\theta), \theta) \frac{1 - F(\theta)}{f(\theta)} f(\theta) d\theta.$$

Thus, the principal solves the following relaxed problem:

$$\max_{x(\cdot)} E[\Phi(x(\tilde{\theta}), \tilde{\theta})],$$

where we have set $\Phi(x, \theta) = s(x(\theta), \theta) - u_\theta(x(\theta), \theta)(1 - F(\theta))/f(\theta)$. This maximization can be performed by pointwise maximizing the argument of the expectation operator. MCS of x in θ are obtained, if Φ is supermodular in (x, θ), i.e., if

$$\Phi_{x\theta} \geq 0.$$

Since

$$\Phi_{x\theta} = s_{x\theta} - \frac{u_{x\theta\theta}}{h} + u_{x\theta} \frac{h'}{h},$$

where $h = f/(1-F)$ is the *hazard rate*, it is sufficient to assume that $u_{x\theta} > 0$, $u_{x\theta\theta} \leq 0$, and $h' \geq 0$.[9] Examining the first-order necessary optimality condition, $\Phi_x = s_x - u_{x\theta}/h = 0$ or equivalently

$$(u_x - c_x) f = (1 - F) u_{x\theta},$$

it becomes apparent that for $\theta = \bar{\theta}$, the instrument x is provided at the efficient level (since $u_x(x(\bar{\theta}), \bar{\theta}) = c_x(x(\bar{\theta}))$), i.e., there is no second-best distortion at the top of the type scale. Lower types, as in the two-type model, are however underprovided with the instrument (e.g., quality) at inefficient levels.

6 Notes

The implementation theorem is due to Mirrlees (1971); the version presented here is by Guesnerie and Laffont (1984). The treatment of mechanism design in Section 3 and 4 is based on Laffont (1989, pp. 153–163) and a set of unpublished lecture notes by Lars Stole. Following are some remarks on

[9]The increasing-hazard rate condition is satisfied for many important distributions, such as uniform, exponential, or normal. It excludes distributions, which for instance have "strong bimodal bumps."

the literature on product differentiation. – The notion of product differentiation as such can be traced back at least to Launhardt (1885; pp. 141–189), who examines the impact of transportation costs on competition and thus predates the classical analysis in Hotelling's (1929) seminal paper on horizontal competition. Since then numerous contributions have been made to product differentiation.[10] The corresponding literature can be divided into locational models in the tradition of Hotelling, where each firm is attributed an "address" in product space, and into so-called "non-address" models in the spirit of Chamberlins (1933) monopolistic competition, where a representative consumer exhibits (probabilistic) preferences for different products.[11] An important distinction between the two groups of models is that in the latter group, each product is competing with each other, while in the former consumers are truly heterogeneous in their preferences, and some products may have no overlap, i.e., may never be in direct competition. The locational approach tends to capture consumer heterogeneity well and allows for an explicit consideration of participation constraints that inevitably arise when dealing with a spatial distribution of endowed unobservable consumer characteristics. In fact, Lancaster (1966) first realized,

> "[t]he good, per se, does not give utility to the consumer; it possesses characteristics, and these characteristics give rise to utility (...) In general a good will possess more than one characteristic, and many characteristics will be shared by more than one good." (p. 65)

We refer to these Lancasterian characteristics as product attributes, and naturally products contain a number of different such attributes, which facing a heterogeneous consumer base of unknown types allows a monopolist (the principal) to screen the agents. The product attributes can be used as instruments in the screening process. Using multiple instruments to screen consumers of a one-dimensional type has been earlier examined by Matthews and Moore (1987), whereas the inverse case of a single instrument (price) given consumers of multidimensional types has been considered among others by Laffont, Maskin, and Rochet (1987). This line of work on second-degree price discrimination dates back to Mussa and Rosen (1978), based

[10]For a good bibliography see Anderson et al. (1992), Beath and Katsoulacos (1991, pp. 194–199), as well as Tirole (1988, pp. 166-168, 302–303).

[11]A notable exception in this dichotomy, is the model by Perloff and Salop (1985) that combines characteristics from both groups, driven by symmetry assumptions in the preferences of a representative consumer who is faced with localized products.

on methods developed earlier by Mirrlees (1971) in the context of optimal income taxation, who treat the case for consumers of a single characteristic and single vertical-attribute products. Wilson (1993) and Armstrong (1996) provide generalizations for fully nonlinear pricing models in the multiproduct case. A multidimensional screening model generalizing these approaches has been advanced by Rochet and Choné (1998). Rochet and Stole (2003) provide an excellent overview of recent results.

7 Problems

Problem 1 (Monopolistic Price Discrimination) An airline serves a certain resort destination (that is also a popular conference haven!) exclusively. The airline would like to distinguish between tourists (θ_T) and conference travellers (θ_C) as their sensitivities to price seem to be quite different. The airline executive committee asks you for an appropriate approach and you suggest advance purchase time τ as a possible instrument. You suggest the following quasi-linear model for the utility of a traveller of type $\theta \in \{\theta_C, \theta_T\}$ in case he or she buys a ticket at price p:

$$u(\tau, p, \theta) = \bar{u} - \theta p - \tau,$$

where $\theta_T > \theta_C > 0$ and $\bar{u}$ is a positive constant. You think that the ratio of conference travellers to tourists is about $\lambda = \mu/(1-\mu) \in (0, 1)$ and the executive committee suggests you assume that the cost for transporting a passenger is equal to $c > 0$. (i) Determine the optimal mechanism $\{(\tau_C, p_C), (\tau_T, p_T)\}$ the airline should adopt to segment the market. (ii) Assume that the market has a size of one. What is the expected profit increase of the scheme proposed in (i) over a one-product solution? (iii) When does the solution in (i) imply shutdown (i.e., only one type of traveller is served) and what does this imply? Are there any practical problems with a shutdown solution?

Problem 2 (Adverse Selection in Insurance) A *risk-averse* individual of initial wealth $w > 0$ faces a probability θ of suffering a loss of size $L \in (0, w)$. You work for a monopolist insurance company and would like to offer the individual an insurance contract of the form (c_1, c_2), where c_1 is the individual's leftover wealth in case there is no loss and c_2 is the amount the individual has left to consume, if a loss occurs. The insurance premium in the event of no loss is therefore $w - p_1$ and the net payment received from the insurance if a loss occurs, is $c_2 - (w - L)$. The individual's preferences over wealth are represented by a smooth utility function $u : \mathbb{R} \to \mathbb{R}$. (i) Characterize the set of first-best insurance contracts (each of the form $(c_1^{\mathrm{FB}}, c_2^{\mathrm{FB}})$), when the loss probability θ is observable by all parties. (ii) Unfortunately you have no way of exactly knowing θ as it pertains to the individual's private information. For simplicity assume that $\tilde{\theta} \in \{\theta_L, \theta_H\}$ with $\theta_H > \theta_L > 0$ and a prior $\mathrm{Prob}(\tilde{\theta} = \theta_H) = \mu \in [0, 1]$. What are the optimal second-best contract offerings of the form (c_1^*, c_2^*) that you can offer? Draw a picture in (c_1, c_2)-space. How does your solution in (ii) vary with μ? (iii) Consider your solution in (ii). Is one type of individual being "rationed" in terms of being able to acquire insurance? Describe. Comment also on the externality between the two types. (iv) Determine the information rent in (ii) as a function of μ. Who gets it?

Problem 3 (Multidimensional Screening) Consider a monopolistic firm that can produce special utility vehicles of a certain performance $q \in [0, 1]$ and of a certain color $z \in [0, 2]$. Consumers all agree on the performance ranking (=vertical characteristic) for vehicles of the same color. However, they differ in their tastes for a specific color (=horizontal characteristic). Each consumer is characterized by her ideal color, $x \in \mathcal{X} = \mathbb{R}/2\mathbb{Z}$ (i.e., $\mathcal{X}$ is isomorph to a circle of radius $1/\pi$), and her marginal utility for performance, $\theta \in \Theta = [0, 1]$. Consumers are of type (x, θ) and have a utility that is quasi-linear in wealth: at price p for a product of characteristics (z, q) a consumer's net utility is

$$u(z, q, p; x, \theta) = \theta q - |z - x| - p,$$

if she buys the product, and zero otherwise. The firm assumes that consumer types (x, θ) are uniformly distributed on the outside of a cylinder $\mathcal{X} \times \Theta$. The unit cost for a vehicle of performance q is cq^2, where c is a positive constant. (i) If the firm can produce only one vehicle model, determine its optimal choice for (z, q, p). (ii) You are a consultant and recommend that the firm produce two different vehicle types (z_i, q_i, p_i), $i \in \{1, 2\}$, to segment the market. Determine the profit-maximizing choice of (z_i, q_i, p_i). Draw a picture showing how the market is segmented by this product offering. (iii) Describe the gains and losses of the approach in (ii) when compared to the first-best solution. (iv) Can your solution in (ii) be easily extended to three or more products, or even a fully nonlinear pricing model? Why or why not? [Hint: be careful to examine the shutdown solutions as a function of c.]

Problem 4 (Multiattribute Versioning) Observing the remarkable success in the special utility vehicle industry achieved by your recommendations discussed in Problem 3, the CEO of a little software startup in Silicon Valley asks you if such a scheme would work for him. In the software industry, he explains, the cost structure is a little different though: it typically costs cq^2 to create a product of quality q (where q measures the number of features contained in the product; a lower-q product having less features). But this only holds for the highest-quality product in a product portfolio. Once a high-quality ("flagship") product has been established it becomes cheap for the firm to create lower quality versions. In fact it is sufficient to simply disable features of the flagship product to obtain lower-quality versions, which can be done at a small (but positive) versioning cost c_V per *additional* product.[12] (i) Neglecting the taste dimension and assuming uniformly distributed consumers of type $\theta \in \Theta = [0, 1]$ with net utility $\theta q - p$ in case they buy the product (otherwise zero), what are the optimal qualities and prices you would recommend for an n-product portfolio? In other words, determine an optimal menu, $\{(q_i, p_i)\}_{i=1}^n$, of sale contracts. (ii) Does the solution obtained in (i) depend on the fact in which order products are introduced into the market. If yes, how? If no, why? (iii) "This is all very interesting," says the CEO after you presented him with your conclusions in (i) and (ii), "but" he continues, "our market seems to have a second dimension that is more like 'taste': some people want their software for small businesses, some for home use, and others work somewhere in between." Can you help him out? Introduce a taste dimension into the model and solve for a simple (but nontrivial) case. (iv) From a practical point of view, why might a manager object to introducing n differentiated versions of the same product, even though the model indicates that this could increase profits substantially?

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[12]The *versioning cost* c_V (similar to the upfront investment cq^2) is incurred with the creation of a new product through versioning. The unit cost for selling the product in the market (of total size one) is for simplicity assumed to be zero.

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Appendix

Let us briefly discuss *dynamic optimization*, which is useful for many practical problems. The single most important achievement in dynamic optimization within the last century was no doubt Pontryagin's maximum principle, which provides necessary optimality conditions for a large class of so-called *optimal control problems* (OCPs).[13] We first formulate the standard OCP and then state the maximum principle. Let $x \in \mathcal{X}$ be the *state* of a system, where $\mathcal{X}$ is a nonempty convex and compact subset of $\mathbb{R}^n$. The choice variable or *control* $u \in \mathcal{U}$ of the system is a bounded measurable function $u : \mathbb{R} \to \mathcal{U}$ with values in the nonempty compact convex set $\mathcal{U} \subset \mathbb{R}^m$. At time $t \in [0, T] \subset \mathbb{R}$ (for some $T > 0$), the evolution of the state of the system is governed by the state equation,[14]

$$\frac{dx}{dt} = f(x, u, t), \tag{36}$$

where $f : \mathcal{X} \times \mathcal{U} \times \mathbb{R} \to \mathcal{X}$ is a function, continuous with respect to u and continuously differentiable with respect to x. The initial condition of the system at time $t = 0$ is captured by the condition

$$x(0) = x_0, \tag{37}$$

for some given $x_0 \in \mathcal{X}$. The OCP consists in solving the problem

$$\max_{u(\cdot)} \int_0^T h(x(t), u(t), t)dt, \tag{38}$$

where $h : \mathcal{X} \times \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ is a function continuous with respect to u, t and continuously differentiable with respect to x. The function $h(x, u, t)$ describes instantaneous payoffs that accrue through the use of the (possibly costly) control u while traversing state x at time t. Define

$$H(x, u, \psi(t), t) = h(x, u, t) + \psi(t) \cdot f(x, u, t),$$

[13]Optimal control theory is widely used in economics. Seierstad and Sydsæter (1987) provide a rigorous introduction with regards to problems in economics. Sethi and Thompson (2000) is a useful volume that contains many applications of control theory to managerial problems. A word of caution: despite the beauty of its methods, optimal control theory is not easy to apply to *nonlinear* systems (and many economic systems are highly nonlinear!); often the level of modelling detail must be kept very low due to the analytical complexities that solving the model entails.

[14]The state equation in the screening problem (on page 15) is particularly easy; it features neither "time" (i.e., the type) nor the "state" (i.e., the instrument) on the RHS.

as the *Hamiltonian* of the system, where the function $\psi : \mathbb{R} \to \mathbb{R}^n$ is called the *adjoint variable*. The n-dimensional adjoint variable $\psi(t)$ is somewhat of a dynamic analogue to Lagrange multipliers in standard constrained optimization problems. It represents the "shadow value" of the system in motion at time t. Proposition 4 provides necessary optimality conditions for problem (38) subject to (36)–(37) and $u \in \mathcal{U}$.

Proposition 4 (Pontryagin's Maximum Principle) *Let $(x^*(t), u^*(t))$ describe an optimal state-control trajectory for the OCP for all $t \in [0, T]$. Then the following conditions are satisfied:*

$$\psi'(t) = -\partial_x H(x^*(t), u^*(t), \psi(t), t), \tag{39}$$

for all $t \in [0, T]$, with the boundary condition

$$\psi(T) = 0. \tag{40}$$

Moreover,

$$u^*(t) \in \arg\max_{\hat{u} \in \mathcal{U}} H(x^*(t), \hat{u}, \psi(t), t), \tag{41}$$

for all $t \in [0, T]$.

Proof. See Pontryagin et al. (1962, pp. 79–114; see Theorem 8 on p. 81 and Theorem 3 on p. 50 for a statement of the maximum principle together with the transversality conditions somewhat more general than here).

REMARK If the Hamiltonian H is strictly concave in the control variable u, the optimal control u^* is unique and the conditions in Pontryagin's maximum principle become also sufficient for optimality. – Pontryagin's maximum principle provides a full set of conditions to identify candidates for optimal state-control trajectories. Its use in actual problems requires sometimes a little analytical ingenuity, since the two-point boundary problem for the system of differential equations governing the evolution of $(x(t), \psi(t))$ is not easy to solve, if those equations are coupled. Note also that the optimal control $u^*(t)$ does not have to be continuous, even for problems in which all the primitives (i.e., f and h) are arbitrarily smooth. – For problems in which $T \to \infty$, the *transversality condition* (40) is not guaranteed (even in the limit), and the necessary optimality conditions implied by Pontryagin's maximum principle are not enough to determine candidates for optimal state-control trajectories. By careful extension of the maximum principle to the infinite horizon via successive approximations it is sometimes possible to find bounds on the adjoint variable, which can be used to construct a full set of optimality conditions, cf. Weber (2006).