MGT-621 Microeconomics

Risk Aversion in the Small and in the Large

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Problem Statement. Consider an individual, Joe, who has a decreasing marginal utility for money. In other words, Joe has a concave utility function $u : \mathbb{R} \to \mathbb{R}$. Assume that Joe's current wealth level is w. (i) Show that if Joe *always* (i.e., for *any* wealth level w) turns down a 50/50 bet of losing \$10 vs. winning \$11, then Joe also always turns down a 50/50 bet of losing \$1,000 vs. winning any positive amount x (that could even be infinity!). [Hint: Show first that $u(w+11) - u(w) \leq u(w) - u(w-10)$ for all w implies that dollar $w_0 + 11$ is worth about 10/11 times dollar $w_0 - 10$ for any given reference wealth level w_0. Similarly, dollar $w_0 + 32$ is worth about $(10/11)^2$ times dollar $w_0 - 10$, etc.] (ii) The last part shows that a consistent moderate risk aversion against small bets can imply an enormous risk aversion against even very lucrative large bets. Interpret this "paradox."

Remark. These problems with the concept of risk aversion have been pointed out by Paul Samuelson ("Risk and Uncertainty: A Fallacy of Large Numbers," *Scientia*, 1963, Vol. 98, pp. 108–113) and Matthew Rabin ("Risk Aversion and Expected Utility Theory: A Calibration Theorem," *Econometrica*, 2000, Vol. 68, No. 5, pp. 1281–1292); there is no need for you to look up these papers.

Solution.

(i) Since Joe always turns down a 50/50 bet of losing \$10 vs. winning \$11, we have

$$0.5[u(w+11) + u(w-10)] \leq u(w), \tag{1}$$

or equivalently

$$u(w+11) - u(w) \leq u(w) - u(w-10) \tag{2}$$

for all $w \in \mathbb{R}$. Joe's marginal value of the w-th dollar is measured by $u'(w)$ (i.e., the slope of his utility function at w). Fix any wealth level $w_0 \in \mathbb{R}$. Since u is concave, its slope is decreasing in wealth, so that

$$u'(w_0 + 11) \leq \frac{u(w_0 + 11) - u(w_0)}{11},$$

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and

$$u'(w_0-10) \geq \frac{u(w_0)-u(w_0-10)}{10}.$$

By inequality (1) it is

$$\frac{u(w_0+11)-u(w_0)}{u(w_0)-u(w_0-10)} \leq 1.$$

Therefore, we have

$$\frac{u'(w_0+11)}{u'(w_0-10)} \leq \frac{10}{11} \cdot \frac{u(w_0+11)-u(w_0)}{u(w_0)-u(w_0-10)} \leq \frac{10}{11}. \tag{3}$$

When Joe's wealth increases by \$21 to w_0+21, by virtue of inequality (1) he will still turn down a 50/50 bet of losing \$10 vs. winning \$11. Thus, by (2) we have

$$u(w_0+21+11)-u(w_0+21) \leq u(w_0+21)-u(w_0+21-10),$$

i.e.,

$$u(w_0+32)-u(w_0+21) \leq u(w_0+21)-u(w_0+11).$$

Then using the same method as before, we find that

$$\frac{u'(w_0+32)}{u'(w_0+11)} \leq \frac{10}{11} \tag{4}$$

Combining (3) and (4) yields

$$\frac{u'(w_0+32)}{u'(w_0-10)} = \left(\frac{u'(w_0+11)}{u'(w_0-10)}\right)\left(\frac{u'(w_0+32)}{u'(w_0+11)}\right) \leq \left(\frac{10}{11}\right)\left(\frac{10}{11}\right) = \left(\frac{10}{11}\right)^2.$$

Continuing in this manner, we get

$$0 \leq \frac{u'(w_0+n\cdot 21+11)}{u'(w_0-10)} \leq \left(\frac{10}{11}\right)^{n+1}, \tag{5}$$

for all $n \in \{0,1,2,\ldots\}$. Hence, by taking the limit for $n \to \infty$ on both sides of the last inequality,

$$\lim_{n\to\infty} \frac{u'(w_0+n\cdot 21+11)}{u'(w_0-10)} = \lim_{n\to\infty} \left(\frac{10}{11}\right)^{n+1} = 0,$$

which means that the value of any very large added wealth must be negligible compared with the value of original wealth. By rewriting (5) in terms of $\hat{w}_0 = w_0 - 10$ and then relabelling $\hat{w}_0$ back to w_0, we can see that (5) is equivalent to

$$\frac{u'(w_n)}{u'(w_0)} \leq \left(\frac{10}{11}\right)^n, \tag{6}$$

where $w_n = w_0 + 21n$ for any integer n. Note first that

$$u(w_n) \leq u(w_0)+21\sum_{n=0}^{\infty} u'(w_n) \leq u(w_0)+21\sum_{n=0}^{\infty}\left(\frac{10}{11}\right)^n u'(w_0) = u(w_0)+21\cdot 11\cdot u'(w_0)$$

for all n. On the other hand, it is $u(w) \geq u(w_{-47}) + 21\,(u'(w_{-46}) + \cdots + u'(w_0))$, and also $u'(w_{-n}) \geq (11/10)^n u'(w_0)$. Hence,

$$u(w_0) \geq u(w_{-47}) + 21 \sum_{n=0}^{46} \left(\frac{11}{10}\right)^n u'(w_0) > u(w_{-47}) + 21 \cdot 11 \cdot u'(w_0).$$

By taking the limit for $n \to \infty$ we have therefore shown that

$$u(\infty) - u(w_0) \leq 21 \cdot 11 \cdot u'(w_0) < u(w_0) - u(w_0 - 1000),$$

whence

$$0.5\,[u(\infty) + u(w_0 - 1000)] < u(w_0),$$

i.e., Joe always turns down a 50/50 bet of losing \$1,000 vs. winning any positive amount x.

(ii) This 'paradox' occurs, because Joe is averse to small bets for all wealth levels. Since the utility function is concave, the marginal utility for money has to decrease very quickly, which implies that the value of large amounts of money has to deteriorate very fast, too. The increased utility from winning a very large amount of money will not be able to offset the deceased utility from losing a sizable (\$1,000) bet even though the expected gain from this gamble can be arbitrarily high, such that the expected utility will drop far below the original utility level. In particular the utility function is bounded. As a result, Joe will reject such a bet.