MGT 621 – MICROECONOMICS

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8. (Optional) *General Equilibrium, Part II*

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AGENDA

Some Preliminaries

Fundamental Welfare Theorems

Existence of a Competitive Equilibrium

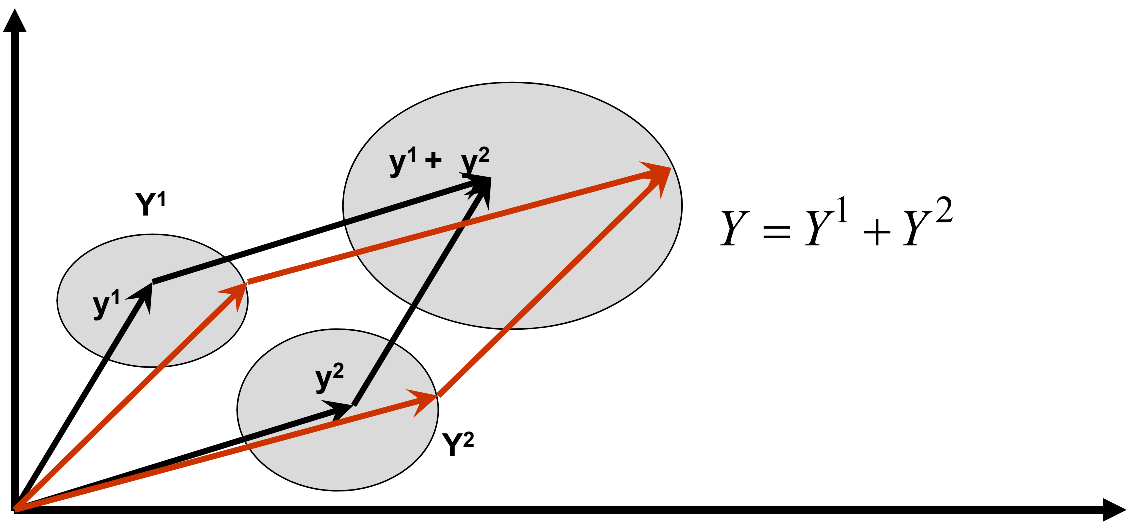
General Equilibrium vs. Partial Equilibrium

Key Concepts to Remember

CONCEPT OF SET SUMMATION

Set summation of set Y¹ and set Y²: $Y = Y^1 + Y^2 = \{y : y = y^1 + y^2, y^1 \in Y^1, y^2 \in Y^2\}$

Intuition. Choose any point, y^1 from set Y^1 and any point y^2 from set Y^2; the set Y consists of the set of all points $y^1 + y^2$.



FEASIBLE TOTAL OUTPUT IN THE ECONOMY

Initial Endowment of commodities by consumer c: ω^c

Net output by firm f: $y^f \in Y^f$

→ Total Supply: $\sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} y^f$

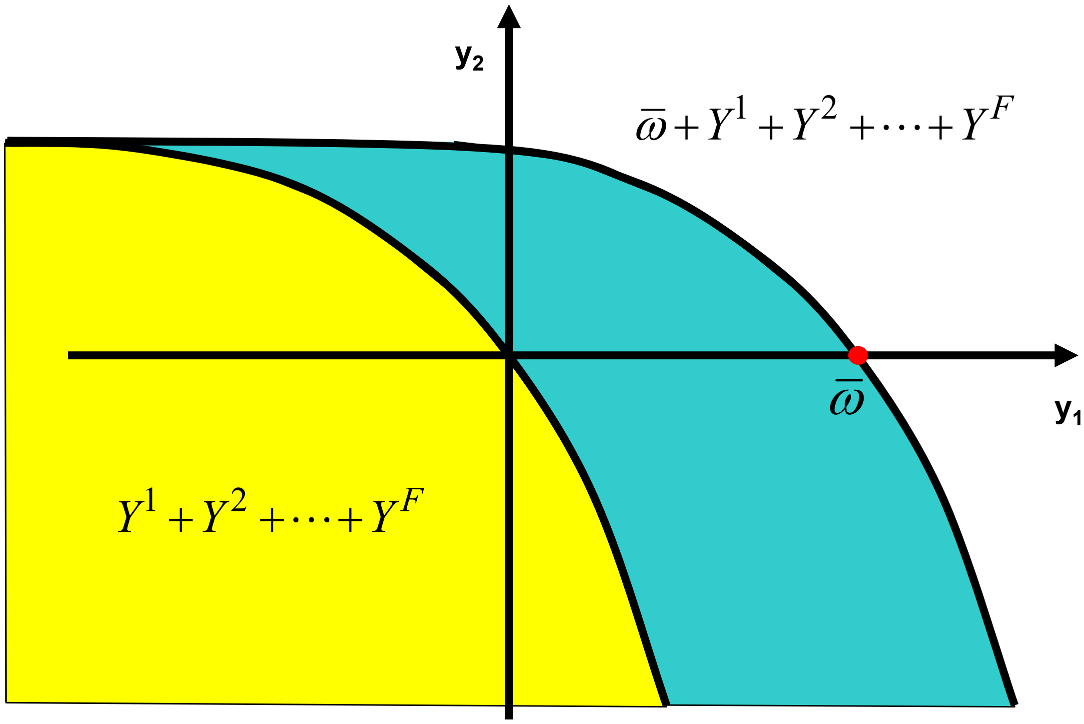
Define total Initial Endowments $\bar{\omega} = \sum_{c=1}^{C} \omega^c$

Then, feasible set of total outputs is

$$Y = \bar{\omega} + Y^1 + Y^2 + \cdots + Y^F$$

Remark. Note that this depends on the assumption that there are no externalities in production. If there are externalities, then the feasible set of total outputs is not set summation of individual production sets.

TOTAL FEASIBLE OUTPUT IN THE ECONOMY (Cont'd)

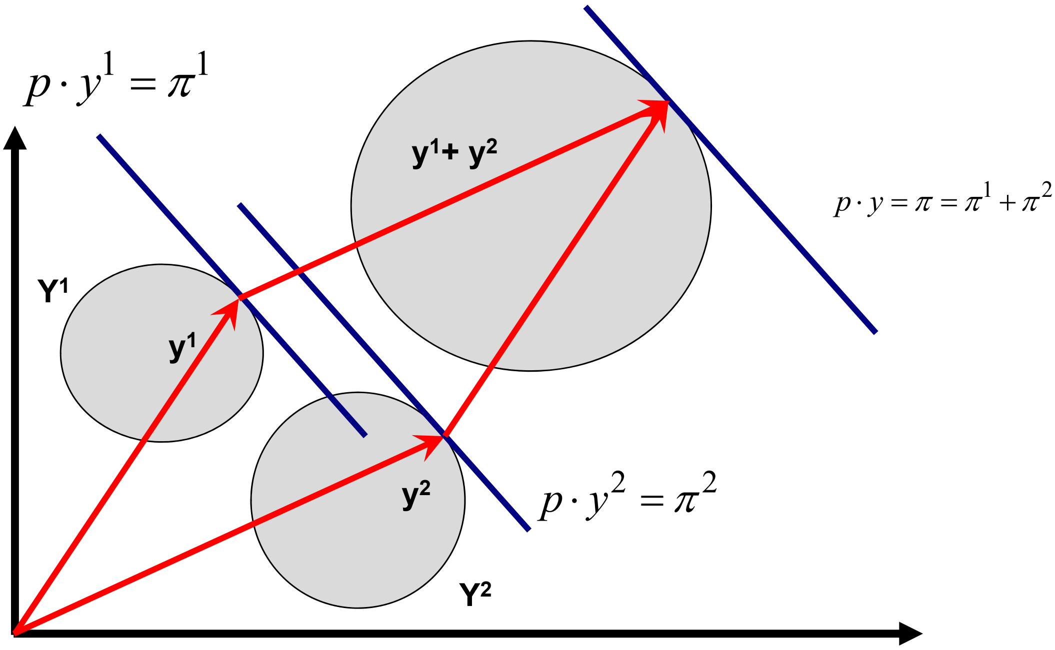


VALUING TOTAL OUTPUT AT MARKET PRICES

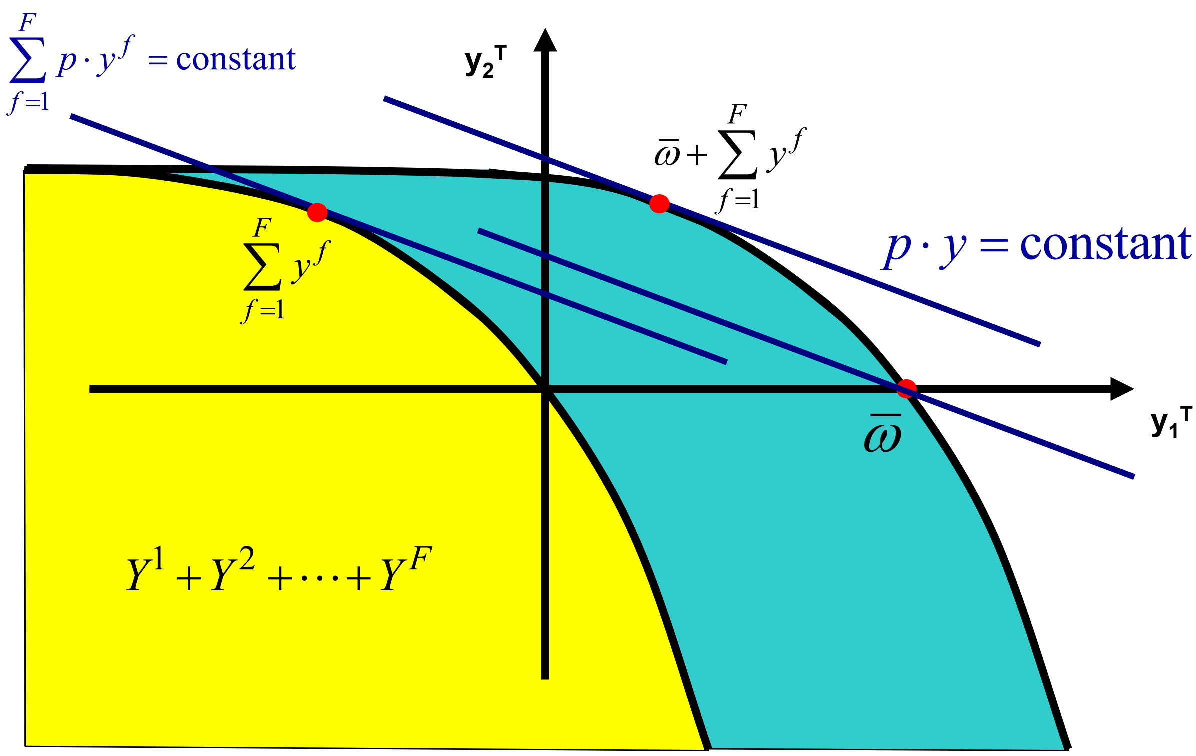
$$p \cdot y = \sum_{c=1}^{C} p \cdot \omega^c + \sum_{f=1}^{F} p \cdot y^f$$

Value of Total Output in Economy = Value of Initial Endowments + Sum of Firms' Profits

INDIVIDUAL MAXIMIZATION IMPLIES GLOBAL MAXIMIZATION



MAXIMUM VALUE OF TOTAL OUTPUT



CONSUMER CHOICE

Consumer maximizes utility, subject to budget constraint

$$u^{c*} = \max \ u^c(x^c)$$

$$\text{s.t. } p \cdot x^c \le w^c$$

Equivalently, consumer minimizes expenditure for achieving a certain utility level

$$w^{c*} = \min \ p \cdot x^c$$

$$\text{s.t. } u^c(x^c) \ge U^{c*}$$

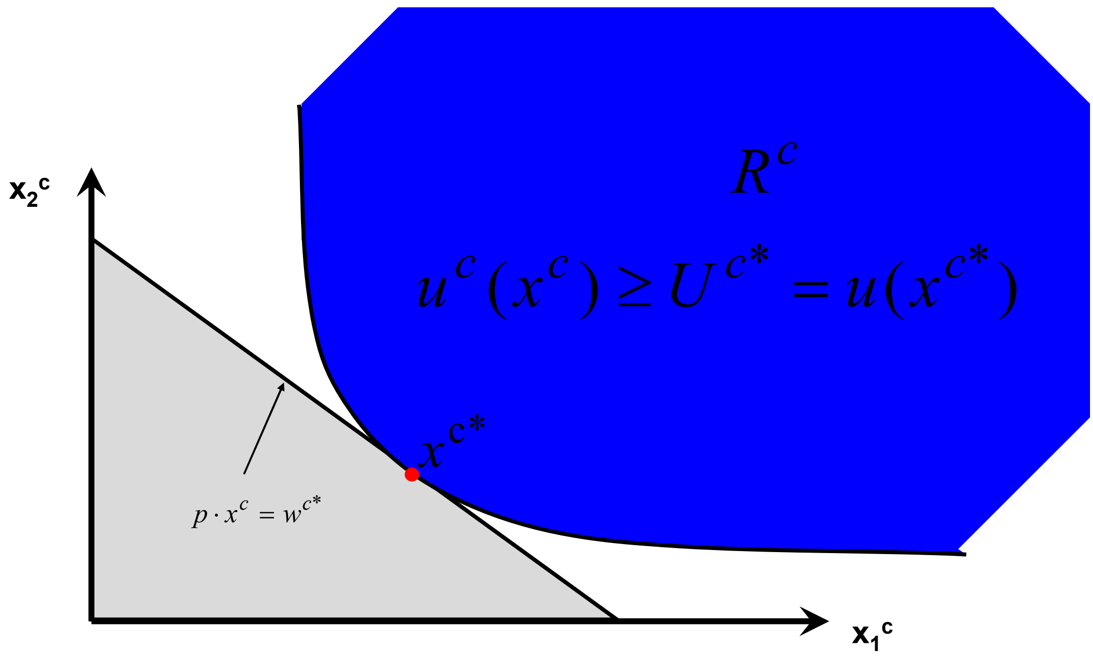
CONSUMER CHOICE (Cont'd)

Define set of consumption bundles weakly preferred to optimal choice as preference set for consumer c: R^c

Then consumer minimizes expenditure, given x^c in R^c

$$\min_{x^c \in R^c} \ p \cdot x^c$$

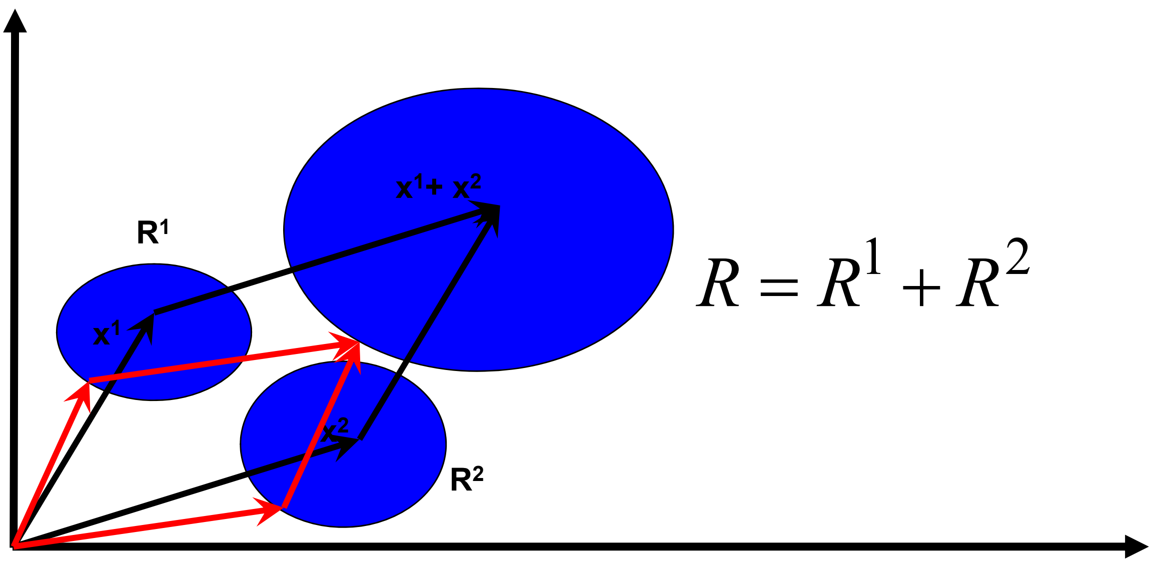
CONSUMER CHOICE (Cont'd)



SET SUMMATION OF INDIVIDUAL PREFERENCE SETS

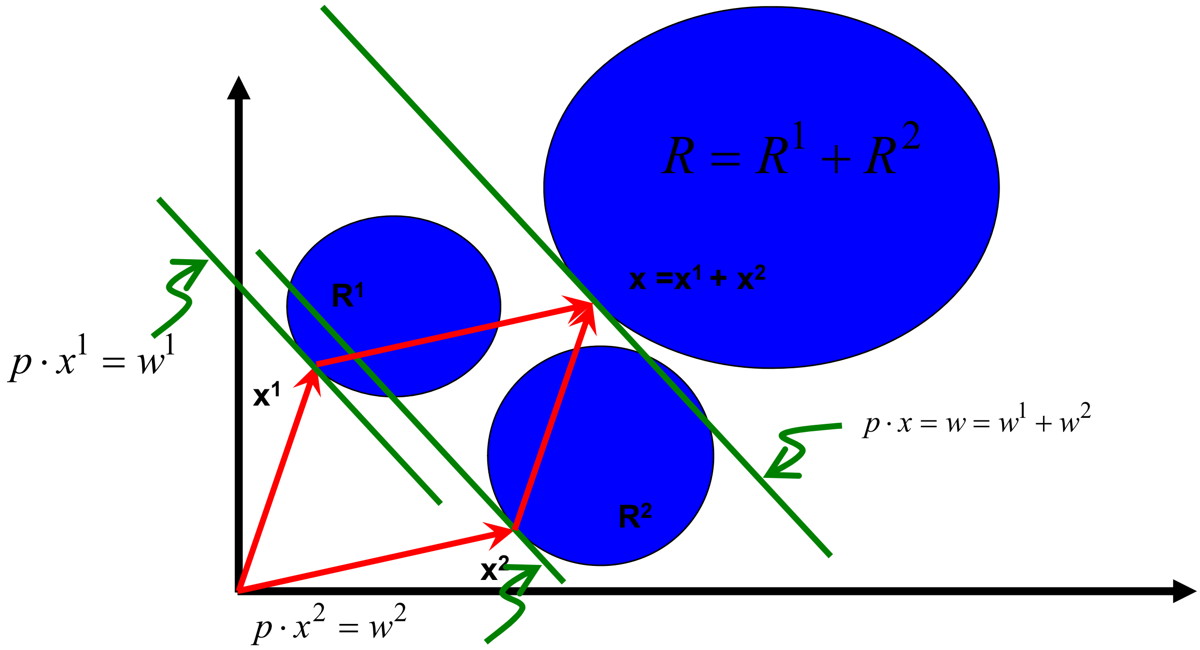
Set summation of individual preference sets is the set of total consumption bundles that allows each consumer to have utility at least as high as his/her U^{c*}. R is the aggregate preference set.

Total consumption in interior of R could allow Pareto-superior allocations to consumers.

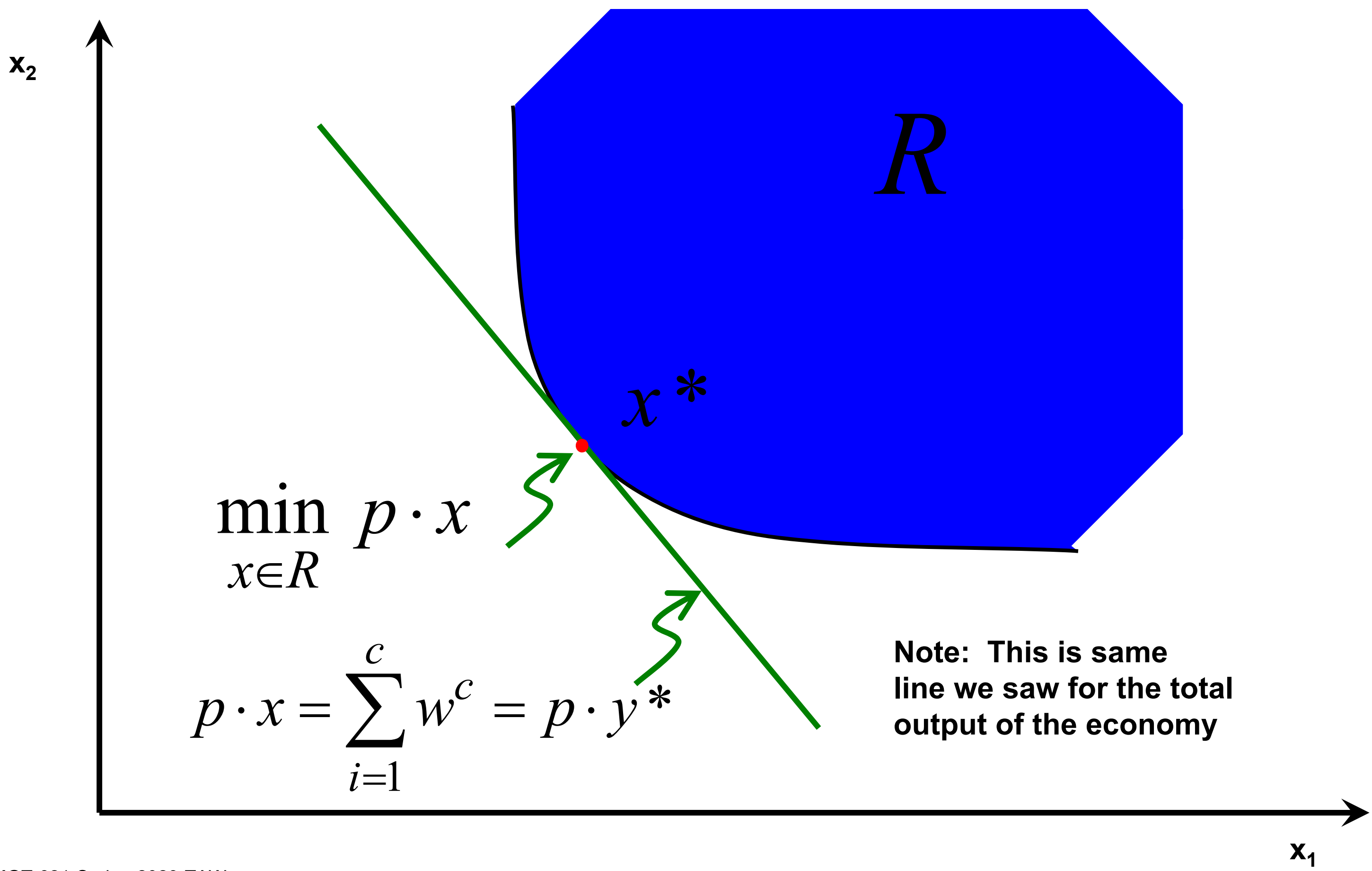


Remark. Note that this depends on the assumption that there are no externalities in consumption. If there are externalities, then the aggregate preference set is <u>not</u> set summation of individual preference sets.

INDIVIDUAL MINIMIZATION IMPLIES GLOBAL MINIMIZATION



MINIMIZATION OF TOTAL EXPENDITURE



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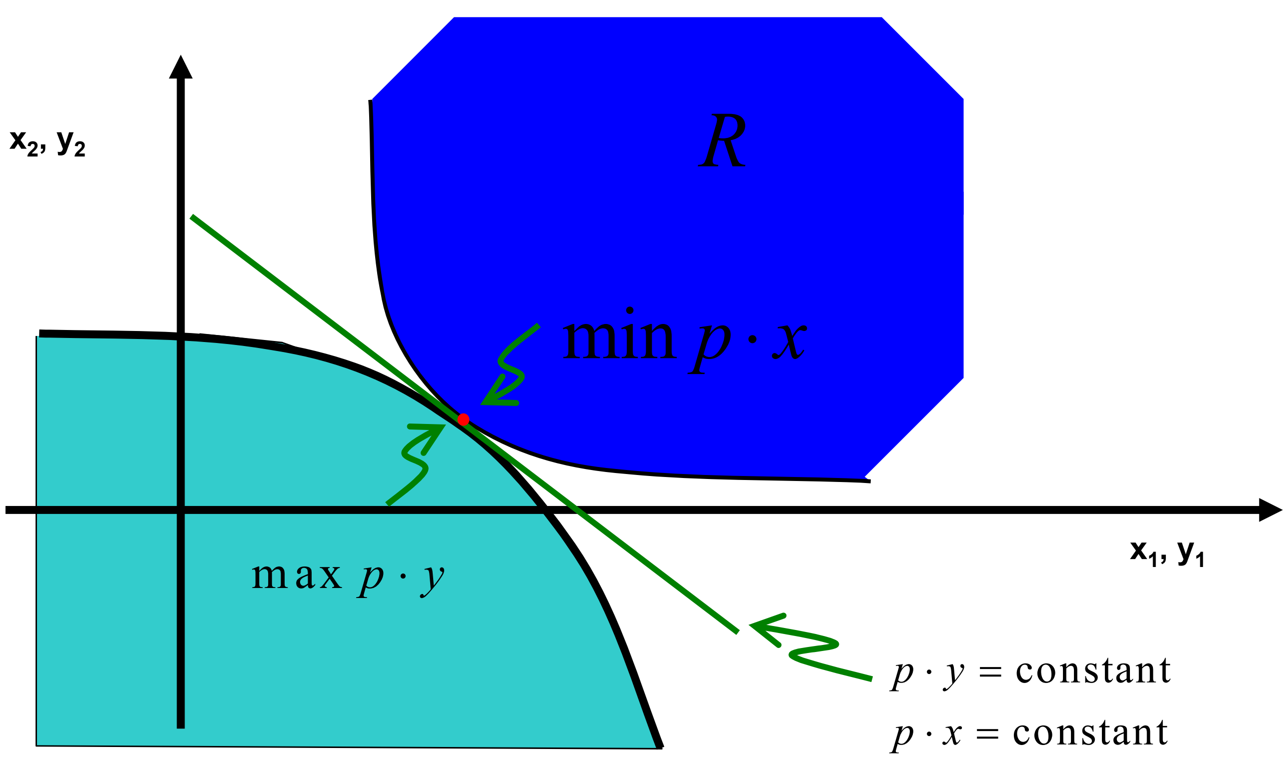
General Equilibrium vs. Partial Equilibrium

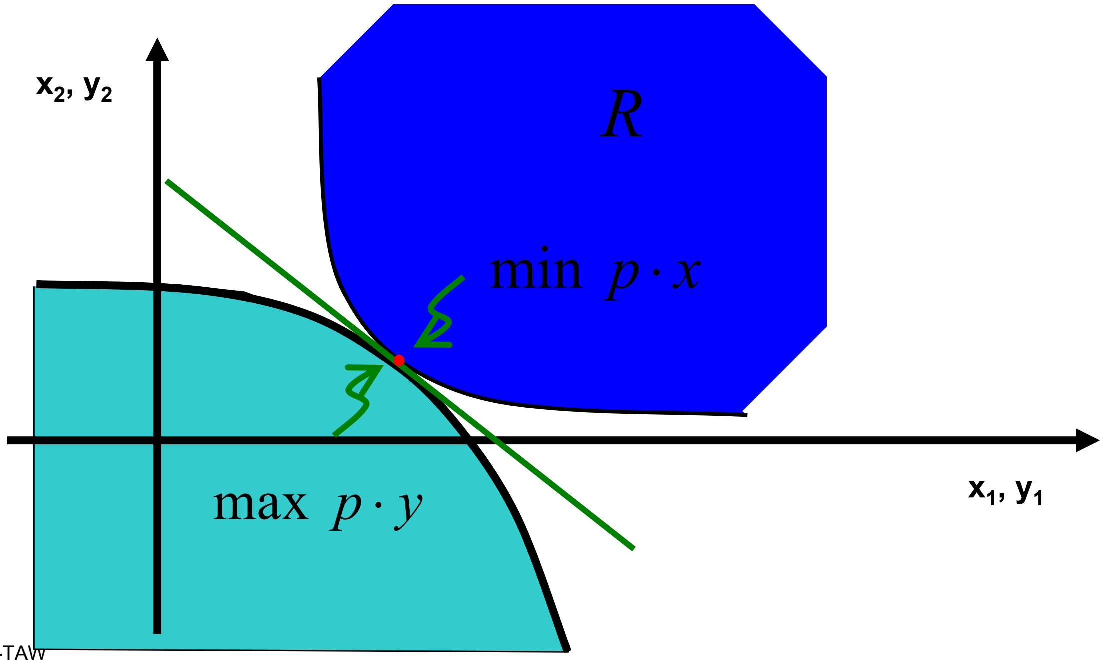
Key Concepts to Remember

COMPETITIVE EQUILIBRIUM MATCHES SUPPLY AND DEMAND

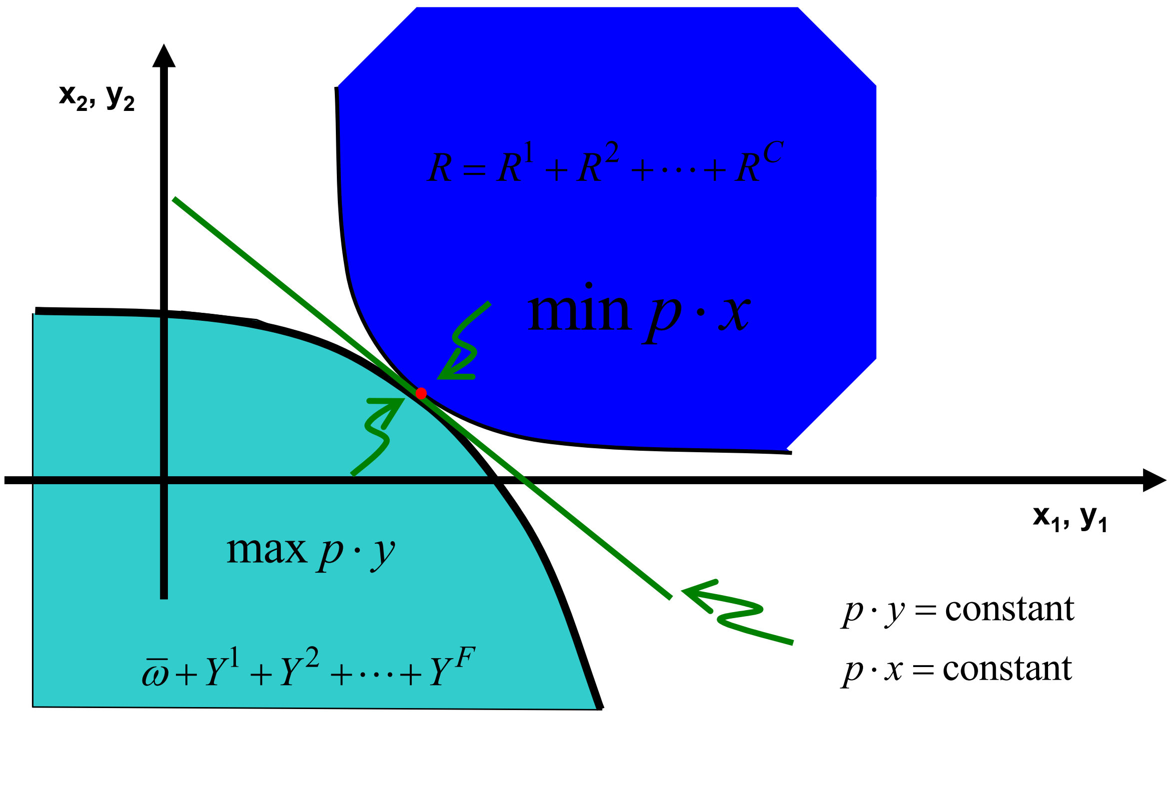


FIRST FUNDAMENTAL WELFARE THEOREM

Competitive equilibrium implies set of economy-wide feasible outputs is separated from aggregate preference set, the set of points that allow Pareto-dominant allocation (neither set includes interior point of other set). Therefore, the competitive market equilibrium must be a Pareto-optimal allocation



COMPETITIVE EQUILIBRIUM IS PARETO OPTIMAL



FIRST FUNDAMENTAL WELFARE THEOREM

Definition: Assume that consumer c's preferences are representable by a continuous utility function $u_c(\cdot)$. His preferences are locally nonsatiated *if for any feasible consumption vector* $x^c \in \Re_+^N$ *and any* $\varepsilon > 0$ *there exists another feasible consumption vector* $\hat{x}^c \in U_\varepsilon(x^c) = \{y \in \Re_+^N : \| y - x^c \| < \varepsilon\}$ *such that* $u_c(\hat{x}^c) > u_c(x^c)$.[(1)]

Theorem (1st FWT): Assume that for all consumers $c \in \{1,\dots,C\}$ the utility function is locally nonsatiated. If $(p,(\hat{x}^1,\dots,\hat{x}^C),(\hat{y}^1,\dots,\hat{y}^F))$ is a Walrasian equilibrium, then the allocation $((\hat{x}^1,\dots,\hat{x}^C),(\hat{y}^1,\dots,\hat{y}^F))$ is Pareto optimal.

(1) *Local nonsatiation is implied by strict monotonicity of consumer c's utility function*. The converse is not true since some of the components of the consumption vector may not be desirable, i.e., it may contain "bads" instead of "goods". (However, it is not possible that all consumption goods are "bads," since then at 0 would become a (global) *satiation point*.)

FIRST FUNDAMENTAL WELFARE THEOREM
Proof

Proof: [*by contradiction*]

Suppose that $((x^1,\dots,x^C),(y^1,\dots,y^F))$ is a feasible allocation, such that for all $c \in \{1,\dots,C\}$:

$$u_c(x^c) \geq u_c(\hat{x}^c) \qquad \textbf{(1)}$$

and for some c, say $c = c'$, we have a strict inequality. Then, necessarily (by utility maximization), it is

$$p \cdot x^{c'} > p \cdot \hat{x}^{c'} \qquad \textbf{(2)}$$

and local nonsatiation implies that as a consequence of (1), for all $c \in \{1,\dots,C\}$:[(1)]

$$p \cdot x^c \geq p \cdot \hat{x}^c$$

Hence, using (2),

$$p \cdot \left(\sum_{c=1}^{C} x^c \right) = \sum_{c=1}^{C} (p \cdot x^c) > \sum_{c=1}^{C} (p \cdot \hat{x}^c) = p \cdot \left(\sum_{c=1}^{C} \hat{x}^c \right) \qquad \textbf{(3)}$$

(1) Otherwise, any $\breve{x}^c \in \Re_+^N$ sufficiently close to x^c must satisfy $p \cdot \breve{x}^c < p \cdot \hat{x}^c$. But by local nonsatiation, there must exist at least one such $\breve{x}^c$ for which also $u_c(\breve{x}^c) > u_c(x^c)$. By transitivity this implies $u_c(\breve{x}^c) > u_c(\hat{x}^c)$, which contradicts the assumption that $\hat{x}^{c}$ solves the utility maximization problem as part of a Walrasian equilibrium. If some consumers were local satiated one may be able to transfer small amounts of money from consumers that are locally indifferent to a consumer that cares at the margin.

FIRST FUNDAMENTAL WELFARE THEOREM
Proof (cont'd)

Feasibility of the WE (i.e., demand = supply) implies

$$\sum_{c=1}^{C} x^c = \sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} y^f \quad \textbf{(4)}$$

and

$$\sum_{c=1}^{C} \hat{x}^c = \sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} \hat{y}^f \quad \textbf{(5)}$$

Combining (3)—(5) we obtain

$$p \cdot \left(\sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} y^f \right) > p \cdot \left(\sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} \hat{y}^f \right)$$

whence

$$p \cdot \left(\sum_{f=1}^{F} y^f \right) > p \cdot \left(\sum_{f=1}^{F} \hat{y}^f \right) \quad \textbf{(6)}$$

FIRST FUNDAMENTAL WELFARE THEOREM
Proof (cont'd)

Since the allocation $((x^1,\ldots,x^C),(y^1,\ldots,y^F))$ is by assumption feasible, we have that $y^f \in Y_f$ for all $f \in \{1,\ldots,F\}$. Profit maximization implies that for all $f \in \{1,\ldots,F\}$:

$$p \cdot \hat{y}^f \geq p \cdot y^f$$

But then it must be true that

$$p \cdot \left(\sum_{f=1}^{F} \hat{y}^f \right) \geq p \cdot \left(\sum_{f=1}^{F} y^f \right)$$

which contradicts (6). QED

SECOND FUNDAMENTAL WELFARE THEOREM

The common point is the competitive equilibrium, since

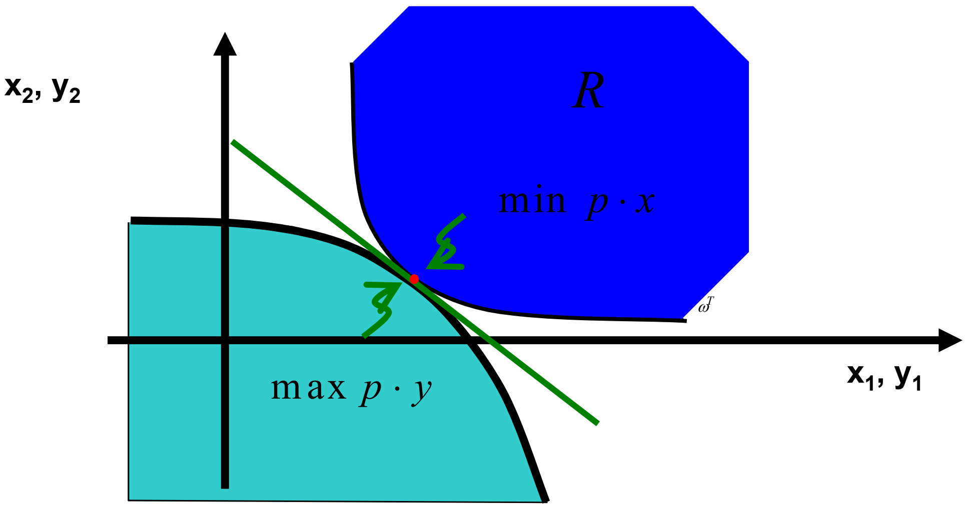
(1) it minimizes expenditure,

(2) it maximizes profits,

(3) it has all supplies equal to all demands, and

(4) it has all profits allocated to consumers.

However, wealth is not necessarily consistent with initial endowments. Thus, a lump-sum wealth redistribution is likely to be required.



SECOND FUNDAMENTAL WELFARE THEOREM

Theorem (2nd FWT): Assume that for all consumers $c \in \{1,...,C\}$ the utility function is locally nonsatiated, continuous, and has convex upper contour sets. Let $\overline{\omega} \in \Re_+^N$ be some vector of initial resources (endowments). (i) If, *starting from* $\overline{\omega}$, the allocation $((\hat{x}^1,...,\hat{x}^C),(\hat{y}^1,...,\hat{y}^F))$ is Pareto optimal, then there exists a price vector $p \in \Re_+^N$ such that

- **for all $c \in \{1,...,C\}$:** $u_c(x^c) \geq u_c(\hat{x}^c) \Rightarrow p \cdot x^c \geq p \cdot \hat{x}^c$
- **for all $f \in \{1,...,F\}$:** $y^f \in Y_f \Rightarrow p \cdot \hat{y}^f \geq p \cdot y^f$

(ii) If, *in addition*, for all $c \in \{1,...,C\}$ there exists a vector $\overline{x}_c \in \Re_+^N$ such that $p \cdot \hat{x}^c > p \cdot \overline{x}^c$, then there is a division of initial resources $\overline{\omega}$, $(\omega^1,...,\omega^C)$, and of firm ownership shares, $(\theta^1,...,\theta^C)$, such that $(p,(\hat{x}^1,...,\hat{x}^C),(\hat{y}^1,...,\hat{y}^F))$ is a Walrasian equilibrium relative to $(\omega^1,...,\omega^C)$ and $(\theta^1,...,\theta^C)$.

SEPARATING HYPERPLANE THEOREM

Definition: **A plane** $P = \{x \in X : f(x) = 1\}$ **separates two sets** $A, B \subset X$**, if**

$$x \in A \Rightarrow f(x) \le 1$$
$$x \in B \Rightarrow f(x) \ge 1$$

Hahn-Banach Theorem: Let A **and** B **be *two disjoint nonempty convex sets in a vector space*** X**. If** A **has an inner point, then there exists a plane** P **separating** A **and** B**.**[(1)]

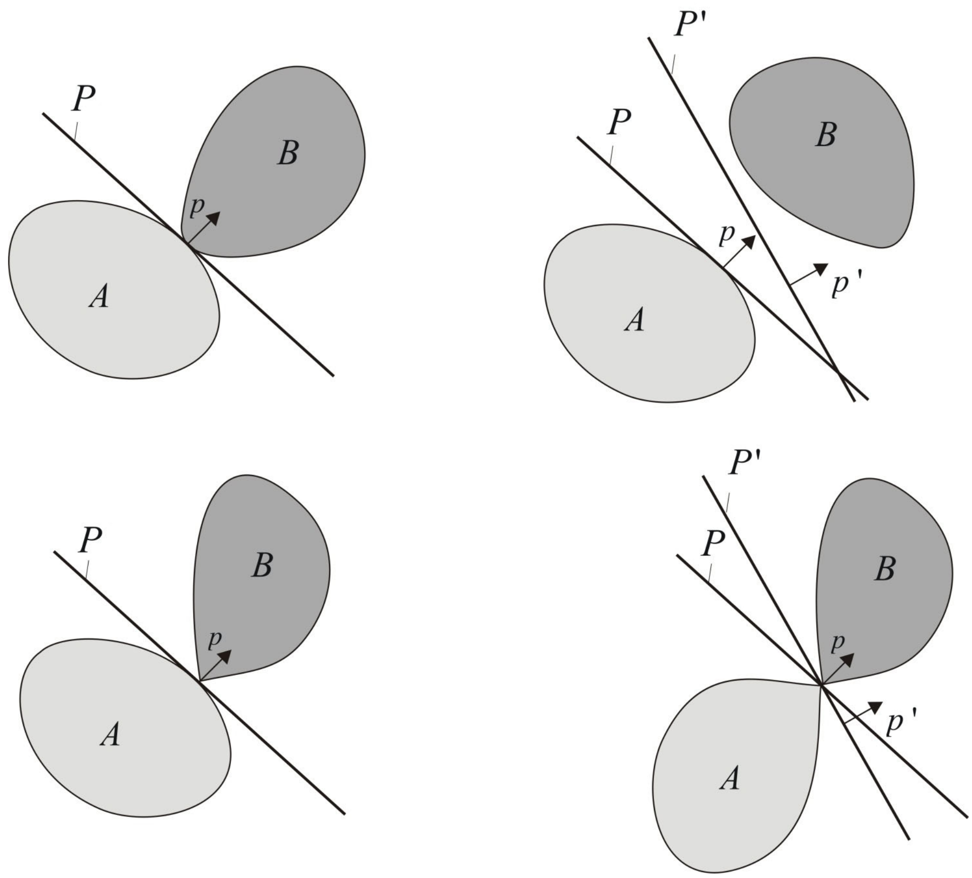
Separating Hyperplane Theorem: Let $A, B \subset \Re^N$ **be *two disjoint nonempty convex sets*. Then there exists a nonzero vector** $p \in \Re^N$ **and a scalar** $\alpha \in \Re$ **such that**

$$p \cdot x \le \alpha \le p \cdot y$$

for any $(x, y) \in A \times B$**.**[(2)]

(1) For a proof of this theorem, see e.g., Berge, C. (1963), "Topological Spaces," Oliver & Boyd, Edinburgh and London, UK, pp. 154—157. Reprinted by Dover Publications in 1997.
(2) In other words, it is possible to select a *linear form* f in the Hahn-Banach theorem. For a proof of that theorem, see MWG, p. 948.

SEPARATING HYPERPLANE THEOREM
Geometric Interpretation



SECOND FUNDAMENTAL WELFARE THEOREM
Proof

Proof: [*proceeds in 7 steps*]

***Step 1*: Apply the Separating Hyperplane Theorem**

For all consumers $c \in \{1,...,C\}$, the set of preferred allocations (upper contour set),

$$V^c(\hat{x}^c) = \{x^c \in \Re_+^N : u_c(x^c) > u_c(\hat{x}^c)\}$$

is convex. As a result, $V = \sum_{c=1}^{C} V^c(\hat{x}^c)$ is convex. Similarly, convexity of the production set Y_f for all $f \in \{1,...,F\}$ implies that

$$Y = \sum_{f=1}^{F} Y_f + \{\overline{\omega}\}$$

is convex. By assumption we know that the allocation $((\hat{x}^1,...,\hat{x}^C),(\hat{y}^1,...,\hat{y}^F))$ is Pareto optimal, i.e.,

$$\left(\sum_{c=1}^{C} V^c(\hat{x}^c)\right) \cap \left(\sum_{f=1}^{F} Y_f + \{\overline{\omega}\}\right) = V \cap Y = \varnothing$$

In other words, *there is nothing that the economy can produce that makes everybody better off*.

SECOND FUNDAMENTAL WELFARE THEOREM
Proof (cont'd)

The separating hyperplane theorem implies that for any $(x,y) \in V \times Y$ there exists a vector p and a scalar α, such that $p \cdot y \le \alpha \le p \cdot x$

***Step 2*: Show that** $p \cdot \left(\sum_{f=1}^{F} \hat{y}^f + \overline{\omega}\right) = p \cdot \left(\sum_{c=1}^{C} \hat{x}^c\right) = \alpha$

Since $((\hat{x}^1,...,\hat{x}^C),(\hat{y}^1,...,\hat{y}^F))$ is feasible, we have $\sum_{c=1}^{C} \hat{x}^c = \sum_{f=1}^{F} \hat{y}^f + \overline{\omega} \in Y$, so that by Step 1:

$$\alpha \ge p \cdot \left(\sum_{f=1}^{F} \hat{y}^f + \overline{\omega}\right) = p \cdot \left(\sum_{c=1}^{C} \hat{x}^c\right)$$

Now, for each $c \in \{1,...,C\}$ and $n \ge 1$, let

$$\widehat{x}^c(n) = \hat{x}^c + \left(\frac{1}{n},...,\frac{1}{n}\right)$$

By local nonsatiation(1) it is $\widehat{x}^c(n) \in V^c(\hat{x}^c)$ and thus $\sum_{c=1}^{C} \widehat{x}^c(n) \in V$. Hence by Step 1:

$$p \cdot \left(\sum_{c=1}^{C} \widehat{x}^c(n)\right) \ge \alpha$$

(1) Actually we are using monotonicity here. For a justification see Step 3, where the same construction is used.

SECOND FUNDAMENTAL WELFARE THEOREM
Proof (cont'd)

Taking the limit for $n \to \infty$ **gives thus** $\alpha \le \lim_{n\to\infty} p \cdot \left(\sum_{c=1}^{C} \hat{x}^c(n) \right) = p \cdot \left(\sum_{c=1}^{C} \hat{x}^c \right) = p \cdot \left(\sum_{f=1}^{F} \hat{y}^f + \overline{\omega} \right)$

***Step 3*: Show that** $x \in \overline{V} \Rightarrow p \cdot \left(\sum_{c=1}^{C} \hat{x}^c \right) \le p \cdot x$**, where** $\overline{V} = \sum_{c=1}^{C} \underbrace{\{x^c \in \Re_+^N : u_c(x^c) \ge u_c(\hat{x}^c)\}}_{\overline{V}^c(\hat{x}^c)}$

For simplicity, let us assume here that all the commodities are desirable, so that local nonsatiation is equivalent to monotonicity of the consumers' utility functions. For any $x^c \in \overline{V}$ **let**

$$x^c(n) = x^c + \left(\frac{1}{n}, \ldots, \frac{1}{n} \right)$$

so that by monotonicity, $x^c(n) \in V^c(\hat{x}^c)$ **and** $\sum_{c=1}^{C} x^c(n) \in V$

Hence, by Step 1, $p \cdot \left(\sum_{c=1}^{C} x^c(n) \right) \ge \alpha$**, so that after taking the limit for** $n \to \infty$ **we obtain**

$$p \cdot \left(\sum_{c=1}^{C} \hat{x}^c \right) = \alpha \le \lim_{n\to\infty} p \cdot \left(\sum_{c=1}^{C} x^c(n) \right) = p \cdot \left(\sum_{c=1}^{C} x^c \right) = p \cdot x$$

SECOND FUNDAMENTAL WELFARE THEOREM
Proof (cont'd)

***Step 4*:** $y \in Y \Rightarrow p \cdot \left(\sum_{f=1}^{F} \hat{y}^f + \overline{\omega} \right) \ge p \cdot y$ **and thus(1)** $y^f \in Y_f \Rightarrow p \cdot \hat{y}^f \ge p \cdot y^f$

***Step 5*:** $x \in \overline{V} \Rightarrow p \cdot \left(\sum_{c=1}^{C} \hat{x}^c \right) \le p \cdot x$ **and thus(1)** $x^c \in \overline{V}^c(\hat{x}^c) \Rightarrow p \cdot \hat{x}^c \le p \cdot x^c$

***Step 6*: Show that: if for all consumers** $c \in \{1, \ldots, C\}$ **there exists a vector** $\overline{x}^c \in \Re_+^N$ **such that** $p \cdot \overline{x}^c < p \cdot \hat{x}^c$**, then** $u_c(x^c) > u_c(\hat{x}^c) \Rightarrow p \cdot \hat{x}^c < p \cdot x^c$ **.**

By Step 5, $p \cdot \overline{x}^c < p \cdot \hat{x}^c$ **implies that** $u_c(\hat{x}^c) > u_c(\overline{x}^c)$ **and (since** $u_c(x^c) > u_c(\hat{x}^c)$**) also** $p \cdot \hat{x}^c \le p \cdot x^c$ **. Thus,** $p \cdot \overline{x}^c < p \cdot (\beta x^c + (1-\beta)\overline{x}^c) < p \cdot x^c$ **for any** $\beta \in (0,1)$ **.**

By the continuity of $u_c(\cdot)$ **there is a** $\beta \in (0,1)$ **such that** $u_c(\hat{x}^c) = u_c\left(\beta x^c + (1-\beta)\overline{x}^c\right)$**.**

But $\beta x^c + (1-\beta)\overline{x}^c \in \overline{V}^c(\hat{x}^c)$ **, so that by Step 5** $p \cdot \hat{x}^c \le p \cdot (\beta x^c + (1-\beta)\overline{x}^c)$**, and thus** $p \cdot \hat{x}^c \le p \cdot (\beta x^c + (1-\beta)\overline{x}^c) < p \cdot x^c$ **as claimed.**

(1) Just consider the inequality for each consumer/producer individually by setting all other components to $\hat{x}^c$ or $\hat{y}^f$, so that they cancel out.

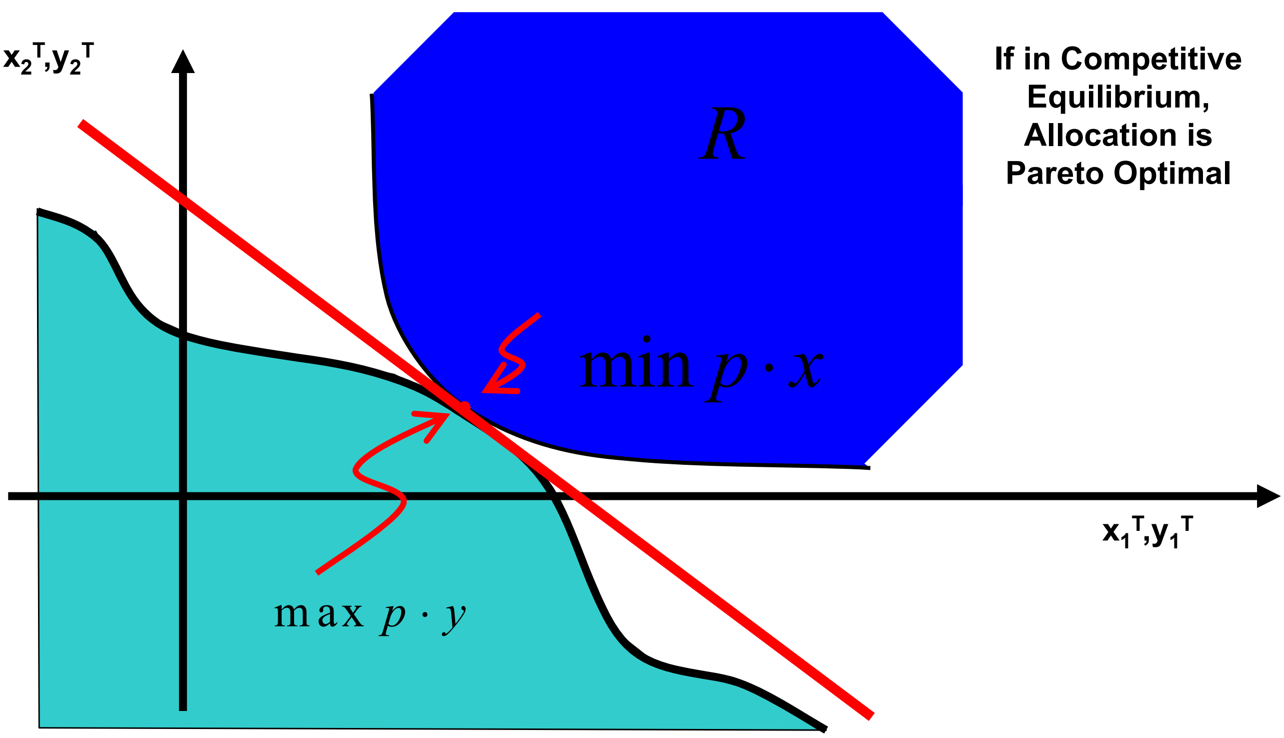
SECOND FUNDAMENTAL WELFARE THEOREM
Proof (cont'd)

Step 7: **It is now enough to choose a division of the initial endowment** $\omega, (\omega^1, ..., \omega^C)$**, and of firm ownership shares,** $(\theta^1, ..., \theta^C)$**, such that**

$$p \cdot \hat{x}^c = p \cdot \omega^c + \sum_{f=1}^{F} \theta_f^c (p \cdot \hat{y}^f)$$

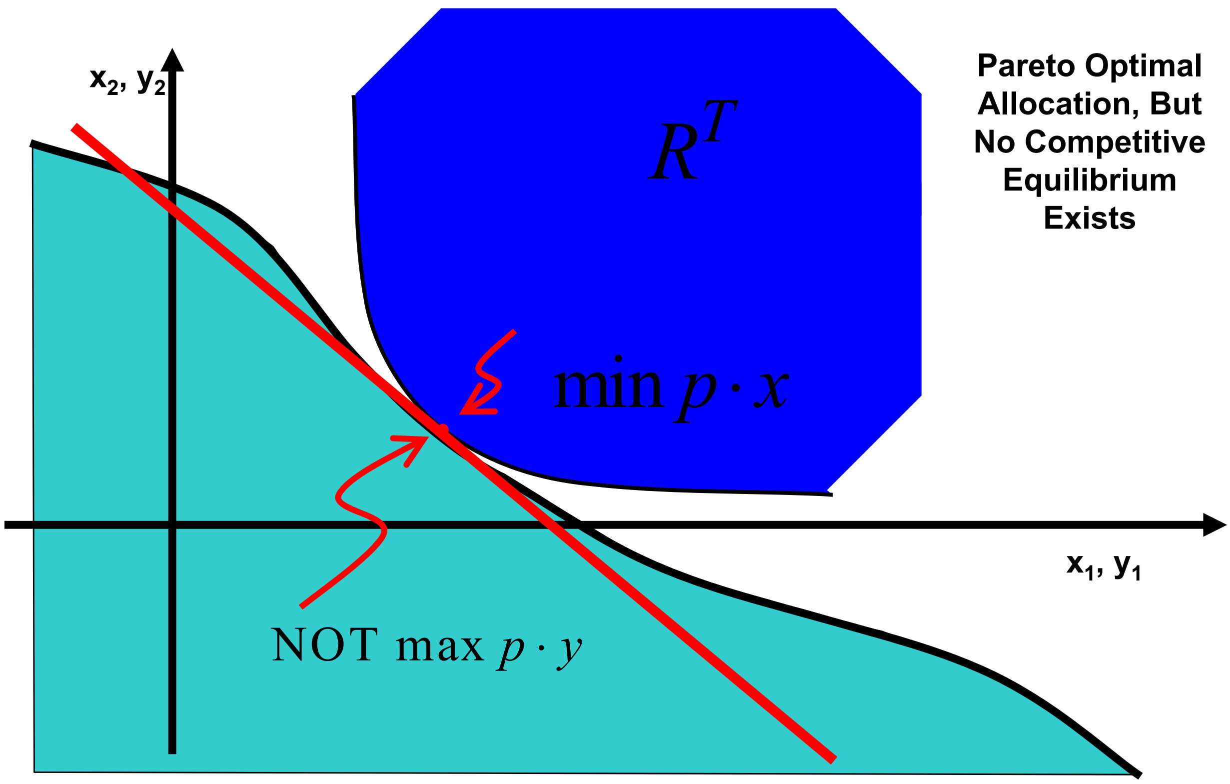
which completes our proof. QED

NONCONVEXITY (1st FWC)



If in Competitive Equilibrium, Allocation is Pareto Optimal

NONCONVEXITY (2nd FWC)



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HOMOGENEOUS FUNCTIONS

Definition: A function $g : \Re^N_+ \to \Re$ is homogeneous of degree k, if for any $\lambda > 0$ and $x \in \Re^N_+$:

$$g(\lambda x) = \lambda^k g(x)$$

Examples:

- **The supply function** $y^f(p) = \arg\max_{y \in Y_f} p \cdot y$

  ***is homogeneous of degree zero*. Indeed, for any $\lambda > 0$ we have that**

$$y^f(p) = \arg\max_{y \in Y_f} p \cdot y = \arg\max_{y \in Y_f} \{(\lambda p) \cdot y\}$$

- **The profit function** $\pi^f(p) = \max_{y \in Y_f} p \cdot y$ **is also *homogeneous of degree one*, since**

$$\lambda \pi^f(p) = \lambda \max_{y \in Y_f} p \cdot y = \max_{y \in Y_f} \{(\lambda p) \cdot y\} = \pi^f(\lambda p)$$

EXCESS DEMAND

Definition: The excess demand function for consumer $c \in \{1, \ldots, C\}$ is

$$z^c(p) = x^c(p, I^c) - \omega^c$$
$$= x^c(p, p \cdot \omega^c + \sum_{f=1}^{F} \theta^c_f \pi^f(p)) - \omega^c$$

Summing up over all consumers and subtracting the firms' production, the function

$$z(p) = \sum_{c=1}^{C} z^c(p) - \sum_{f=1}^{F} y^f(p)$$

denotes excess market demand (also referred to *aggregate excess demand function*).

Exercise: Show that the excess demand function *and* the excess market demand are homogeneous of degree zero.

WALRAS' LAW

Proposition (Walras' Law): For any price vector p the value of excess market demand is zero, i.e.,

$$p \cdot z(p) = 0$$

Proof: Consumer c's budget constraint implies that

$$p \cdot z^c(p) = p \cdot x^c(p, p \cdot \omega^c + \sum_{f=1}^{F} \theta_f^c \pi^f(p)) - p \cdot \omega^c = \sum_{f=1}^{F} \theta_f^c \pi^f(p)$$

Adding up over all consumers and subtracting the firms' production yields

$$p \cdot z(p) = p \cdot \left[\sum_{c=1}^{C} z^c(p) - \sum_{f=1}^{F} y^f(p) \right] = \sum_{f=1}^{F} \sum_{c=1}^{C} \theta_f^c \pi^f(p) - \sum_{f=1}^{F} p \cdot y^f(p)$$

$$= \sum_{f=1}^{F} \pi^f(p) - \sum_{f=1}^{F} p \cdot y^f(p) = \sum_{f=1}^{F} p \cdot y^f(p) - \sum_{f=1}^{F} p \cdot y^f(p) = 0$$

QED

EXISTENCE OF A WALRASIAN EQUILIBRIUM

Proposition: Assume that the supply function $y^f(p)$ and the (finite) demand function

$$x^c(p, p \cdot \omega^c + \sum_{f=1}^{F} \theta_f^c \pi^f(p)) < \infty$$

exist for all $c \in \{1,\ldots,C\}$, $f \in \{1,\ldots,F\}$, and all $p \in \Delta = \{\hat{p} \in [0,1] : \sum_{i=1}^{N} \hat{p}_i = 1\}$ Suppose further that

- **The production sets Y_f are closed, bounded, and strictly convex**
- **The utility functions $u_c(\cdot)$ are continuous, locally nonsatiated, and with strictly convex upper contour sets $V^c(\cdot)$**

Then there exists a price vector $p^* \in \Delta$ such that excess market demand is zero, i.e., $z(p^*) = 0$ (*this price supports a WE*)

EXISTENCE OF A WALRASIAN EQUILIBRIUM
Proof

Proof: The supply function $y^f(p)$ and the (finite) demand function

$$x^c(p, p\cdot\omega^c + \sum_{f=1}^{F}\theta_f^c \pi^f(p))$$

exist for all $c \in \{1,...,C\}$, $f \in \{1,...,F\}$, are unique as a consequence of the imposed convexity/concavity assumptions, and are continuous(1) in $p \in \Delta$. Hence the market excess demand function $z(p) = (z_1(p),...,z_n(p))$ is unique and continuous on Δ.

Let us now define

$$z_i^+(p) = \max\{z_i(p), 0\}$$

and the corresponding vector

$$z^+(p) = (z_1^+(p),...,z_n^+(p))$$

Then the mapping $h: \Delta \to \Delta$ with

$$h(p) = \frac{p + z^+(p)}{\sum_{i=1}^{n}(p_i + z_i^+(p))}$$

is well-defined and continuous on Δ.

(1) This can be concluded e.g., from Berge's maximum theorem (cf. an earlier lecture).

EXISTENCE OF A WALRASIAN EQUILIBRIUM
Proof (cont'd)

Brower's fixed-point theorem implies that the mapping h possesses a fixed point p* in Δ, i.e.,

$$h(p^*) = \frac{p^* + z^+(p^*)}{\sum_{i=1}^{n}(p_i^* + z_i^+(p^*))} = p^*$$

Using Walras' Law we find

$$0 = p^* z(p^*) = h(p^*) z(p^*) = \frac{\overbrace{p^* z(p^*)}^{=0} + z^+(p^*) z(p^*)}{\sum_{i=1}^{n}(p_i^* + z_i^+(p^*))} \overset{\sum_{i=1}^{n} p_i^* = 1}{=} \frac{z^+(p^*) z(p^*)}{1 + \sum_{i=1}^{n} z_i^+(p^*)}$$

and therefore

$$0 = z^+(p^*) z(p^*) = \sum_{i=1}^{n} z_i(p^*) \max\{0, z_i(p^*)\}$$

which implies that

$$z_i(p^*) \le 0$$

(1) This can be concluded e.g., from Berge's maximum theorem (cf. an earlier lecture).

EXISTENCE OF A WALRASIAN EQUILIBRIUM
Proof (Cont'd)

In addition, since $z^+(p^*) = p^* \sum_{i=1}^{N} z_i^+(p^*)$ **, good i can be in excess supply, i.e.,** $z(p^*) < 0$ **, only if it is worthless, that is to say only if** $p_i^* = 0$ **.**

In particular, $0 = p^* \cdot z(p^*) = p_1 z_1(p^*) + \cdots + p_N z_N(p^*)$ **implies that**

$$z_i(p^*) < 0 \;\Rightarrow\; p_i^* = 0$$

One can also show that the fixed point p* needs to occur in the interior of the simplex Δ, (cf. MWG, p. 586) so that the excess demand must vanish in equilibrium,

$$z(p^*) = 0$$

QED

AGENDA

Some Preliminaries

Fundamental Welfare Theorems

Existence of a Competitive Equilibrium

General Equilibrium vs. Partial Equilibrium

Key Concepts to Remember

GENERAL VS. PARTIAL EQUILIBRIUM ANALYSIS

To see how General Equilibrium Theory can yield predictions that are radically different from Partial Equilibrium Theory, consider the following example.

Example: Tax Incidence

Consider an economy with N cities (where N is a large number).

- **In each city there is a single price-taking firm that produces a single consumption good using the increasing, *strictly concave production function* $f(\cdot)$**
- **There are M identical workers. *Each worker is free to move between cities to be paid the highest wage*.**
- ***Each worker derives utility from the single consumption good that is available*. Without loss of generality the price of the consumption good can be normalized to 1.**

Question: If a tax on labor is levied in city 1, who bears the cost (firms or workers)?

GENERAL VS. PARTIAL EQUILIBRIUM (cont'd)

Analysis I (Partial Equilibrium) – Consider only City 1

- ***Before the tax is introduced*, given that workers can move freely, wages must be equal in each city, i.e.,**

$$w_1 = \cdots = w_N = \overline{w} = f'(M/N)$$

which yields each firm's equilibrium profit, $\overline{\pi} = f(M/N) - \overline{w}(M/N)$

- **The supply of workers in city 1 must be completely elastic, and thus the equilibrium wage *after the tax* $t \geq 0$ *is introduced* must still be equal to** $\overline{w}$
- **Hence, we find that *in city 1*, output drops to**

$$\pi_1 = f(L_1) - (\overline{w} + t)L_1 < \overline{\pi}$$

where the labor used in city 1, L_1**, is such that** $f'(L_1) = \overline{w} + t$

- **As a result, since** $L_1 < M/N$**, *some labor moves away from city 1*, but all the tax is borne by producers!**

GENERAL VS. PARTIAL EQUILIBRIUM (cont'd)

Analysis II (General Equilibrium)

- ***Before the tax is introduced*, we obtain the same analysis as before.**
- **Let $w(t) = w_1 = \cdots = w_N$ be the common equilibrium wage in all cities *after the tax* $t \geq 0$ *is introduced***
- **Demand = Supply yields $(N-1)L(t) + L_1(t) = M$, where $L(t)$ is the equilibrium labor demand in cities 2,…,N, and $L_1(t)$ is the equilibrium labor demand in city 1**
- **Profit maximization yields $f'(L_1(t)) = w(t) + t$ and $f'(L(t)) = w(t)$**
- **Using the boundary condition for $t = 0$, when $L(0) = L_1(0) = M/N$, we find by differentiating the optimality conditions and evaluating at $t = 0$:**

$$f''(L_1(0))L_1'(0) = -f''(M/N)(N-1)L'(0) = w'(0) + 1$$

$$f''(M/N)L'(0) = w'(0)$$

so that $w'(0) = -1/N$. In other words, the wage rate in all cities declines with an imposition of a tax on labor

GENERAL VS. PARTIAL EQUILIBRIUM (cont'd)

- **Let us now consider the change of firm profits, which again can be done by differentiating profits with respect to t and evaluating at t=0:(1)**

$$(N-1)\bar{\pi}'(\bar{w})w'(0) + \bar{\pi}'(\bar{w})(w'(0)+1) = \bar{\pi}'(\bar{w})\left(-\frac{N-1}{N} + \frac{N-1}{N}\right) = 0$$

In other words, aggregate profits are very little affected (and in the limit unaffected) by a (small) tax.

- **We therefore find that (at least for small taxes) virtually all of the tax in city 1 is incurred by the workers, which is the *opposite* conclusion of what we obtained using partial equilibrium analysis!**

(1) A more detailed proof of this statement can be given as follows. Let $S(t) = (f(z_1) - (w+t)z_1) + (N-1)(f(z) - wz) = f(z_1) + (N-1)f(z) - w(z_1 + (N-1)z) - tz_1$ be the sum of the firms' profits at the tax rate t. Note that (supply = demand) implies that $z_1 + (N-1)z = M$, so that $S(t) = f(z_1) + (N-1)f(z) - wM - tz_1$. We obtained earlier that $w'(0) = -1/N$, and $z_1(0) = z(0) = M/N$. We can therefore differentiate $S(t)$ with respect to t and evaluate at $t = 0$:

$$S'(0) = f'(M/N)(z_1'(0) + (N-1)z'(0)) - w'(0)M - z_1(0) = f'(M/N)\left.\frac{d}{dt}\right|_{t=0}(z_1(t) + (N-1)z(t)) - \left(-(M/N) + (M/N)\right) = f'(M/N)(M)' - 0 = 0$$

The significance of this result is that small deviations from zero in the tax level have an arbitrarily small impact on aggregate profit, whereas the impact on the workers' utilities has a strictly positive slope. In other words, in the limit a very small positive tax is borne entirely by the workers.

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KEY CONCEPTS TO REMEMBER

- **Set Summation**
- **Walrasian Equilibrium (Competitive Equilibrium) (w/ or w/o transfers)**
- **Fundamental Welfare Theorems**
- **Separating Hyperplane Theorem**
- **Walras' Law**
- **General Equilibrium vs. Partial Equilibrium**