MGT 621 – MICROECONOMICS

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4. Theory of the Firm

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École Polytechnique Fédérale de Lausanne
College of Management of Technology

AGENDA

Introduction

Production Sets

Profit Maximization: Some Intuition

The Firm’s Cost Function

Profit Maximization

Key Concepts to Remember

THEORY OF THE FIRM

Production Side of the Economy

Including …

- **Corporations** **(… General Motors, Microsoft, Virgin Atlantic)**
- **Public utilities** **(… Pacific Gas and Electric Co., Metropolitan Water District)**
- **Partnerships** **(… law firm, McKinsey, start-up firm)**
- **Small businesses** **(… retail store, individual consultant, restaurant, internet radio station)**
- **Home production** **(… home improvement, prepare meals)**
- **Educational institutions** **(… UC Berkeley, Stanford U)**
- **Non-profit organizations** **(… community hospital, YMCA)**

THE FIRM AS AN OPTIMIZER

Possible Objectives

- **Maximize (expected) profit**
- **Maximize (expected) utility (which takes profits as one argument)**
- **Minimize cost (e.g., given fixed outputs)**

Feasible Actions

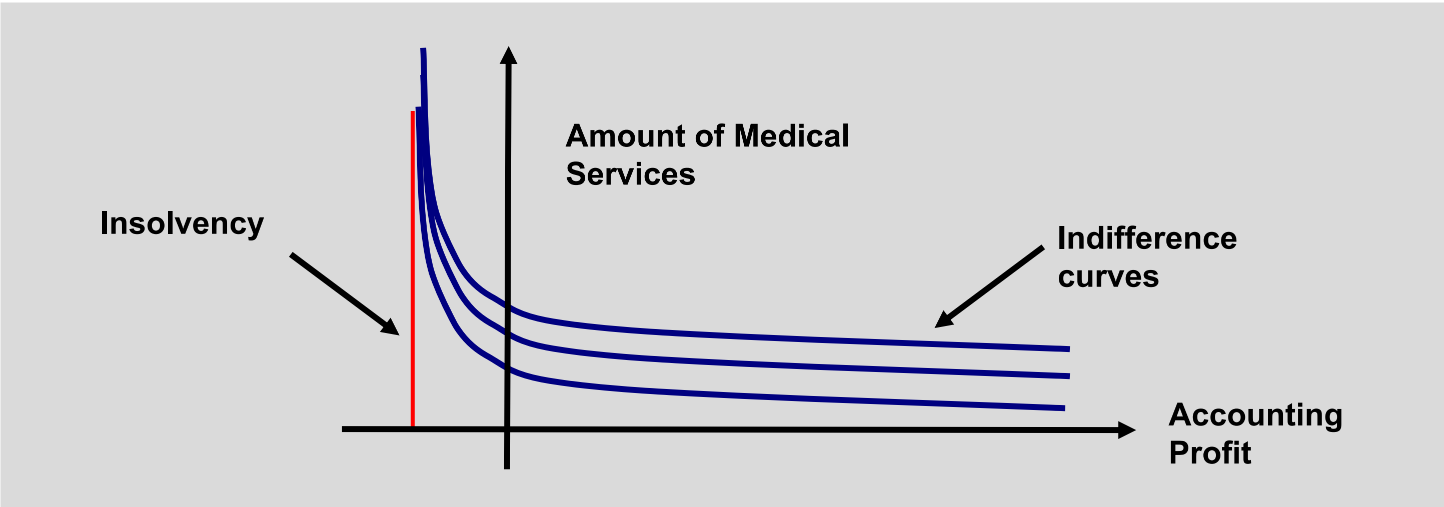
- **Production possibilities**
- **Legal constraints**
- **Competitive necessities**

Firm chooses a most preferred action from the set of all feasible actions.

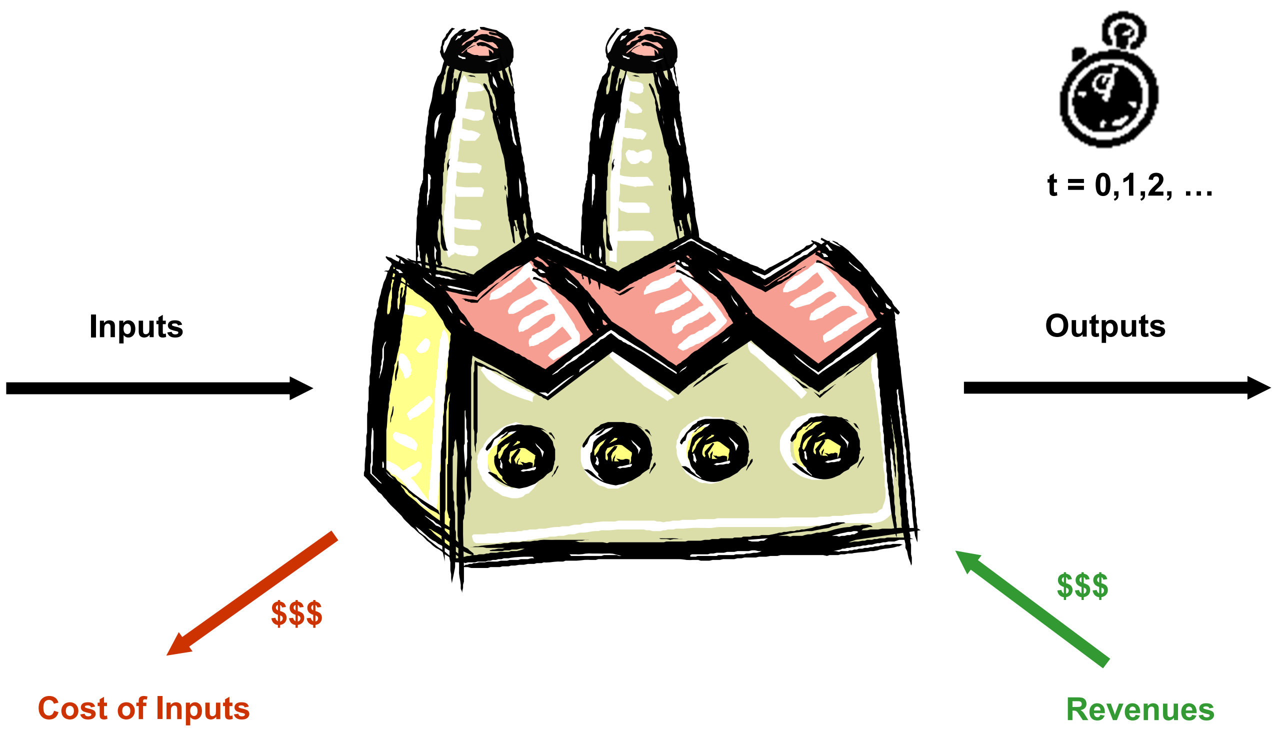
A FIRM'S POSSIBLE OBJECTIVES: EXAMPLE

Maximization of (Expected) "Utility" (Objective Function)

- **Could include accounting profits as only one variable among several others (such as a measure of output)**
  - **Example: a community hospital may put a high value on accounting profits if it is losing money (negative profit), and a high value on the provision of medical services otherwise**



THE FIRM AS A BLACK BOX
Compare Inflows and Outflows

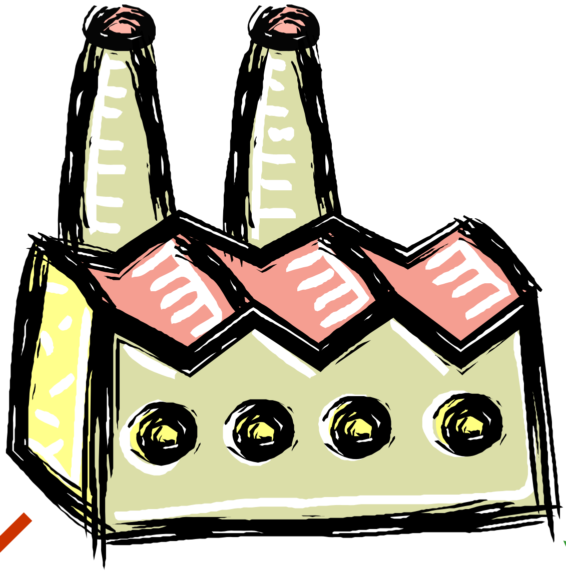


THE FIRM AS A BLACK BOX (Cont'd)
Converts Inputs to Outputs

$t = 0,1,2,\ldots$

Inputs $\underline{z}$

Stream
$\underline{z} = (z^0, z^1, z^2, \ldots)$



Outputs $\underline{q} = F(\underline{z})$

Stream
$\underline{q} = (q^0, q^1, q^2, \ldots)$

$q^t = F^t(\underline{z})$

Production Function F

Costs: $C^t(\underline{q})$

Profit at time t:
$\Pi^t(\underline{q}) = R^t(\underline{q}) - C^t(\underline{q})$

Revenues: $R^t(\underline{q})$

!!!

PRODUCTION FUNCTION

The production function at time t might depend on current and/or (anticipated) future inputs,

$$q^t = F^t(\underline{z}) = F^t(z^0, z^1, z^2, \ldots)$$

Question: Why?

Answer: There are many real-world phenomena which may produce dependencies such as

- **Learning-Curve Effects[(1)]**
- **Demand Effects (e.g., Saturation)**

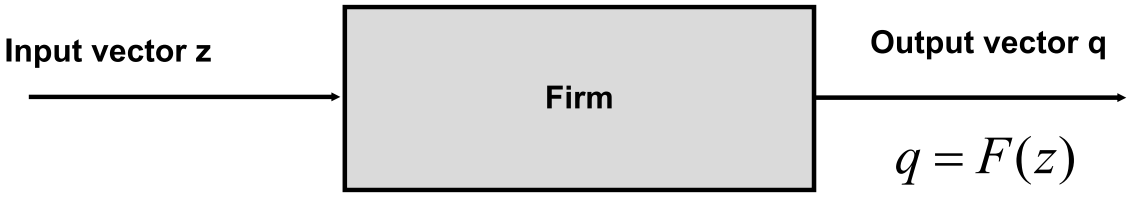
In some practical applications current output depends only on current input, i.e.,

$$q^t = F^t(\underline{z}) = F^t(z^t)$$

(1) The commander of Wright-Patterson Air Force Base in Ohio observed in 1925 that the required direct labor hours for the assembly of a plane decreased in the number of planes built. First academic observations include Hirschman, W.B. (1964) "Profit from the Learning Curve," Harvard Business Review, Jan/Feb, and Arrow, K.J. (1962) "Economic Welfare and the Allocation of Resources to Invention," in: Nelson, R. (ed.) *The Rate and Direction of Inventive Activity: Economic and Social Factors*, Princeton University Press, Princeton, NJ.

A FIRM CONVERTS INPUTS TO OUTPUTS
Simplification: Static Conversion → Omit Time Index

Production function F summarizes conversion



Sometimes it is difficult to decide what exactly is input and what is output, so that it is useful to consider the "general" output (or "production possibility")

$$y = (-z,q)$$

(Net) inputs have negative components and (net) outputs are represented by nonnegative components of y.

GENERAL (STATIC) MODEL OF THE FIRM

Firm chooses vector $z \geq 0$ of inputs to produce vector $q \geq 0$ of outputs = Firm chooses production possibility vector $y = (-z,q)$.

Feasible Set

- **Production Set (or Production Possibility Set) Y**

Objective Function

- **For simplicity, first assume that profit maximization is the objective, given a price p(y) for outputs (it turns out that cost minimization for inputs is a necessary condition)**

→ Profit-Maximization Problem

- **Choose y to maximize the product $\Pi(y) = p(y)\, y$ subject to y being feasible, i.e., solve**

$$\max_{y \in Y} \; \Pi(y)$$

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PRODUCTION SET Y

The firm's production set Y contains all input-output bundles y it might choose.

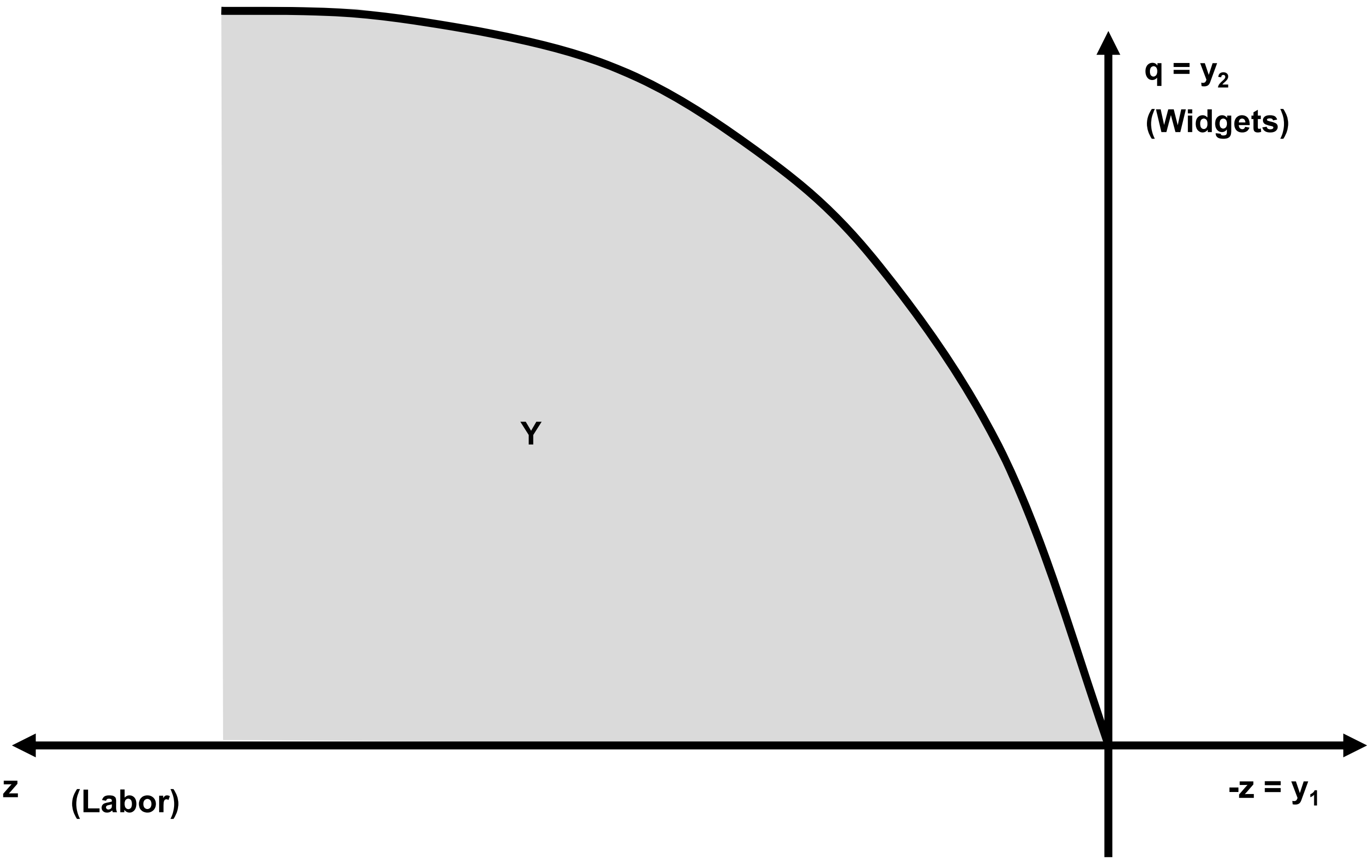
Possible inputs: labor, land, buildings, raw materials, capital equipment, services, intermediate goods (produced by other firms), etc.

Possible outputs: finished goods, services, knowledge, intermediate goods (to be used by other firms), etc.

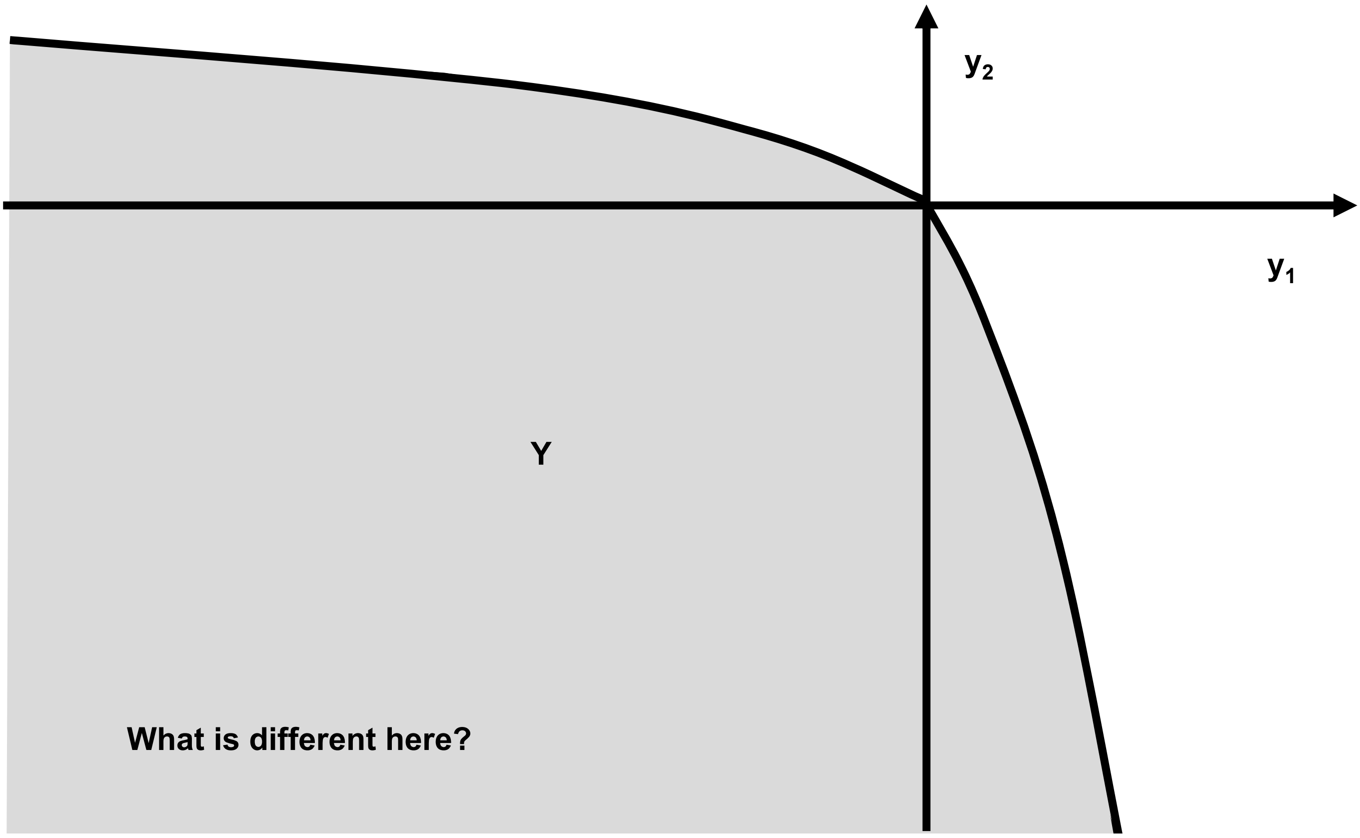
Shape of Y implied by

- technological possibilities
- legal constraints

PRODUCTION SET: EXAMPLE



PRODUCTION SET: ANOTHER EXAMPLE



PROPERTIES OF PRODUCTION SETS

We now discuss standard assumptions on a firm's production set $Y \subset \Re^L_+$.

A1 (Non-Emptiness). The set Y contains at least one element.

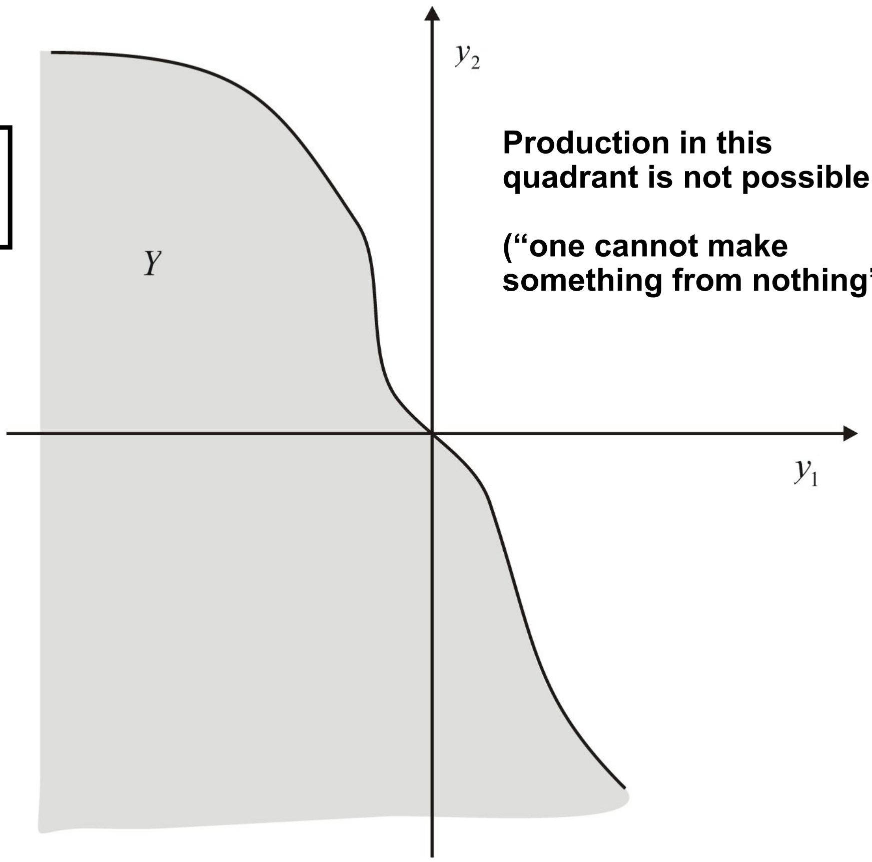
A2 (Closedness). It is not possible to construct a sequence of elements of Y with limit outside Y.

A3 (No Free Lunch). $Y \cap \Re^L_+ \subseteq \{0\}$: it is not possible to have a positive net output y > 0 (i.e., one cannot produce at least one positive net output and not use any positive net inputs).

A1 – A3 are fundamental requirements which we assume are always satisfied.

NO FREE LUNCH

$$Y \cap \Re^L_+ \subseteq \{0\}$$



Production in this quadrant is not possible!

("one cannot make something from nothing")

PROPERTIES OF PRODUCTION SETS (Cont'd)

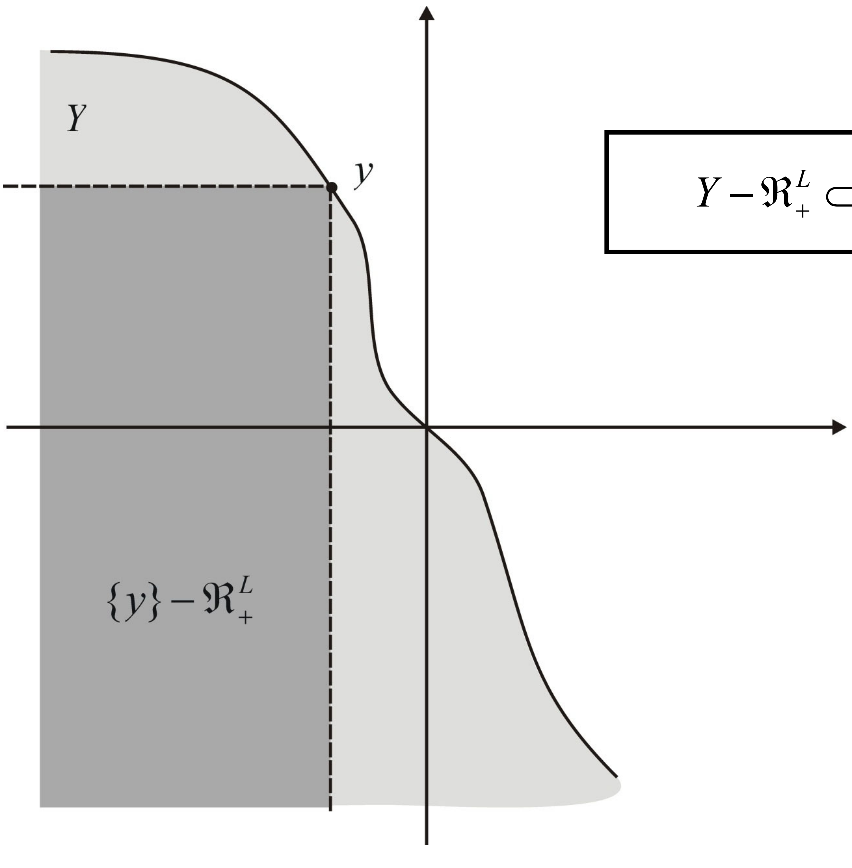
A4 (Possibility of Inaction). $0 \in Y$ **, i.e., it is possible to do nothing.**

A5 (Free Disposal). $Y - \Re_+^L \subset Y$ **, i.e., it is always possible to use more net inputs for the same net output.**

A6 (Irreversibility). $y \in Y \Rightarrow -y \notin Y$ **: it is not possible to obtain the inputs back once the outputs have been created.**

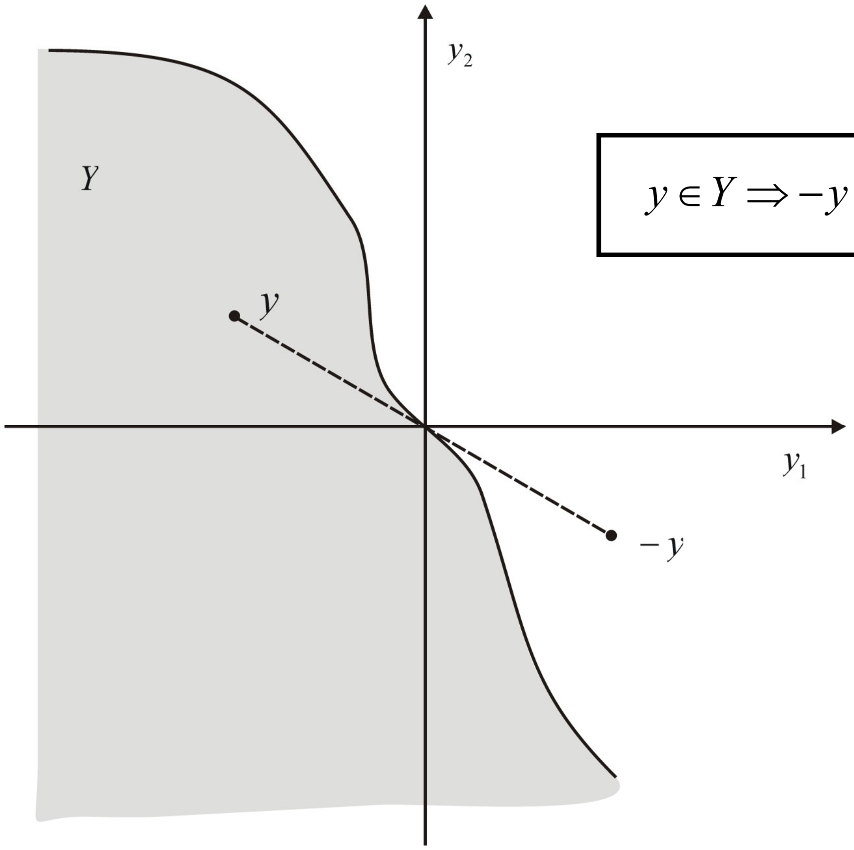
A4 – A6 are basic properties of production sets that may sometimes be violated.

FREE DISPOSAL



$$\boxed{Y - \Re_+^L \subset Y}$$

IRREVERSIBILITY



$$y \in Y \Rightarrow -y \notin Y$$

PROPERTIES OF PRODUCTION SETS (Cont'd)

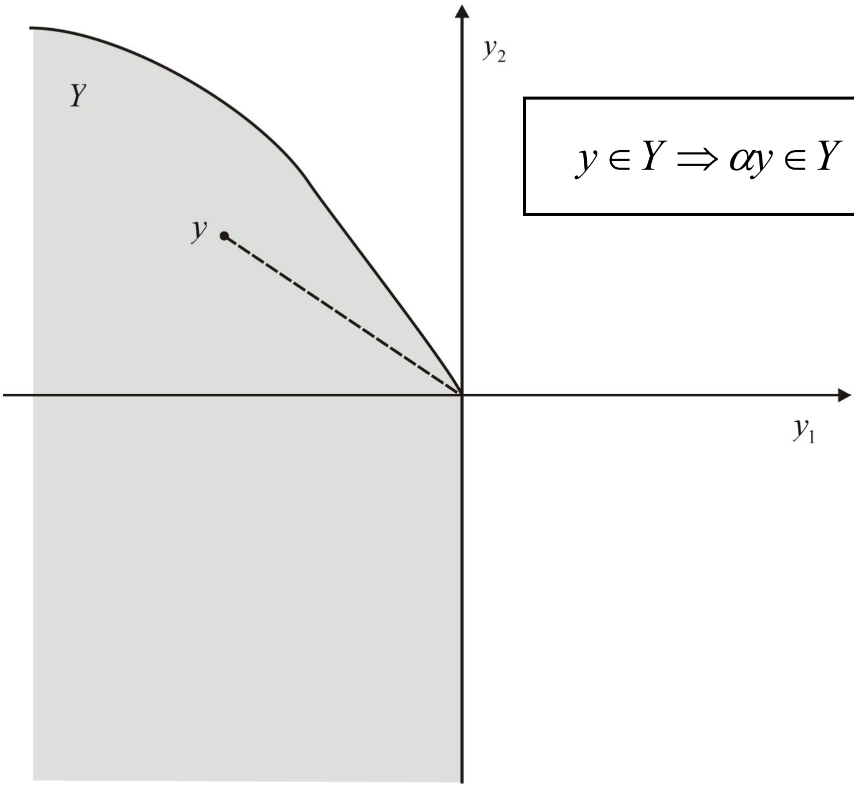
A7 (Nonincreasing Returns to Scale). $y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \in [0,1]$ **, i.e., it is possible to scale down any feasible production vector. Mathematically, this means that Y is "star-shaped" with respect to the origin.**

A8 (Nondecreasing Returns to Scale). $y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \geq 1$ **, i.e., it is always possible to scale up any feasible production vector.**

A9 (Constant Returns to Scale). $y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \geq 0$ **, i.e., any feasible production vector is completely scalable. Mathematically, this means that Y is a cone.**

A7 – A9 are useful properties of production sets that may or may not be satisfied.

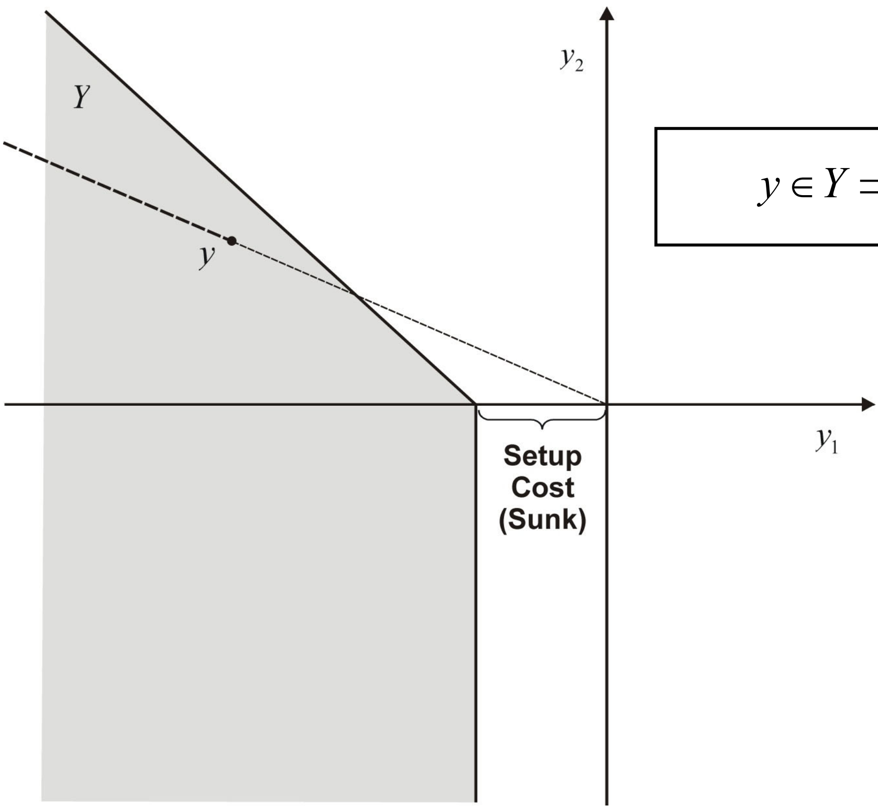
NONINCREASING RETURNS TO SCALE



$$y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \in [0,1]$$

Remark: The set Y exhibits *decreasing* returns to scale if $y \in Y \Rightarrow \alpha y \in \operatorname{int} Y \quad \forall \alpha \in (0,1)$

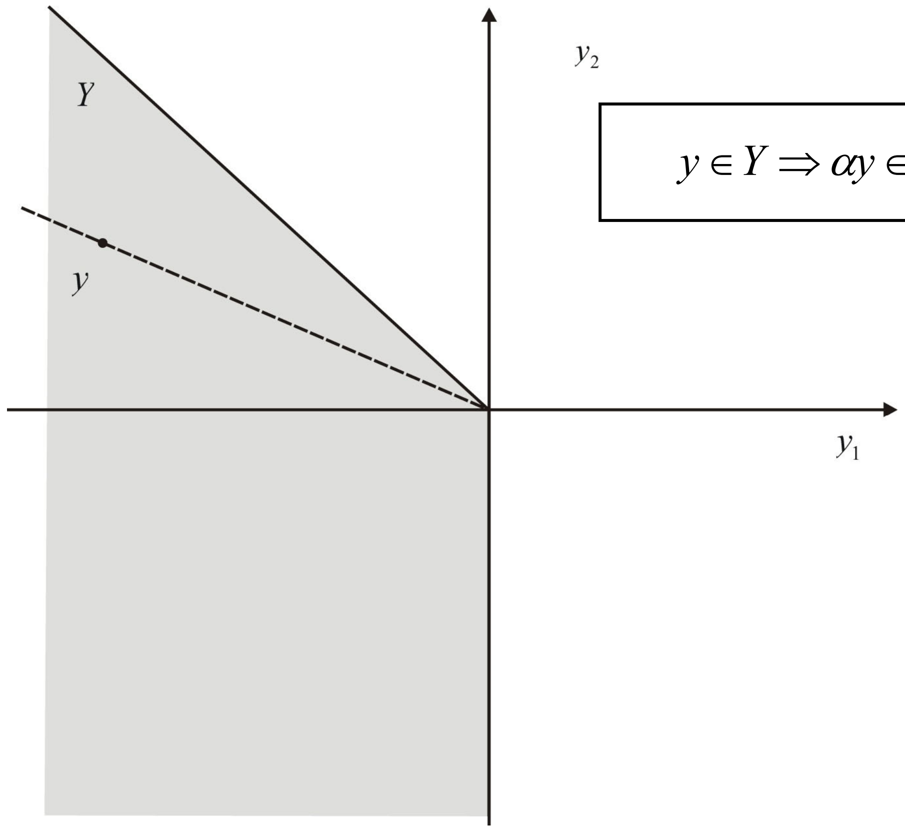
NONDECREASING RETURNS TO SCALE



$$y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \geq 1$$

Remark: The set Y exhibits *increasing* returns to scale if $y \in Y \Rightarrow \alpha y \in \operatorname{int} Y \quad \forall \alpha > 1$.

CONSTANT RETURNS TO SCALE



$$y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \geq 0$$

PROPERTIES OF PRODUCTION SETS (Cont'd)

A10 (Additivity/Free Entry). $y, \hat{y} \in Y \Rightarrow y + \hat{y} \in Y$ **, i.e., it is possible to combine any feasible production vectors. For an economy this means that the aggregate production possibilities are obtained by summing up the firms' individual production possibilities, provided each firm is free to contribute or not (= free entry).**

A11 (Convexity). $y, \hat{y} \in Y \Rightarrow \alpha y + (1-\alpha)\hat{y} \in Y \quad \forall \alpha \in (0,1)$ **, i.e., any convex combination of feasible production vectors is feasible.**

A12 (Convex Cone Property). $y, \hat{y} \in Y \Rightarrow \alpha y + \beta \hat{y} \in Y \quad \forall \alpha, \beta \geq 0$ **, i.e., any feasible production vectors can be combined and scaled. The convex cone property is equivalent to the combination A9 and A11.**

A10 – A12 are structural properties of production sets that affect the optimization methods when looking for optimal production vectors.

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Feasible Set

- Production Set (or Production Possibility Set) Y

Objective Function

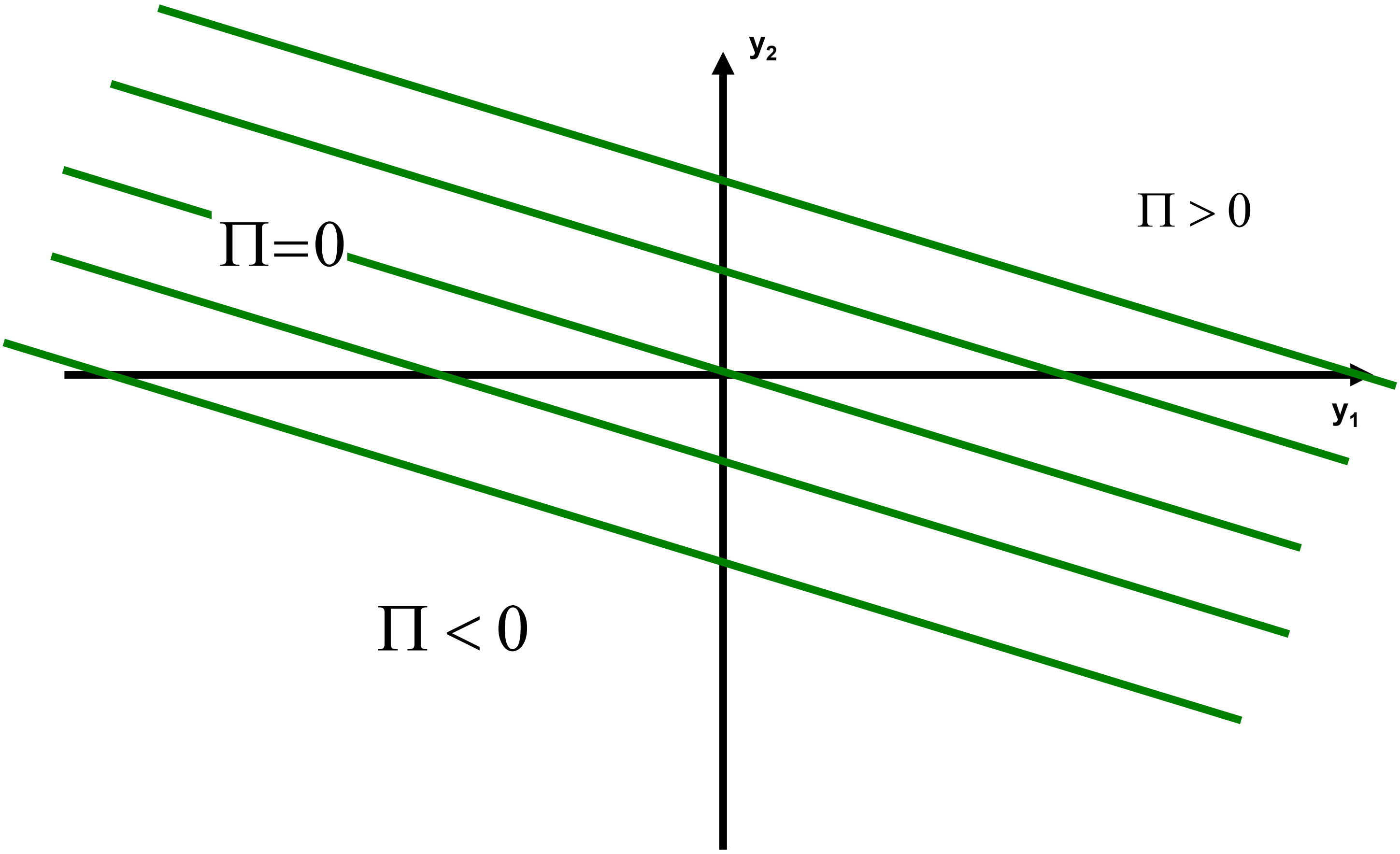
- For simplicity, first assume that profit maximization is the objective, given a price p(y) for outputs
(it turns out that cost minimization for inputs is a necessary condition)

→ Profit-Maximization Problem

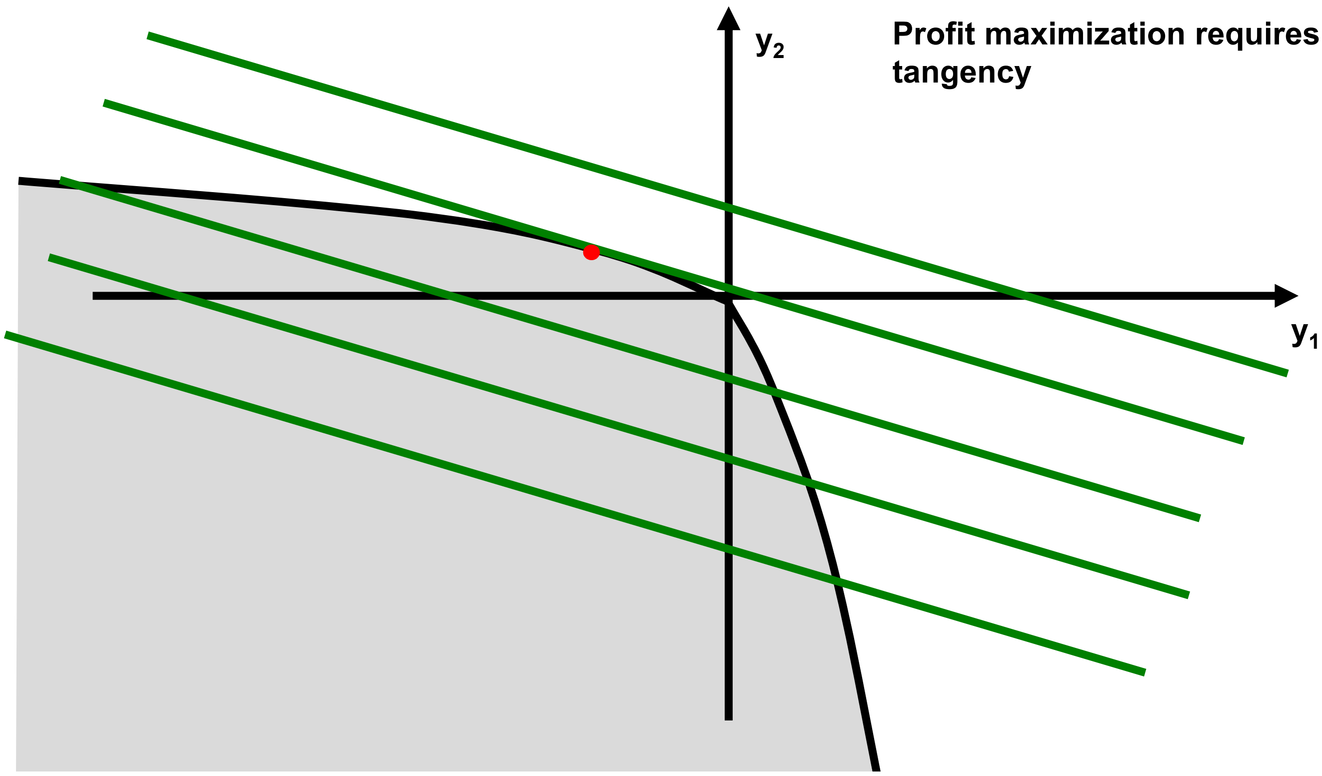
- Choose y to maximize the product $\Pi(y) = p(y)\, y$ subject to y being feasible, i.e., solve

$$\max_{y \in Y} \; \Pi(y)$$

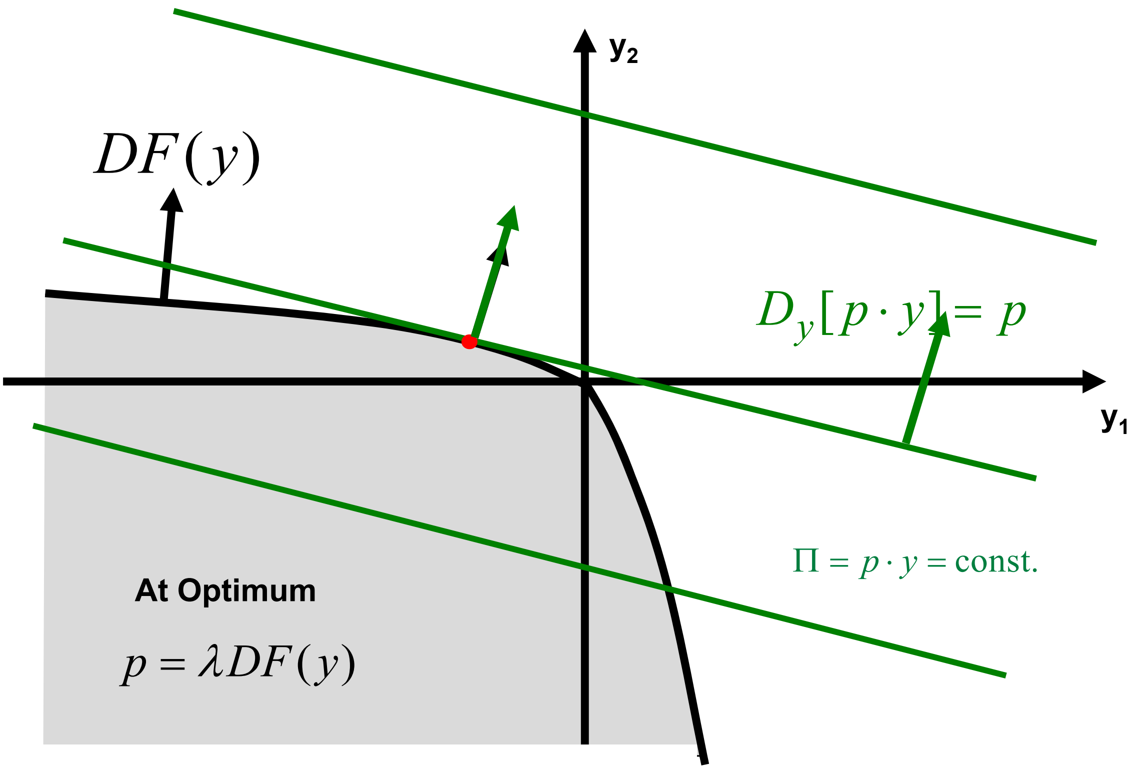
ASSUMPTION THAT p(y)=p (I.E., LINEAR PRICING) … … YIELDS CONSTANT ISO-PROFIT CURVES



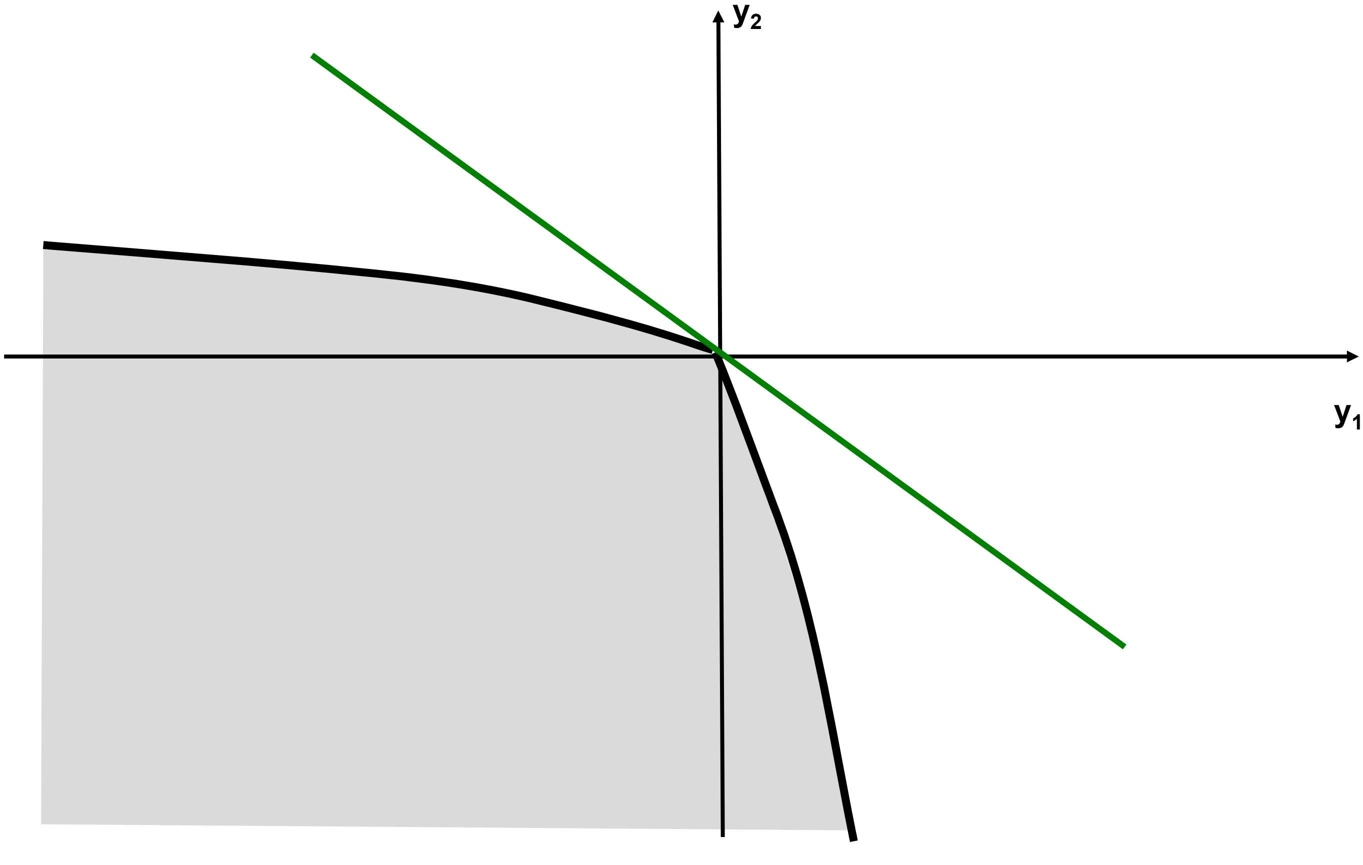
PROFIT MAXIMIZATION OVER PRODUCTION SET



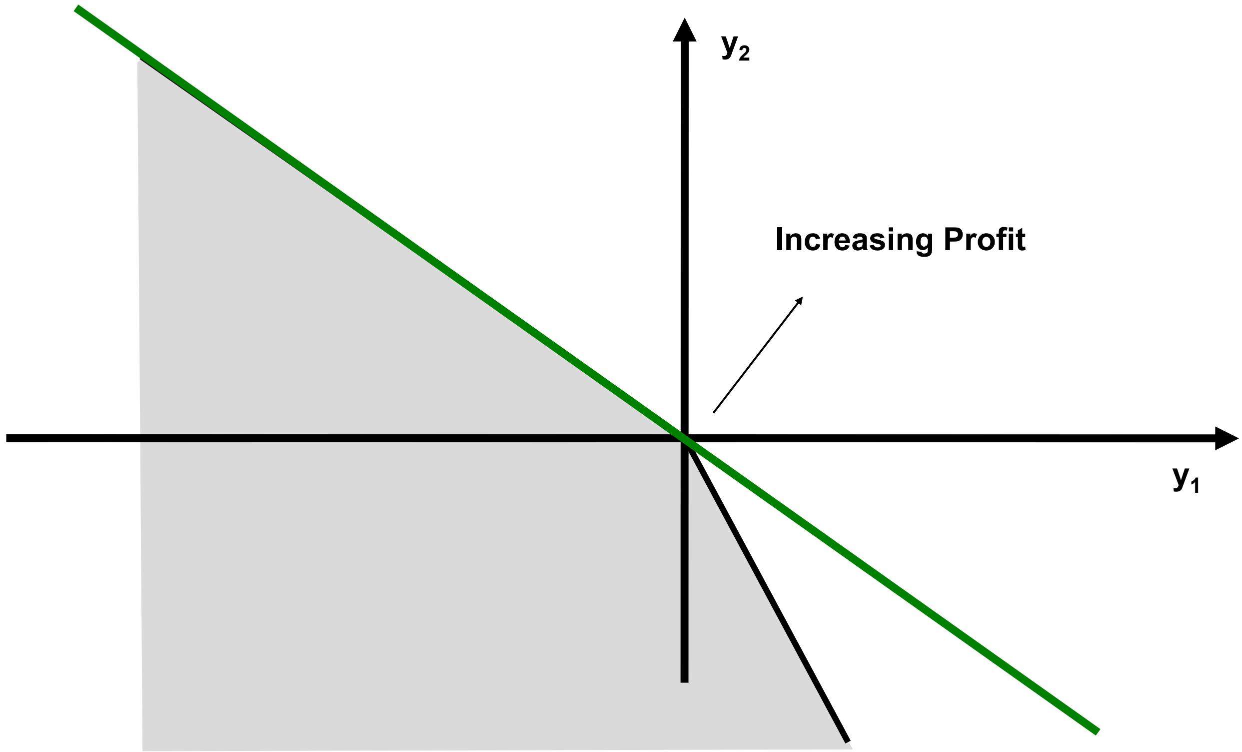
PROFIT MAXIMIZATION OVER PRODUCTION SET (Cont'd)



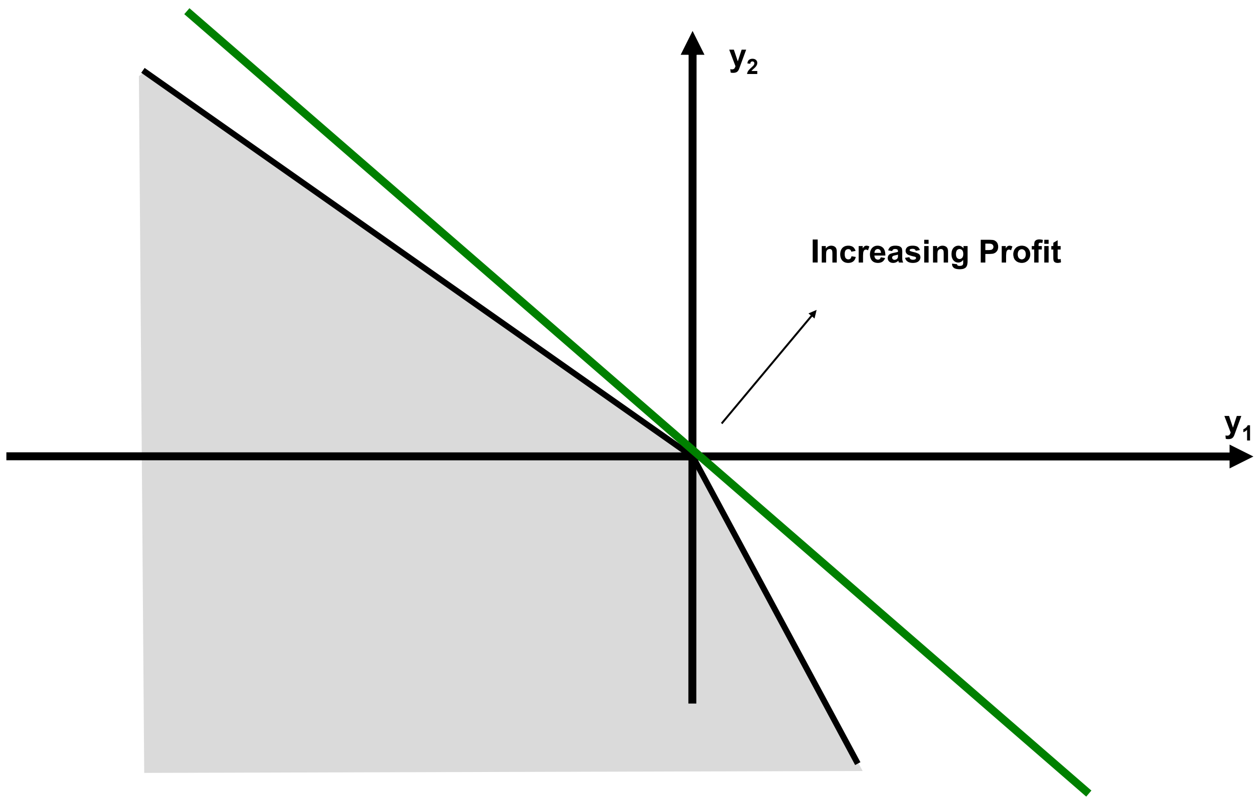
WHAT HAPPENS WHEN THERE ARE KINKS IN PRODUCTION SET?



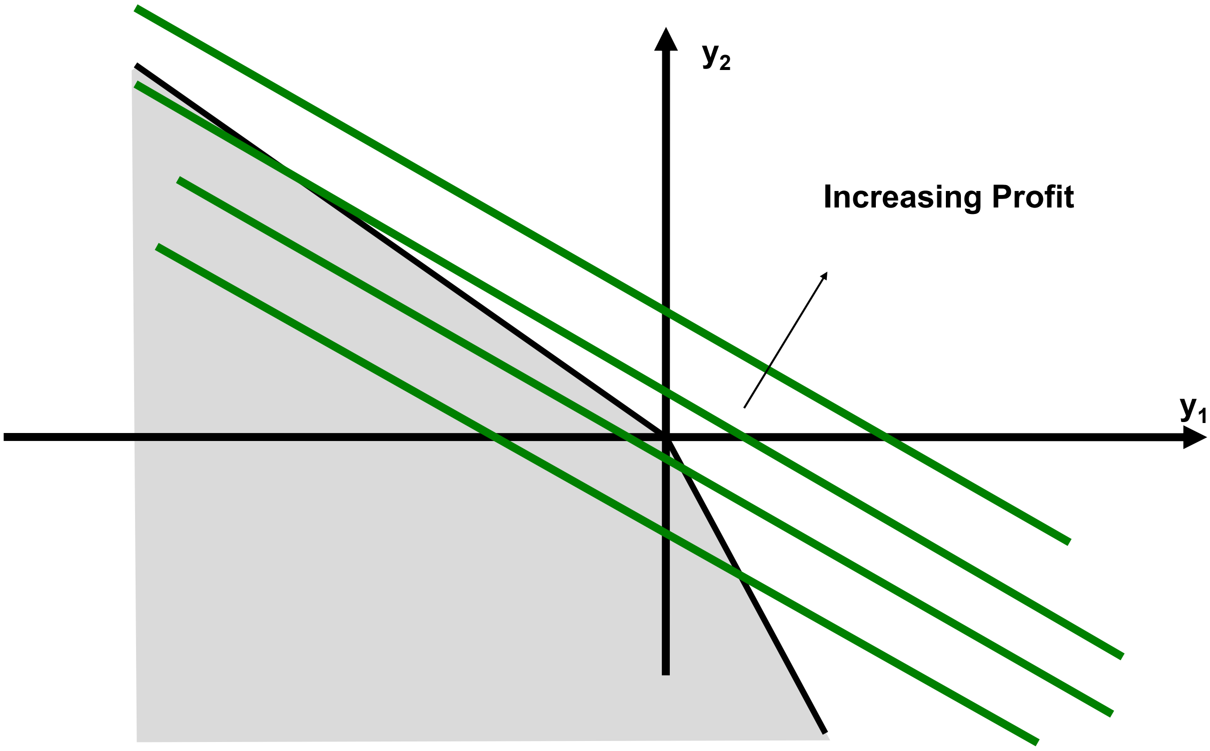
PROFIT-MAXIMIZING FIRM WITH CONSTANT RETURNS TO SCALE



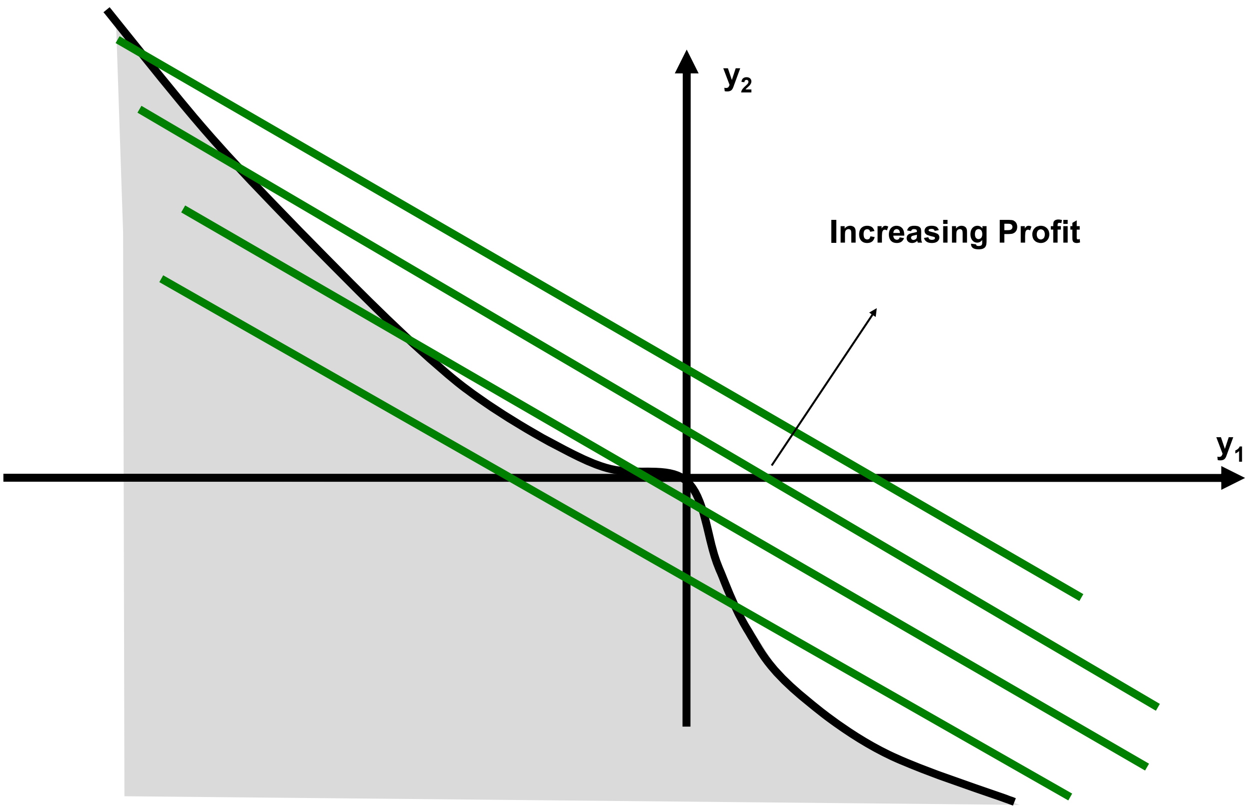
PROFIT MAXIMIZATION WITH CONSTANT RETURNS TO SCALE



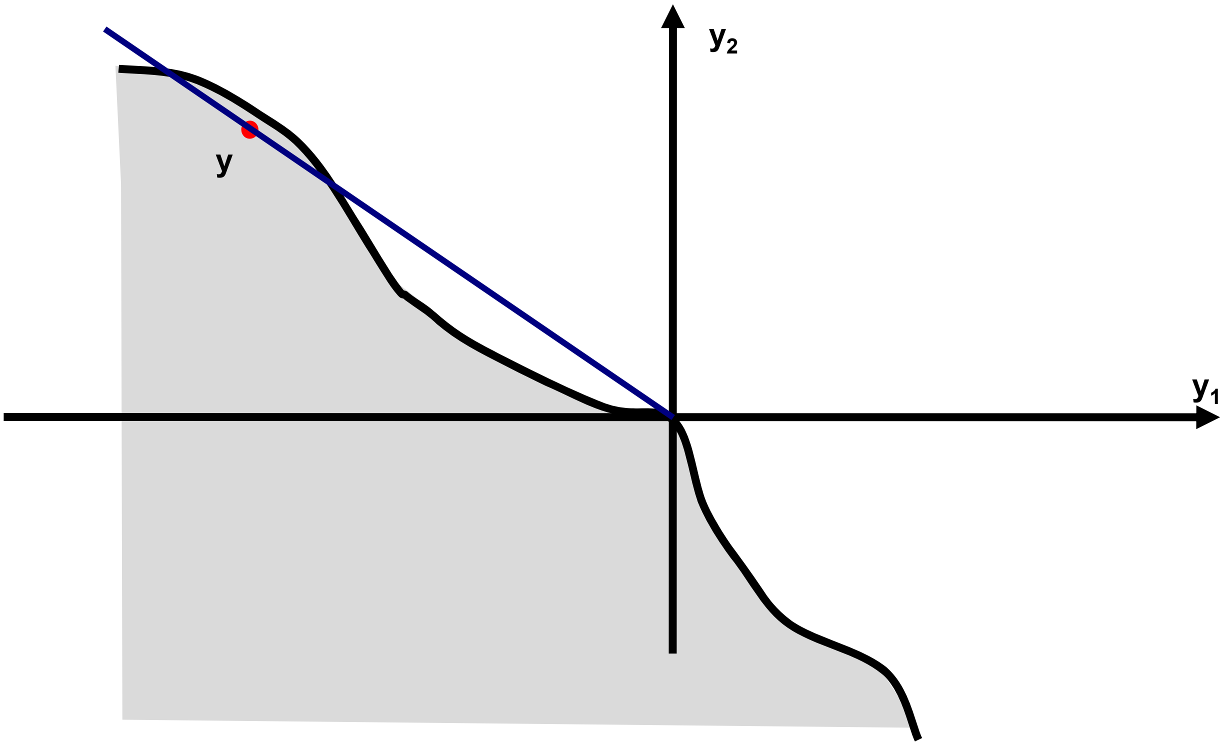
PROFIT MAXIMIZATION WITH CONSTANT RETURNS TO SCALE?



PROFIT-MAXIMIZATION WITH INCREASING RETURNS TO SCALE?



NEITHER INCREASING NOR DECREASING RETURNS TO SCALE



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The Firm's Cost Function

Profit Maximization

Key Concepts to Remember

PRODUCTION FUNCTION

Assume that a firm can clearly distinguish between inputs and outputs, i.e., any y in the production set Y can be written in the form y = (-z,q) where z $\geq$ 0 is the vector of inputs and q $\geq$ 0 is the vector of outputs.(1)

Then it is possible to represent Y in the form

$$Y = \{(-z,q) \in \Re^L : q \leq F(z)\}$$

where F(z) is referred to called the firm's production function.

(1) This holds when the firm's production is "irreversible," i.e., Y satisfies Assumption A6.

COST FUNCTION

Question. Given an increasing production function F(z), determine the firm's cost function C(q), i.e., the firm's minimum cost to produce a (feasible) output vector q $\geq$ 0.

Answer: Given a feasible vector q of outputs, the firm solves the *expenditure minimization problem* (or *'cost minimization problem'* in this context)

$$\min_{y=(-z,q)\in Y} \{w(z)\cdot z\} = \min_{z:F(z)\geq q} \{w(z)\cdot z\}$$

where w(z) is the vector of (positive) input prices. The firm's cost function C(q) is its minimal expenditure,

$$\boxed{C(q) = \min_{z:F(z)\geq q} \{w(z)\cdot z\} = \min\{w(z)\cdot z : F(z) \geq q\}}$$

COST FUNCTION: EXAMPLE

Problem Set. Find the cost function C(q) implied by the production set

$$Y = \left\{(-z_1, -z_2, q) : (z_1, z_2, q) \in \mathbb{R}^3_+, z_1^\alpha z_2^\beta \geq q\right\}$$

where α and β are positive constants with $\alpha + \beta < 1$.

ECONOMIES/DISECONOMIES OF SCALE

A cost function C(q) with a scalar output q > 0 exhibits economies of scale if the average cost decreases in q > 0, i.e.,

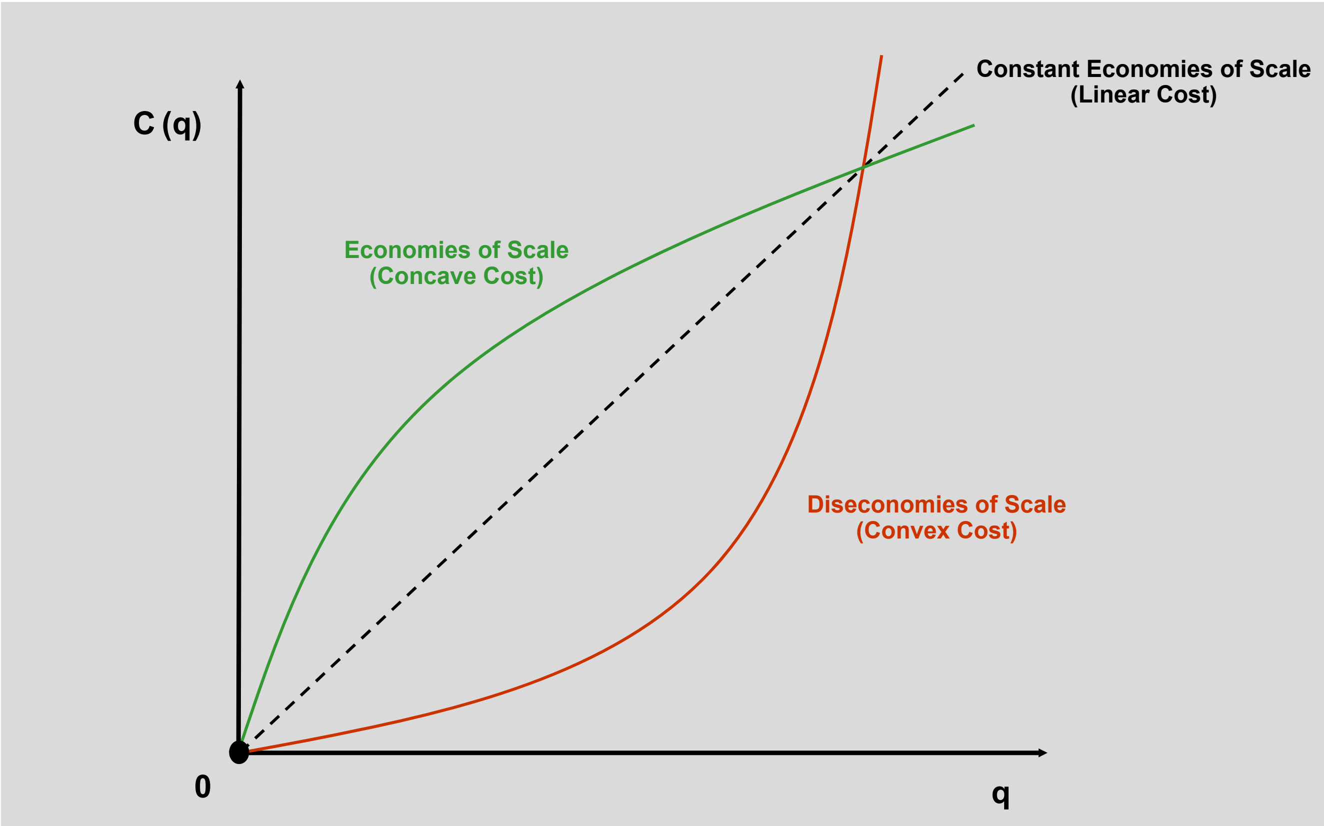
$$AC(q) = \frac{C(q)}{q} \qquad \text{goes down, as q goes up.}$$

If average costs increase in q, then C(.) exhibits diseconomies of scale; if average costs stay constant, then C(.) exhibits constant economies of scale.

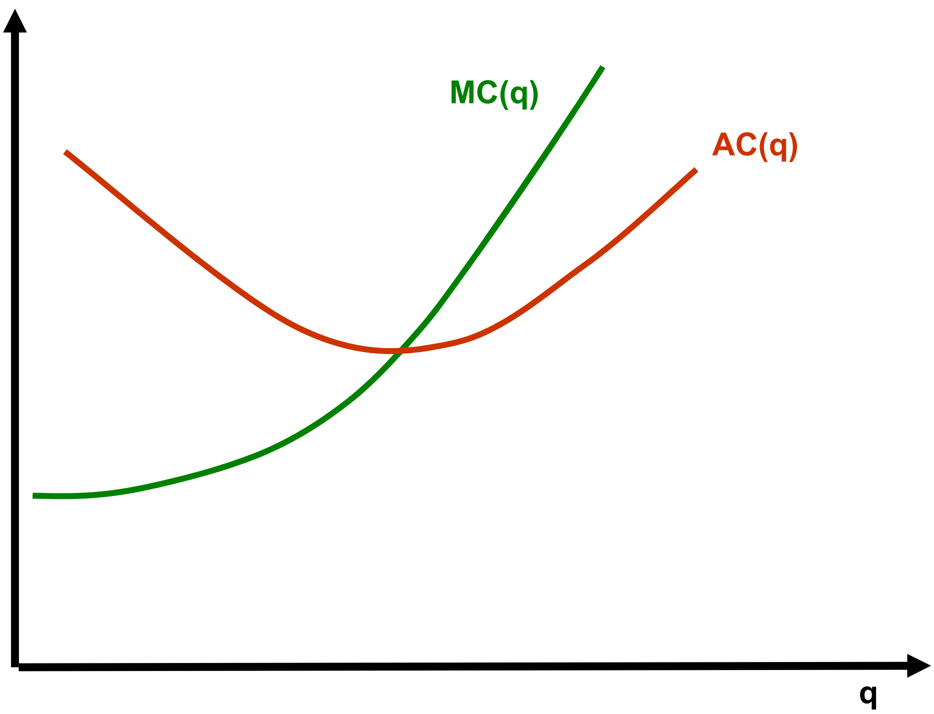
[Remark. Marginal cost is the cost "at the margin," corresponding to the slope of C(q) at q, i.e., MC(q) = C'(q).]

Similarly, a production function F(z) with a scalar input z exhibits economies of scale (*diseconomies of scale/constant economies of scale*) if the conversion rate F(z)/z increases (decreases/stays constant) in z > 0.

ECONOMIES/DISECONOMIES OF SCALE



AVERAGE COST AND MARGINAL COST



MC(q) = AC(q) at the minimizer of the average cost (= "minimum efficient scale")
You should be able to prove this relation yourself

PROFIT

Difference between value to firm of outputs and cost to the firm of inputs.

For firm that sells outputs:

- **Profit Π(q) is difference between total dollar revenues R(q) received by the firm and total dollar costs C(q) the firm incurs, as a function of its output q**

$$\Pi(q) = R(q) - C(q)$$

Revenues include

- **Money for selling the outputs, service fees, royalties, license fees, etc.**

Costs include

- **Expenditures for purchased goods, wages paid for labor, taxes**
- **Normal rate of return on invested capital**
- **Value of firm-owned resources allocated to alternative uses**
- **Value of time of the firm owner, allocated to firm activities**

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PROFIT MAXIMIZATION

$$\Pi(q) = R(q) - C(q)$$

First-order necessary optimality conditions for local unconstrained (or "interior") maximum.

$$\frac{\partial \Pi}{\partial q_i} = \frac{\partial R(q)}{\partial q_i} - \frac{\partial C(q)}{\partial q_i} = 0 \iff MR_i(q) = \frac{\partial R(q)}{\partial q_i} = \frac{\partial C(q)}{\partial q_i} = MC_i(q)$$

Second-order necessary optimality condition for local unconstrained maximum:

$$\left[\frac{\partial^2 \Pi}{\partial q_i \partial q_i} \right]_{i,j=1}^{n} \leq 0$$

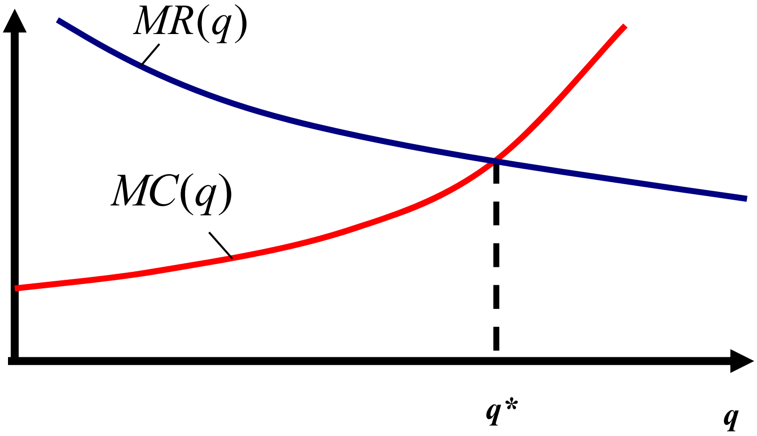
That is, the Hessian $D^2\Pi(q)$ **must be a negative semi-definite matrix at the local maximizer.**

PROFIT MAXIMIZATION (Cont'd)

Necessary optimality conditions in the one-dimensional case

$$\frac{\partial R(q)}{\partial q} = \frac{\partial C(q)}{\partial q}$$

$$R''(q) - C''(q) \leq 0$$



PROFIT MAXIMIZATION (Cont'd)

Necessary optimality conditions for profit-maximization (one-dimensional case)

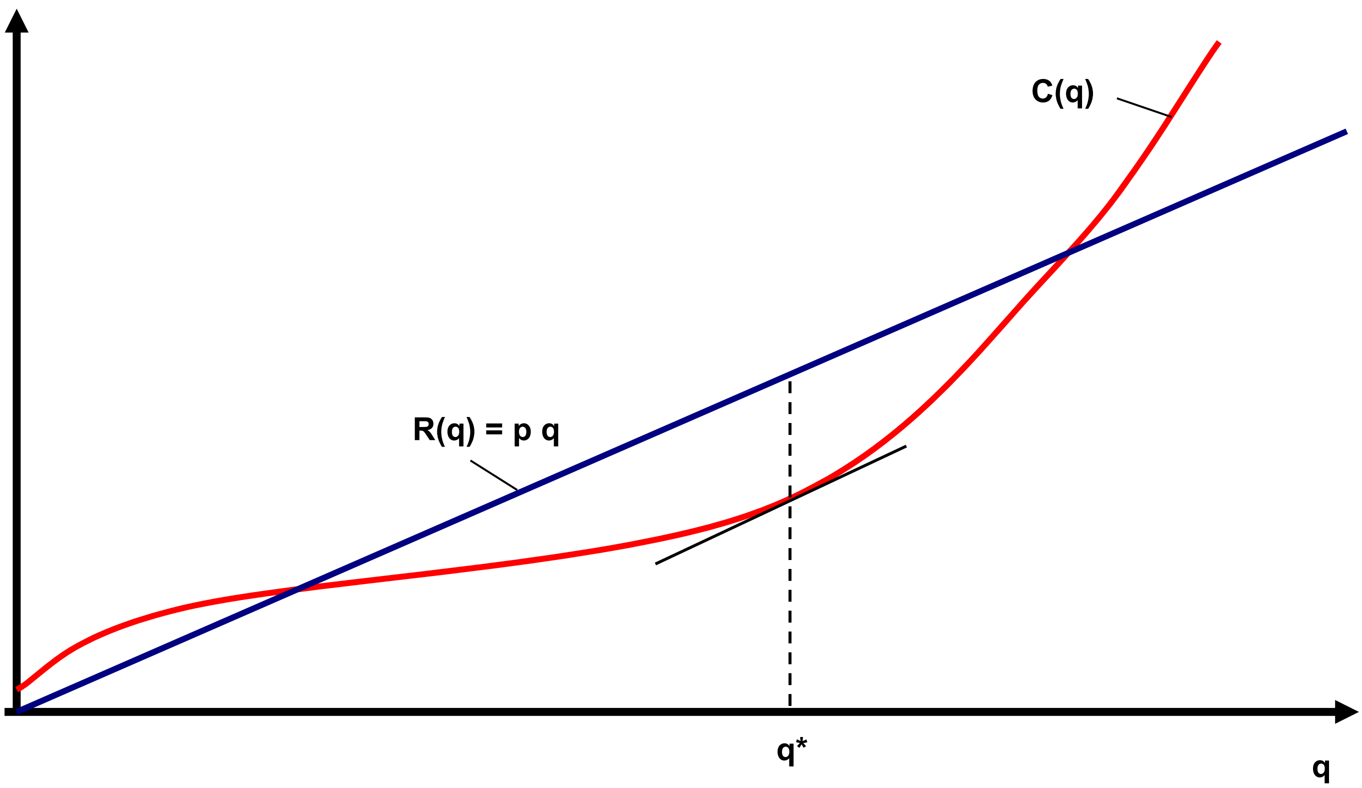
$$MR(q) = MC(q) \qquad\qquad R''(q) - C''(q) \leq 0$$

Different Perspectives

- **External Analyst**
  - Assumes marginal revenue = marginal cost
  - Uses to predict activities of firms
- **Internal Analyst**
  - Tries to see if the marginal revenue is equal to marginal cost, and adjusts levels of activities to bring two into equality.

TOTAL COST AND REVENUE

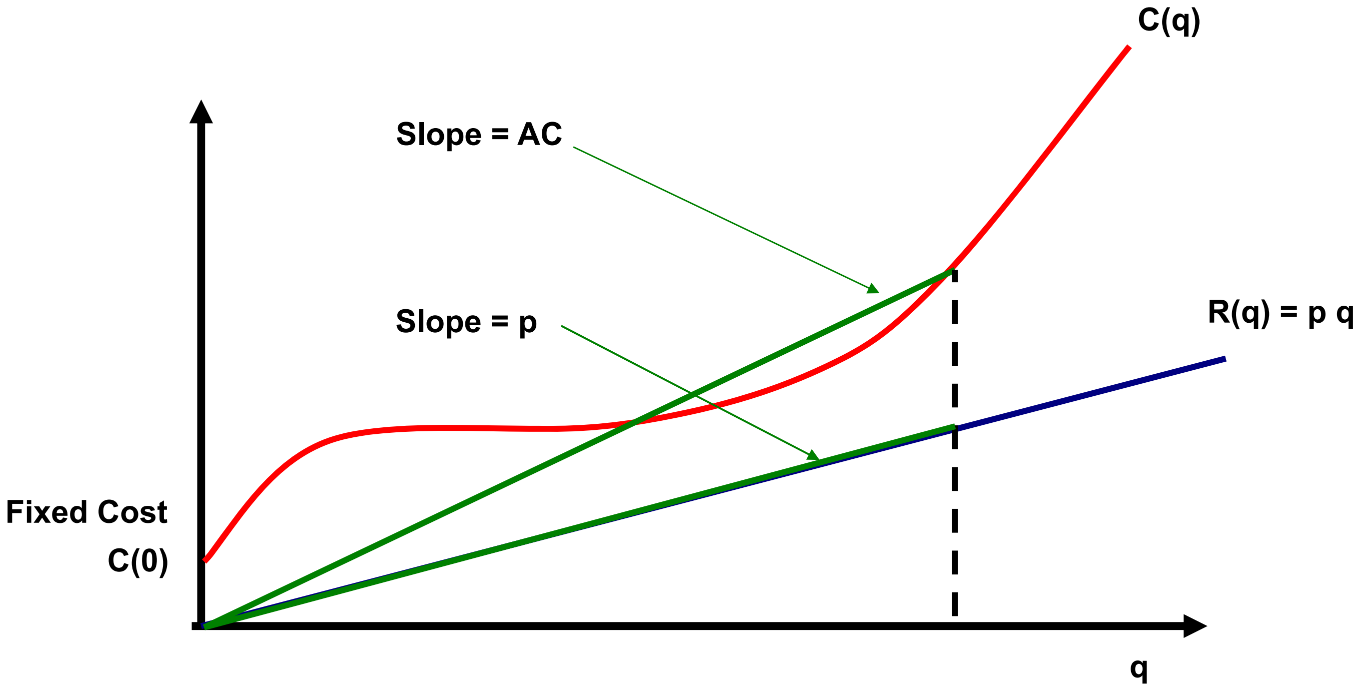
Consider firm in a competitive market, where price p is a given constant



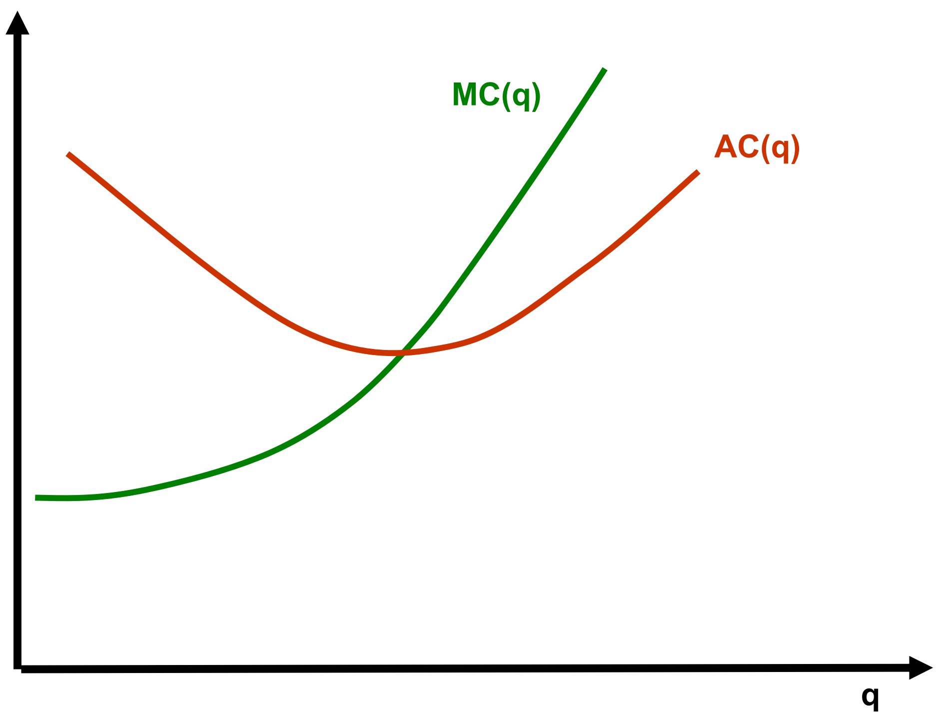
AVERAGE COST & AVERAGE REVENUE

Average Cost: $AC(q) = C(q)/q$

Average Revenue: $AR(q) = R(q)/q = p$



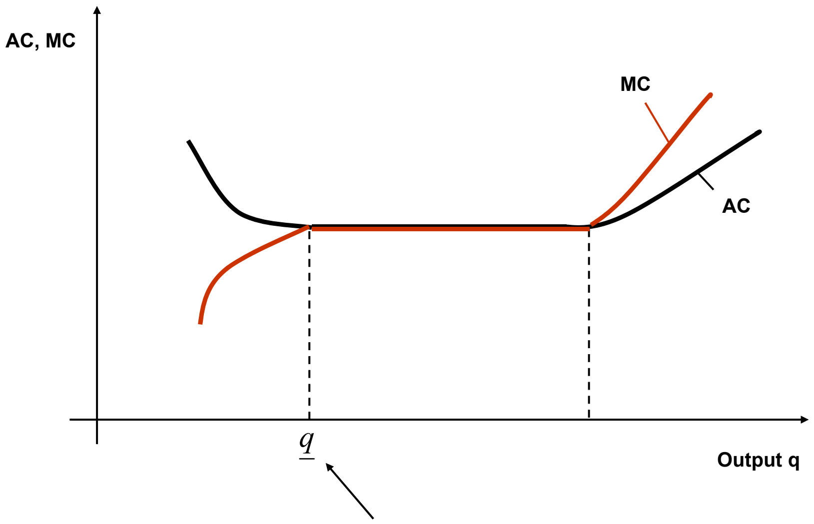
AVERAGE COST AND MARGINAL COST



MC(q) = AC(q) at the minimizer of the average cost (= "minimum efficient scale")
You should be able to prove this relation yourself

MINIMUM EFFICIENT SCALE

Let $C(q)$ be a firm's increasing, smooth, and convex cost function



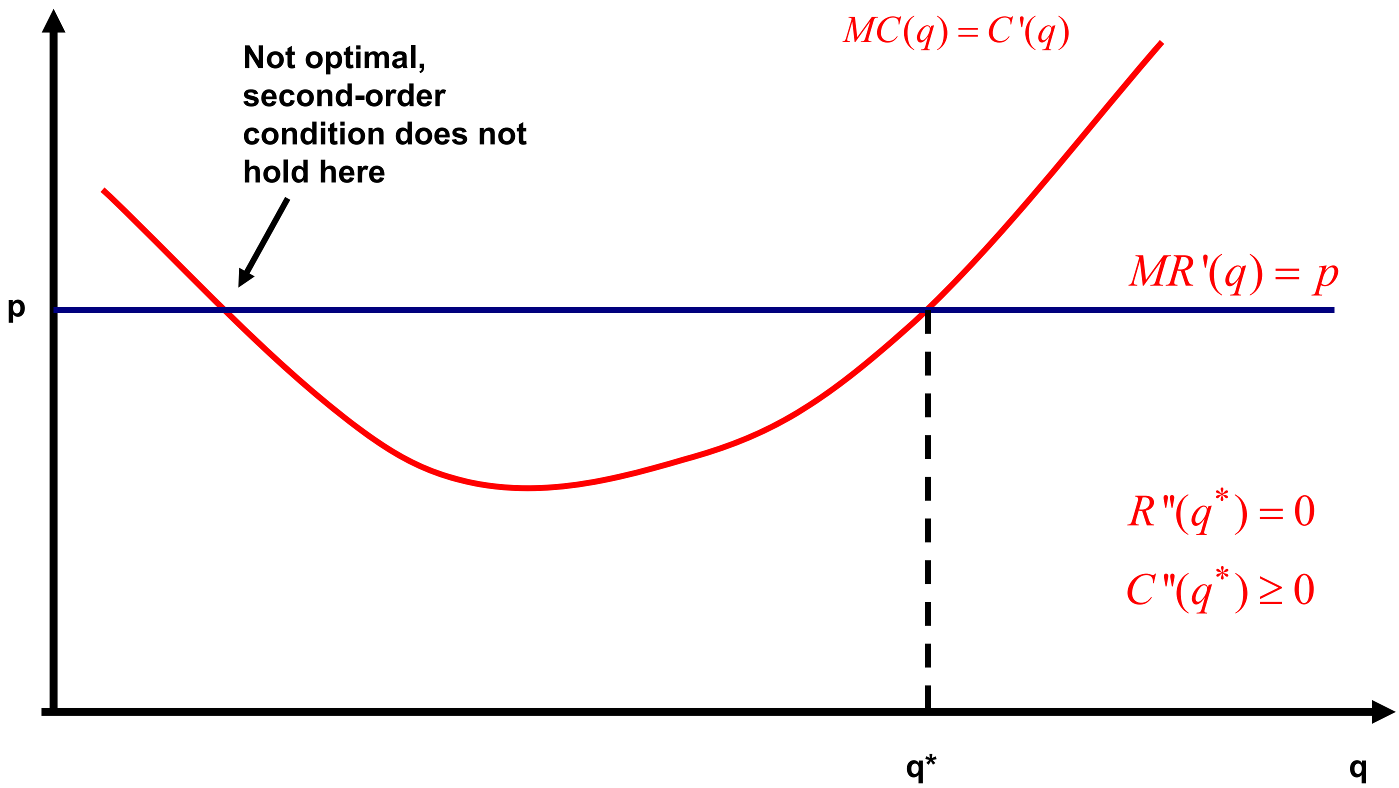
$$AC(q) = C(q)/q$$
$$MC(q) = C'(q)$$

Minimum Efficient Scale [(1)]

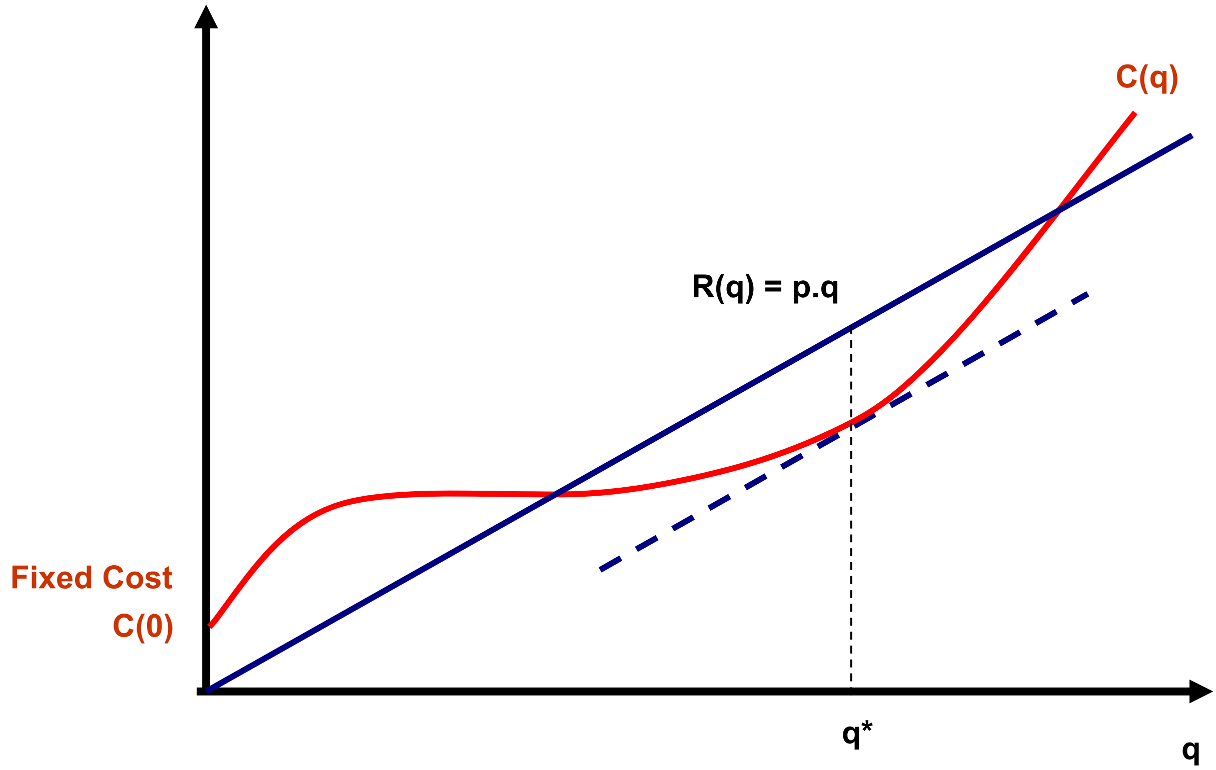
$$\underline{q} = \min\{q \geq 0 : AC'(q) = 0\}$$

(1): More precisely: $\underline{q} = \min\left\{\arg\min_{q\geq 0}\{AC(q)\}\right\}$

OPTIMAL CHOICE OF THE FIRM'S OUTPUT LEVEL



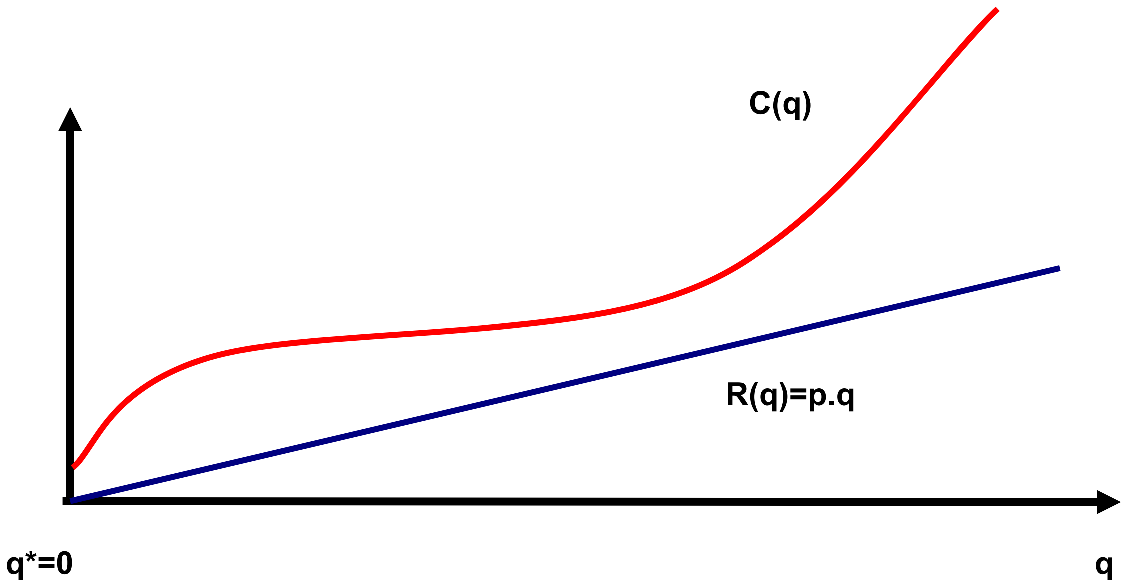
OPTIMAL CHOICE OF THE FIRM'S OUTPUT LEVEL (Cont'd)



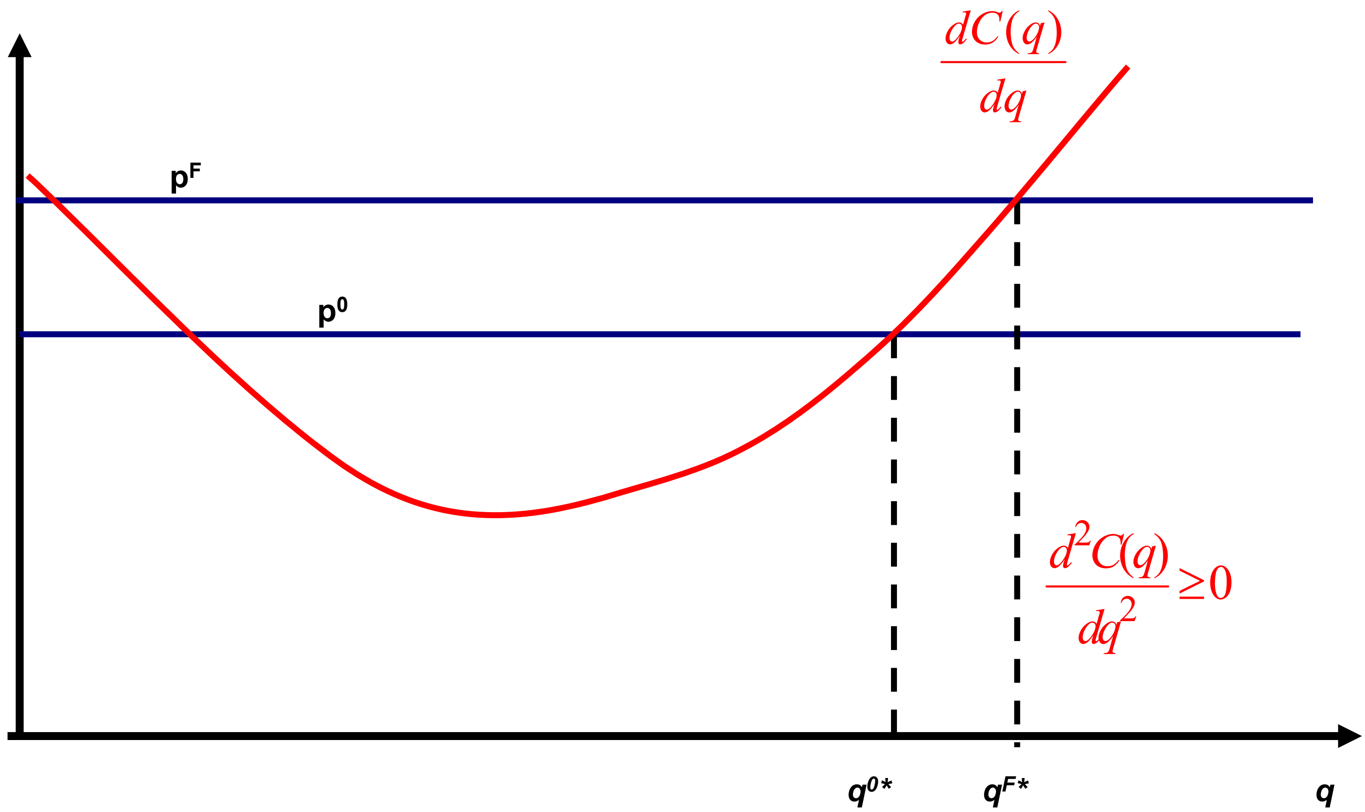
Remark. The firm's variable cost is C(y) – C(0).

OPTIMAL CHOICE OF THE FIRM'S OUTPUT LEVEL (Cont'd)

First and second-order conditions are necessary for an interior optimum. Need to examine the boundaries of the action set as well (here: q* = 0)



COMPARATIVE STATICS OF COMPETITIVE FIRM'S OUTPUT



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KEY CONCEPTS TO REMEMBER

- Profit = Revenue – Cost
- (Net) Inputs vs. (Net) Outputs
- Profit Maximization Problem
- Production (Possibilities) Set
- Properties of Production Sets
- Production Function
- Increasing/Decreasing/Constant Returns to Scale
- Cost Function
- Cost Minimization Problem
- Average Cost, Marginal Cost
- Profit Maximization
- Marginal Revenue = Marginal Cost