MGT 621 – MICROECONOMICS

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3. Choice Under Uncertainty

Autumn 2023

École Polytechnique Fédérale de Lausanne
College of Management of Technology

AGENDA

Elements of Probability

Choice Under Uncertainty

Expected Utility Theory

Risk Aversion and Decision Biases

Key Concepts to Remember

CHOICE UNDER UNCERTAINTY

So far in this course we have assumed that a consumer (decision maker) knows perfectly the consequences of choice.

However, in most practical economic decision situations there is uncertainty.

UNCERTAINTY IN CHOICE
Some Examples

- **How much will my education help me in the job market?**
- **Will I be given valuable assignments if I accept this job offer?**
- **Is the used car I am buying a lemon? Or will it be dependable?**
- **If I put effort into developing a proposal, will it be accepted?**
- **Will an R&D program be successful?**
- **How capable is the person I am considering hiring?**
- **Will my competitors introduce superior new products?**
- **Will potential customers purchase the product I offer?**
- **Will I enjoy the movie?**
- **What will the weather be like in the city I plan to visit?**
- **Will my home catch on fire in the next year and be destroyed?**
- **Will the price of the stock I purchase go up or down?**
- **Will prices for a commodity go up or down? Sign fixed-price contract?**
- **If I take a litigation to trial rather than settling, will I win?**
- **…**

RISK VS. UNCERTAINTY

Distinction between risk and uncertainty due to Frank Knight (1921):

> **"The practical difference between the two categories, risk and uncertainty, is that *in the former the distribution of the outcome in a group of instances is known* (...), *while in the case of uncertainty this is not true*, the reason being in general that it is impossible to form a group of instances, because the situation dealt with is in a high degree unique" (p. 233)**
>
> **"We can also employ the terms "objective" and "subjective" probability to designate the risk and uncertainty respectively, as these expressions are already in general use with a signification akin to that proposed" (ibid.)**

- **Uncertainty: unknowable (e.g., the success probability of your new startup company, or the likelihood of an unforeseen contingency in your project)**
- **Risk: knowable (e.g., the outcome of a die roll)**

In this course, *no explicit distinction between risk and uncertainty*, since in order to formally analyze optimal choice, need to introduce a probability space in either case.

SUBJECTIVE PROBABILITY
Some Examples

- **All would not necessarily agree on the likelihood of events**
  - **Probability relevant for your decision is determined by all your knowledge**
  - **Most economic situations can best be described by subjective probabilities**
- **Will the used car be a lemon? Or will it be dependable?**
  - **Seller knows, i.e., for seller probability of being lemon is 0 or 1**
  - **Buyer does not know, i.e., buyer must assign a probability (= "belief")**
- **What will the weather be like in the city I plan to visit?**
  - **Probability assessment may change after you read weather forecast**
- **How capable is the person I am considering for a job?**
  - **Potential employee has more information about work habits**

SUBJECTIVE PROBABILITY (Cont'd)

New information (e.g., observing other agents' actions) can change beliefs

- **What does this imply? (value of information, … disinformation)**

Can different people have very different probability assessments given the same choice, the same events, and the same information?

- **Agreeing to disagree …**

Most of the time we assume that probabilities are subjective

DISCRETE RANDOM VARIABLES

Definition: A discrete random variable (or lottery) $X = [p_1, x_1; p_2, x_2; \dots; p_n, x_n]$ is a variable x that can take on one of the values

- $x_1, x_2, x_3, \dots, x_n$

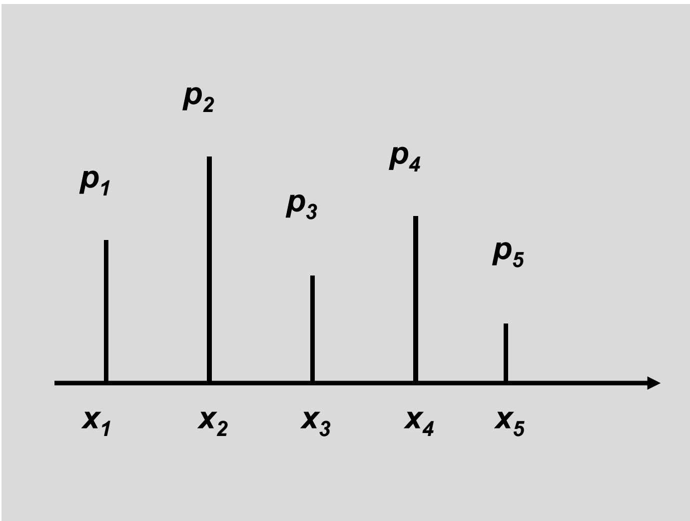
with the respective probabilities

- $p_1, p_2, p_3, \dots, p_n$

where each $p_i \geq 0$ **and**

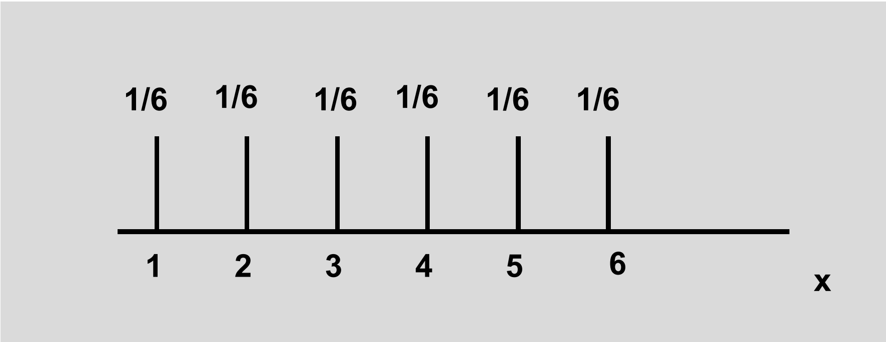
$$\sum_{i=1}^{n} p_i = 1 \qquad \textbf{(normalization)}$$

Using the probability mass function p(.) we can write the probabilities p_i as a function of x_i : $p_i = p(x_i)$.



EXAMPLE: ROLL OF A DIE

	x_i		p_i
x =	1	w/ probability	1/6
	2		1/6
	3		1/6
	4		1/6
	5		1/6
	6		1/6



EXPECTATION

The **expectation** of a (discrete) random variable X is defined as:

$$\overline{X} = E[X] = \sum_{i=1}^{n} p_i x_i$$

Key Property:

Linearity	If X and Y are random variables, and a and b are constants. E[aX+bY] = a E[X] + b E[Y]

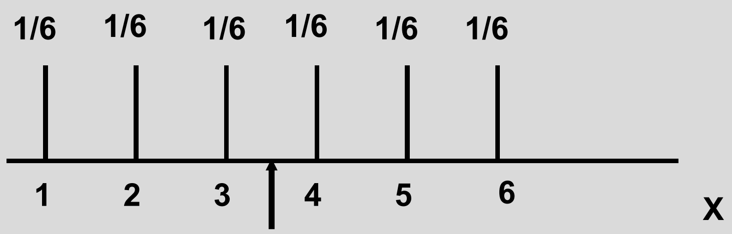
Other Properties:

Sign Preservation	If X can take on only positive values, then E[X]>0.
Certain Value	If X is perfectly known (and equal to x), then E[X]=x.

EXAMPLE (Cont'd)

$$X = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases} \quad \text{w/ probability} \quad \begin{matrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{matrix}$$

$$E[X] = \sum_{i=1}^{6} p_i x_i$$
$$= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \ldots + \frac{1}{6} \cdot 6$$
$$= \sum_{i=1}^{6} \frac{1}{6} \cdot i = 3.5$$



The expectation is where this picture would balance on your finger

VARIANCE AND STANDARD DEVIATION

The variance is a measure of the spread of a random variable around its mean.

$$V[X] = E[(X - \overline{X})^2] = E[(X^2 - 2X\overline{X} + \overline{X}^2)]$$
$$= E[X^2] - 2\overline{X}E[X] + \overline{X}^2 = E[X^2] - \overline{X}^2$$

The standard deviation is the square root of the variance.

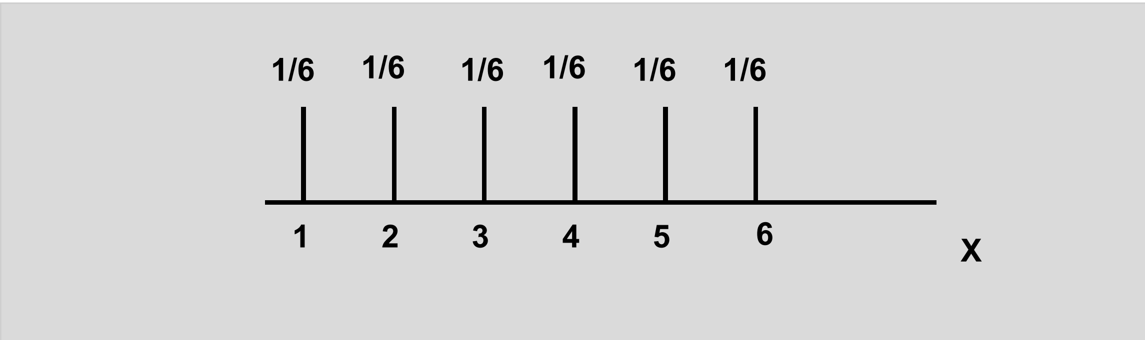
$$\sigma_X = \sqrt{V[X]}$$

It has the same units as X.

EXAMPLE (Cont'd)

$$X = \begin{cases} 1 & \\ 2 & \\ 3 & \\ 4 & \text{w/ probability} \\ 5 & \\ 6 & \end{cases} \quad \begin{matrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{matrix}$$

$$V[X] = E[(X-\overline{X})^2]$$
$$= \frac{1}{6}(1-3.5)^2 + \frac{1}{6}(2-3.5)^2 + \ldots$$
$$+ \frac{1}{6}(6-3.5)^2$$
$$= 2.92$$



CONDITIONAL PROBABILITY

Definition: The conditional probability P(A|B) of event A conditional on event B having realized is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Question. Given a six-sided die what is the probability of rolling a 3 conditional on rolling a 1, 2 or 3?

$$P(\{3\} \mid \{1,2,3\}) = \frac{P(\{3\} \cap \{1,2,3\})}{P(\{1,2,3\})} = \frac{P(\{3\})}{P(\{1,2,3\})}$$

$$= \frac{1/6}{1/2} = 1/3$$

INDEPENDENCE

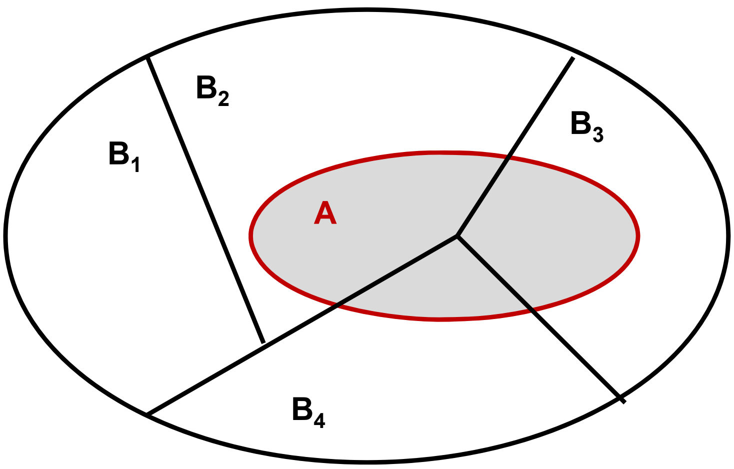
Definition: A and B are independent events, if

- **P(A|B) = P(A)**
- **P(B|A) = P(B)**

$$P(A \cap B) = P(A \mid B)P(B)$$

THE LAW OF TOTAL PROBABILITY

$$P(A) = \sum_{n=1}^{N} P(A \cap B_n) = \sum_{n=1}^{N} P(A \mid B_n)P(B_n)$$



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Key Concepts to Remember

LOTTERIES ARE (DISCRETE) RANDOM VARIABLES

Let X be a random variable with possible outcomes in the set $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ (the "outcome space"). Each outcome x_i occurs with probability p_i.

The random variable X is sometimes also called a lottery and denoted

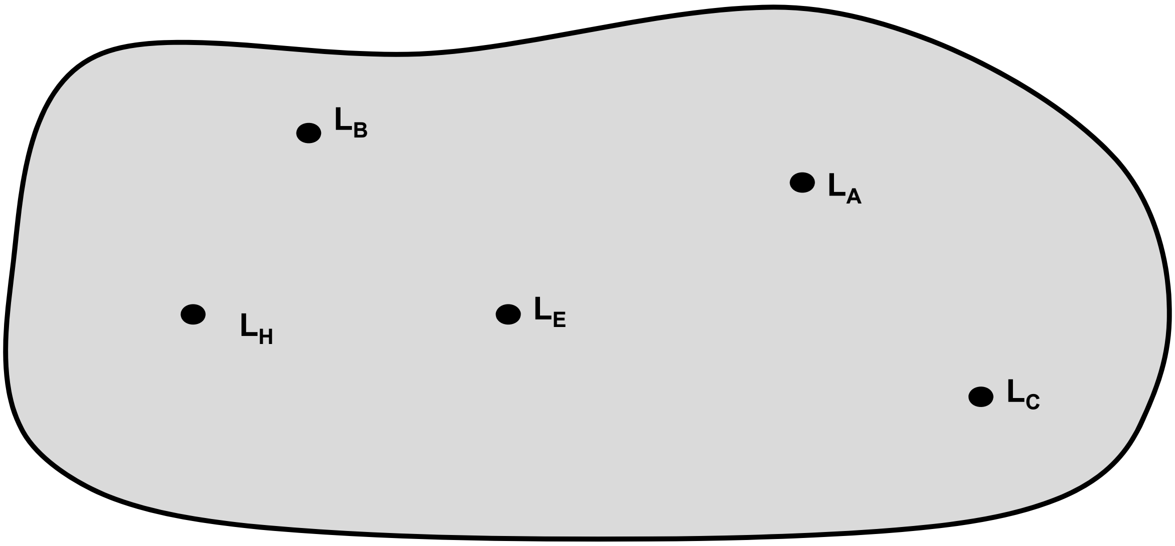
$$X = [p_1, x_1; p_2, x_2; \dots; p_n, x_n]$$

If all outcomes x_i are real and measured in dollars (or any other currency), then X is commonly referred to as a "money lottery."

The set of all lotteries with outcomes in $\mathcal{X}$ is the "lottery space" $L(\mathcal{X})$.

Example: A coin-flip lottery X (with an unbiased coin) pays \$1 if heads and zero if tails. Then X = [0.5,\$1; 0.5, \$0].

THEORY OF CHOICE UNDER UNCERTAINTY



The choice set contains all simple lotteries over the various outcomes.

PREFERENCES OVER LOTTERIES

For a decision maker (DM), *choosing between actions corresponds generally to choosing between lotteries* (given a decision d in the set of possible decisions D, the outcome x_i in *X* occurs with probability $P(x_i|d)$).

Example: wearing a helmet on a motorcycle changes the probability of injury. In a decision tree, each decision node represents a lottery.

Hence, the DM needs to be able to order lotteries according to his preference implying a complete preference (pre-)ordering[(1)] over elements in the lottery space L(*X*).
More specifically, if A and B are elements of L(*X*), then

- **A > B   means DM prefers A to B**
- **A $\geq$ B   means DM does not prefer B to A or DM "weakly" prefers A to B**
- **A ~ B   means DM is indifferent between A and B, i.e., DM will take either one and would play a 50-50 lottery to choose between them.**

(1) Formally, a pre-ordering R is a *binary relation* (i.e., it takes two inputs) that is *reflexive* (i.e., xRx) and *transitive* (i.e., xRy and yRz implies xRz). The pre-ordering is *complete* if for any elements x and y in *X* it is either xRy or yRx (or both).

PREFERENCES OVER LOTTERIES: EXAMPLE

Question. Jane likes to play ping pong and she wonders about how to respond to an opponent's serve.

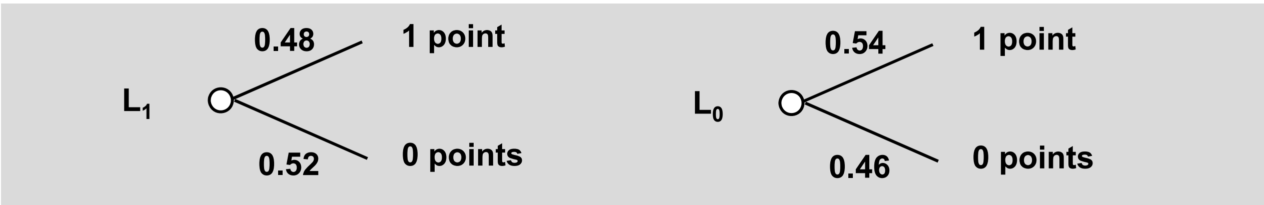
- If she hits a top spin (decision d_1), the ball is going to be on the table with probability 0.6 and given that it is, she is going to score with probability 0.8.
- If she does not play a top spin (decision d_0), the ball is going to land on the table with probability 0.9, but she is only going to score with probability 0.6.

What decision should she take?

Solution:

Jane needs to choose between the following two lotteries:

- L_1 = [P(score|d_1), 1 point; P(don't score|d_1), 0 points]
- L_0 = [P(score|d_0), 1 point; P(don't score|d_0), 0 points]



Thus, she should prefer L_0 (i.e., $L_0 \geq L_1$) which implies **"don't play top spin"** as her decision.

PREFERENCES OVER OUTCOMES
Utility Representation (Reminder)

Preferences over lotteries imply preferences over particular (certain) outcomes in *X*. Indeed, for any x and y in *X* one could just consider the lotteries X and Y that produce the outcomes x and y with probability one respectively.

Thus, $x \geq y$ if the outcome x is (weakly) preferred to the outcome y.

> Definition: A real-valued function u with domain *X* represents the DM's preferences over <u>outcomes</u> in *X*, if for any x,y in *X*:
>
> $$x \geq y \text{ if and only if } u(x) \geq u(y).$$
>
> The function u is called the DM's utility function.

For some preferences no utility function representation exists (e.g., lexicographic preferences). We typically take a utility function as an input for a decision model.(1) A utility function always exists for finite sets of outcomes.

A *utility representation of a DM's preferences is generally* <u>not unique</u>: given any utility function u and a strictly increasing function ϕ (from real numbers to real numbers), the function $v = \phi(u)$ is an equivalent utility representation.

(1) For more details on utility theory see Kreps, D.M. (1988) *Notes on the Theory of Choice*, Westview Press, Boulder, CO, or Fishburn, P.C. (1970) *Utility Theory for Decision Making*, Wiley, New York, NY.

UTILITY REPRESENTATION

Mark has the utility function $u(x) = x^{½}$ for any nonnegative amount of money x (in dollars).

Thus, he prefers \$100 to \$36, since u(100) = 10 > u(36) = 6.

Similarly, for $\phi(y) = y^2$, we have that $\phi(u(100)) = 100 > \phi(u(36)) = 36$, and Mark would have the same preferences for outcomes (amounts of money) for any other ϕ, as long as ϕ is strictly increasing.

- **$x \geq y$ if and only if $u(x) \geq u(y)$ if and only if $\phi(u(x)) \geq \phi(u(y))$**

Thus, in the absence of uncertainty Mark can just maximize x instead of u(x).

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Risk Aversion

Key Concepts to Remember

EXPECTED UTILITY MAXIMIZATION

Given a utility representation u of the DM's preferences over outcomes, we would like to infer his preferences over lotteries of outcomes (random variables), which corresponds to his preferences over actual decisions (e.g., at which speed to drive a car).

Under certain axioms (= assumptions on the DM's preferences over lotteries, typically one uses the Von Neumann-Morgenstern axioms), the DM's expected utility of a particular decision d in D which induces a lottery X(d) with probability distribution P(.|d) is

$$EU(X(d)) = \sum_{x \in X} P(x \mid d) u(x)$$

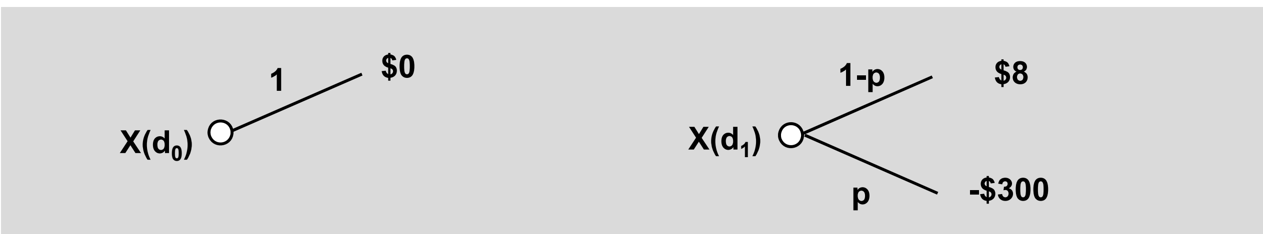
Thus, under uncertainty the DM maximizes expected utility, i.e., he solves

$$\boxed{d^* \in \arg\max_{d \in D} EU(X(d))}$$

EXPECTED UTILITY MAXIMIZATION: EXAMPLE

Question. Joe needs to decide how fast to drive on highway E25 from Lausanne to Geneva. Any minute saved he values at $1. At 120 km/h it takes about 40 min, and at 140 km/h about 32 min. However, if he drives 140 km/h there is a chance p that he gets pulled over and has to pay a ticket worth $280 plus a delay of 20 min (over the 120 km/h time).

Let d_0 : drive 120 km/h, d_1 : drive 140 km/h. He thus needs to choose between the lotteries $X(d_0)$ = [1,$0] and $X(d_1)$= [p, -$300; 1-p, $8].



If Joe's utility function for money is u(x)=-exp(-x/1000), at what detection probability p would he be indifferent between d_0 and d_1?

Answer: $EU(X(d_0)) = u(\$0) = EU(X(d_1)) = pu(-\$300) + (1-p)u(\$8)$

$$\Rightarrow \quad p = \frac{u(\$8) - u(\$0)}{u(\$8) - u(-\$300)} = 2.2\%$$ **(i.e., for p>2.2%, Joe drives 65 mph!)**

VON NEUMANN-MORGENSTERN AXIOMS IMPLY EXPECTED UTILITY REPRESENTATION

Fundamental Justification for Expected Utility Maximization

Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of outcomes and L(X) the corresponding lottery space. Consider arbitrary elements A,B,C of L(X) and x,y,z in X.(1)

1. **Completeness**: $A \geq B$ or $B \geq A$ (can compare any two lotteries)
2. **Reflexivity**: $A \geq A$
3. **Transitivity**: $A \geq B$ and $B \geq C$ implies $A \geq C$ (o/w construct a "money pump")

4. **Continuity**: If x > y > z, then there exists p in (0,1) such that one can achieve indifference between the lottery A = [1, y] and the lottery B(p) = [p, x; 1-p, z], i.e., there is a p such that A ~ B(p)

- **Example: What happens if z is "death"?**

5. **Independence (of Irrelevant Alternatives)**: If x > y, then for any z: [p, x; (1-p), z] > [p, y; (1-p), z]

(1) Note that if x is an outcome in X, then [1, x] is a lottery in L(X). Thus, any outcome can be viewed as a (degenerate) lottery.

KEY IMPLICATION OF THE VNM AXIOMS

Theorem (Von Neumann-Morgenstern, 1944): If a DM's preferences "$\geq$" on L(X) satisfy the Von Neumann-Morgenstern axioms, then there exists an expected utility function EU(.) for that DM which represents his preferences in the sense that for any two lotteries A, B in L(X):

$A \geq B$ if and only if $EU(A) \geq EU(B)$

Proof: See notes on "Risk and Uncertainty" posted on the course website.

In more detail, suppose:

$A = [p_1,x_1; p_2,x_2; \dots ; p_n,x_n]$, where $p_1 + \dots + p_n = 1$

$B = [q_1,x_1; q_2,x_2; \dots ; q_n,x_n]$, where $q_1 + \dots + q_n = 1$

If DM's preferences satisfy the VNM Axioms, then the DM prefers A to B ($A \geq B$) if and only if

$$EU(A) = \sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{n} q_i u(x_i) = EU(B)$$

<u>Note</u>: Two expected utility functions EU(x) and EV(x) represent the same preferences over lotteries if and only if $EU(x) = \alpha EV(x) + \beta$, where $\alpha > 0$ and β is any real number. In the same vein, a "positive affine transformation" of the DM's utility function u (to $v = \alpha u + \beta$, for $\alpha > 0$) does not change the DM's preferences over lotteries.

The original reference for the Von Neumann-Morgenstern result is their immensely readable text Von Neumann, J., Morgenstern, O. (1944) *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ.

KEY IMPLICATION OF THE VNM AXIOMS (Cont'd)

Example: Consider the utility functions $u(x) = 1 - 2e^{-\rho x}$ and $v(x) = 3 - 5e^{-\rho x}$, where $\rho>0$ is some constant.

Since $u(x) = .4v(x) - .2 = \alpha v(x) + \beta$ (with $\alpha > 0$), one can check that for any two lotteries A and B in L(X):

$$EU(A) \geq EU(B) \quad \text{if and only if} \quad EV(A) \geq EV(B)$$

More generally, invariance with respect to positive affine transformations implies that we can fix any two values of a DM's utility function without disturbing his preference ordering over lotteries.

ALLAIS PARADOX

Imagine the following two decision situations—each involving a pair of gambles.

SITUATION I

	P(Winning)	AMOUNT TO WIN
Lottery A	100%	\$1,000,000
Lottery B	10%	\$5,000,000
	89%	\$1,000,000
	1%	-0-

SITUATION II

	P(Winning)	AMOUNT TO WIN
Lottery C	11%	\$1,000,000
	89%	-0-
Lottery D	10%	\$5,000,000
	90%	-0-

ALLAIS PARADOX
Implies Critique of 'Independence of Irrelevant Alternatives'

	Ticket Numbers		
Lottery	1	2-11	12-100
A	$1 million	$1 million	$1 million
B	$0 million	$5 million	$1 million
C	$1 million	$1 million	$0 million
D	$0 million	$5 million	$0 million

NOTE: Ticket numbers 12-100 are the same for A and B so they are irrelevant for choices between them; the same for C and D

If you eliminate these ticket numbers then
A and C are identical and so are B and D

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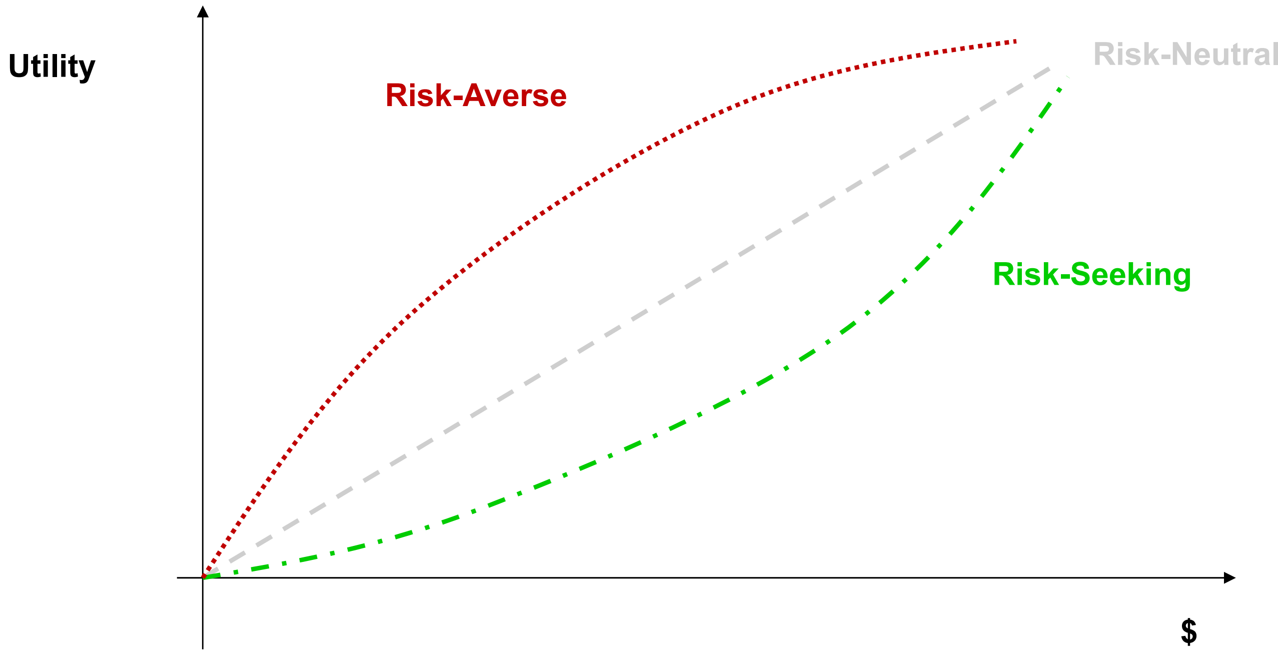
UTILITY FUNCTIONS: SOME COMMON SHAPES



RISK AVERSION: EXAMPLES

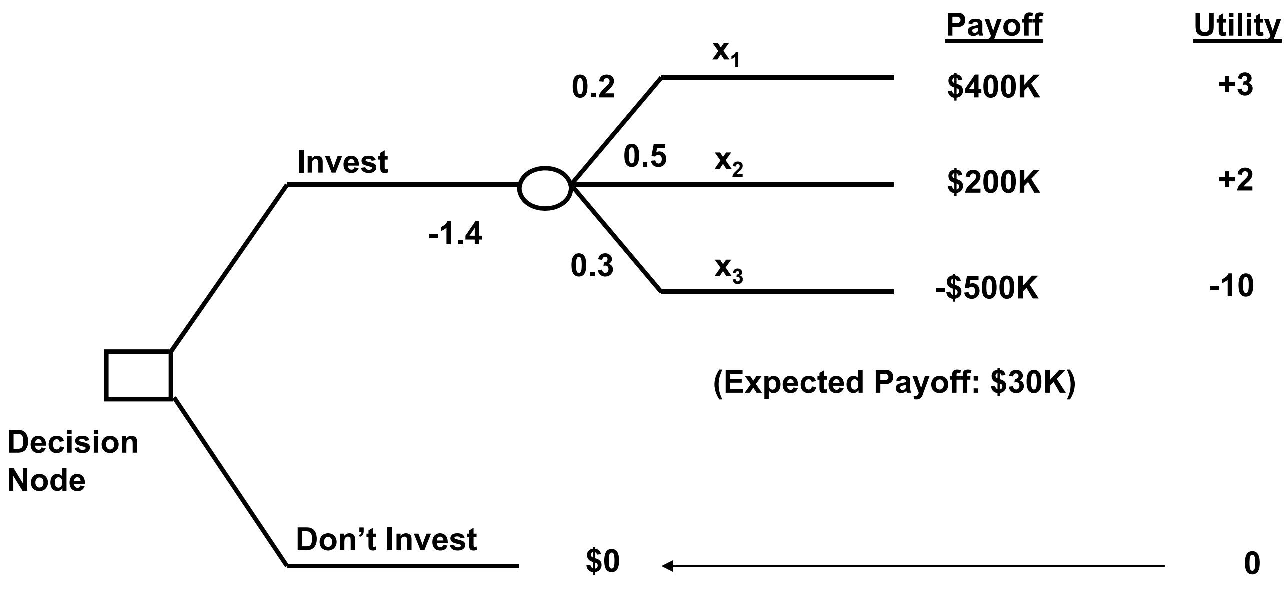
People will buy insurance, even though the expected value of the payment from insurance is smaller than its price. That is the result of risk aversion.

People purchase a portfolio of stocks and bonds, rather than only one. Such diversification reduces risk and is consistent with risk aversion.

People will incur costs to purchase hedges, assets that reduce the risk of the overall portfolio.

Typically the larger the monetary lottery, the greater the degree of risk aversion people exhibit.

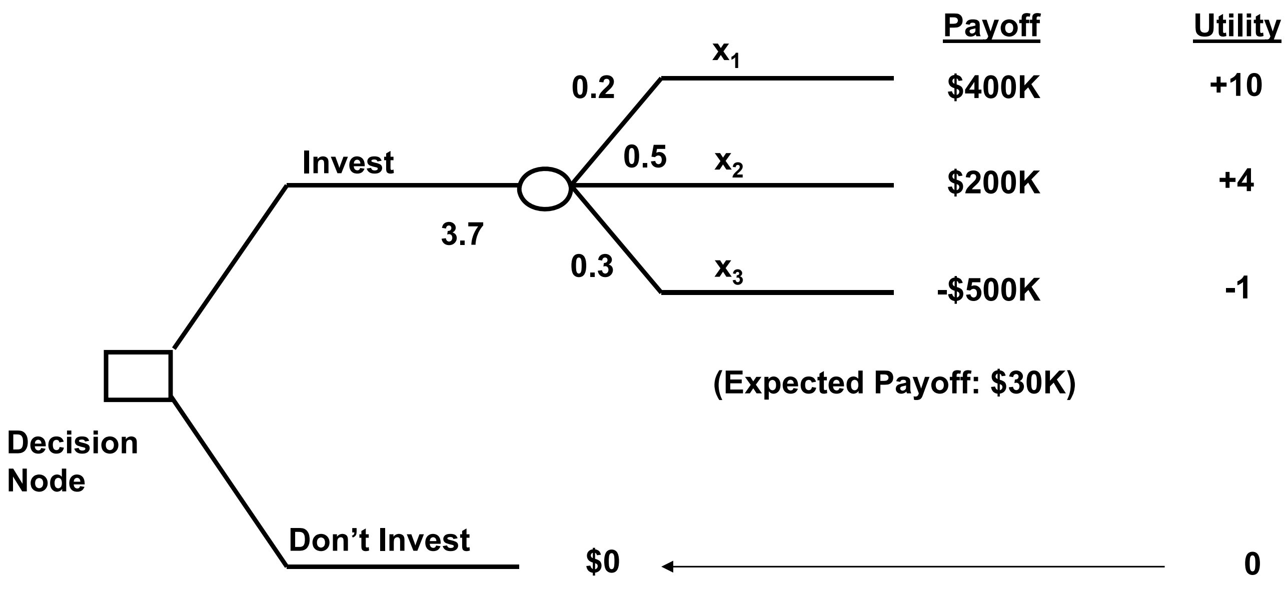
A SIMPLE DECISION: RISK-AVERSE DM



Decision Criterion: Maximize Expected Utility

EU = (0.2)(3)+(0.5)(2)+(0.3)(-10) = -1.4 < 0 → Don't Invest!

A SIMPLE DECISION TREE: RISK-SEEKING DM



Decision Criterion: Maximize Expected Utility

EU = (0.2)(10)+(0.5)(4)+(0.3)(-1) = 3.7. Invest!

ABSOLUTE AND RELATIVE RISK AVERSION DESCRIBE A DM'S RISK ATTITUTE

The level of risk aversion may be measured by the (Arrow-Pratt) absolute-risk-aversion coefficient,

$$R(x) = -\frac{u''(x)}{u'(x)}$$

or the relative-risk-aversion coefficient,

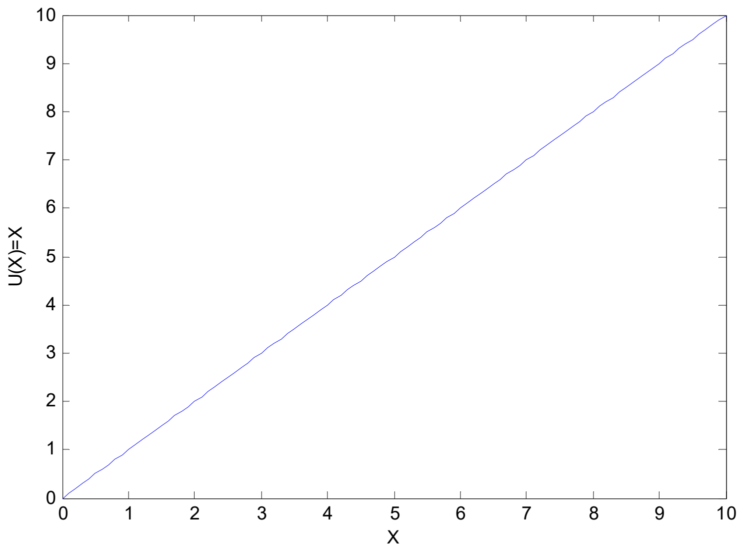
$$r(x) = xR(x)$$

If R(x) > 0, the DM is risk-averse. Similarly, if R(x) < 0, the DM is risk-seeking, while R(x)=0 for a risk-neutral DM.

Both absolute and relative risk aversion are local properties: they can vary for different outcomes.

RISK NEUTRALITY: LINEAR UTILITY

$$u(x) = \alpha\, x \qquad u'(x) = \alpha \qquad u''(x) = 0$$



$$(\alpha > 0)$$

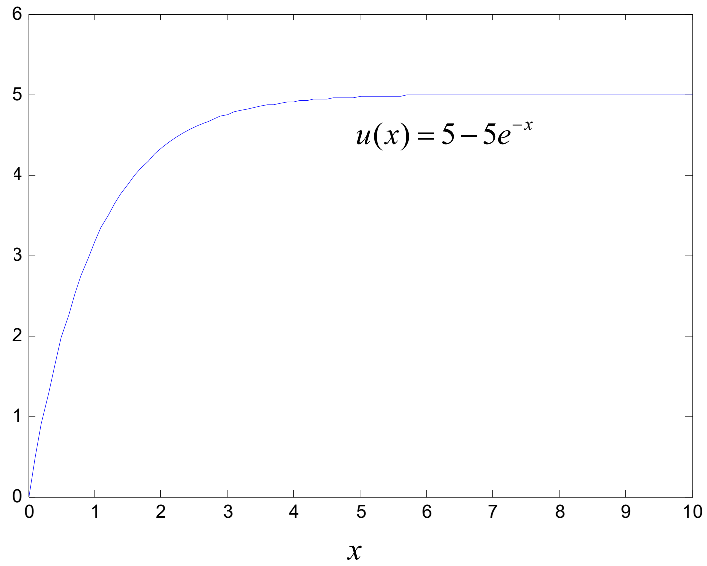
$$R(x) = -\frac{u''(x)}{u'(x)} = 0$$

CONSTANT ABSOLUTE RISK AVERSION

Exponential utility: $u(x) = \alpha - \beta e^{-\rho x}$

$$u'(x) = \rho\beta e^{-\rho x}$$

$$u''(x) = -\rho^2 \beta e^{-\rho x}$$



$$R(x) = -\frac{u''(x)}{u'(x)} = \rho$$

Exponential utility functions exhibit constant absolute risk aversion (CARA).[1]

(1) CARA utility functions are often used in financial modeling, since it allows obtaining conclusions free from wealth effects (adding a constant w to an individual's wealth just amounts to a positive linear transformation and thus leads to the same decisions, since the expected utility representation of the individual's preferences does not change).

CERTAINTY EQUIVALENT

The certainty equivalent of a lottery is a single certain outcome for which the DM is indifferent between receiving the outcome for sure and participating in the lottery.

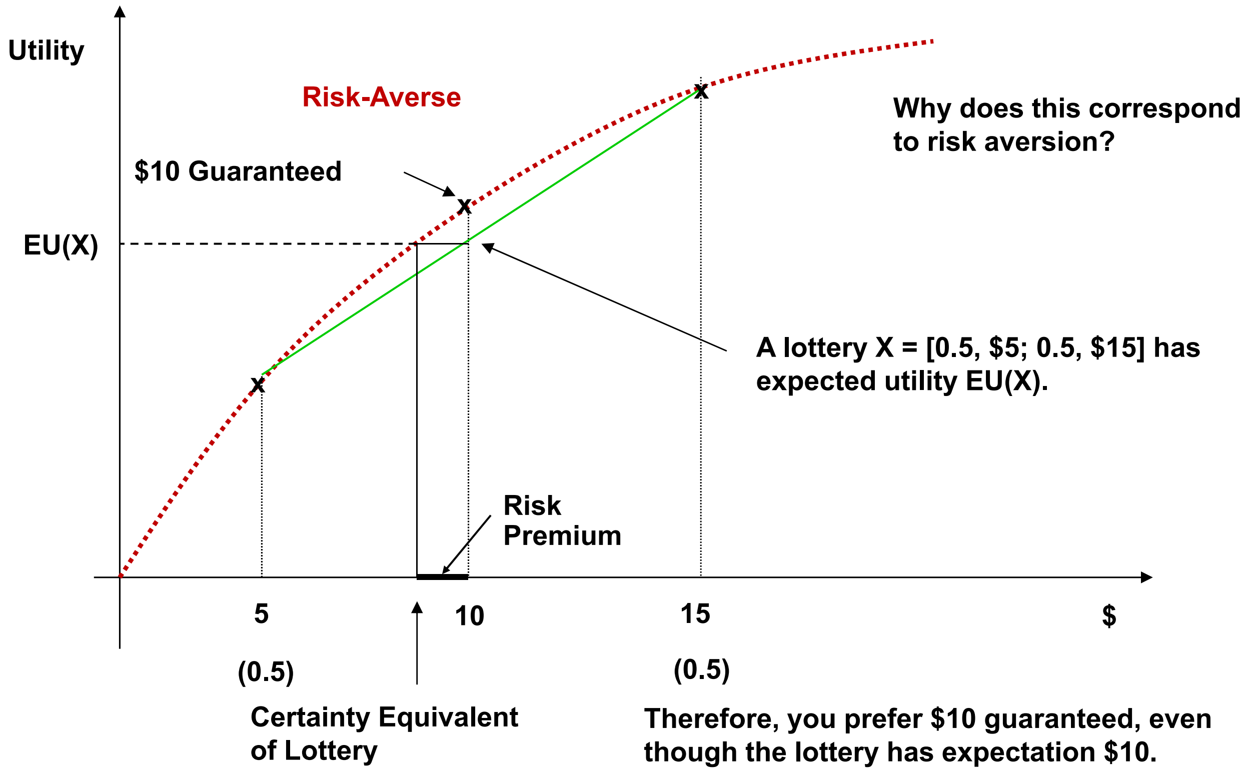
It represents the "selling price" of the lottery.

Denote the certainty equivalent of a lottery X by $CE(X)$

Then: $u(CE(X)) = EU(X)$ **(DM is indifferent)**

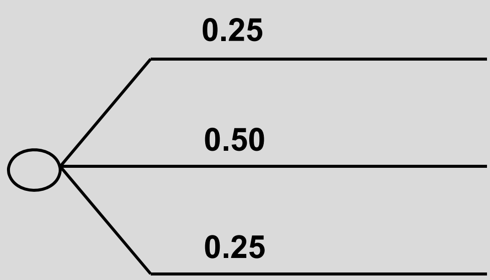
$$\Rightarrow \quad CE(X) = u^{-1}(EU(X))$$

CERTAINTY EQUIVALENT: EXAMPLE



CERTAINTY EQUIVALENT: ANOTHER EXAMPLE

Consider the lottery X = [.25, $100; .5, $49; .25, $0] and the utility function $u(x) = \sqrt{x}$



	Payoff (x)	Utility u(x)
0.25	$100	10
0.50	$49	7
0.25	$0	0

Expected utility: EU(X) = (0.25)(10) + (0.5)(7) + (0.25)(0) = 6

$$EU(X) = u(CE) = \sqrt{CE} \Rightarrow \sqrt{CE} = 6 \Rightarrow CE = 36$$

EXAMPLE: CONSTRUCTING A UTILITY FUNCTION FOR MONEY

Arbitrarily assign utilities to two real-valued outcomes, x_1 and x_2 (say, measured in dollars). For example, x_1 = -\$128 and x_2 = \$128, and

$$u(-\$128) = -100 \text{ and } u(\$128) = 100.$$

Use continuity axiom to specify other utilities.

<u>Certainty Equivalence Method</u>: Fix p and two outcomes x_1 and x_2. Then find an outcome y which makes you indifferent between having y for certain or taking the lottery [p, x_1;1-p, x_2].(1)

The value y is commonly referred to as the certainty equivalent (CE) of the lottery [p,x_1;1-p,x_2]: y = CE,

$$u(CE) = p\, u(x_1) + (1-p)\, u(x_2).$$

(1) In a set of discrete outcomes such an element y might not be available. Then one needs to adjust the probability p accordingly, which by the continuity axiom can always achieve indifference.

A MARKET FOR COIN FLIPS

Consider the following game. You flip a fair coin.

- **If the first flip is heads (H) you win \$2 and you flip the coin again. If it is tails (T), the you win \$0 and the game is over.**
- **If the second flip is H you win \$4 and you flip the coin again. If it is T, then you keep the \$2 you won on the first flip and the game is over.**
- **If the n-th flip is H you win \$$2^n$ and you flip the coin again. If it is T, then you keep the \$$2^{n-1}$ you won on the (n-1)st flip and the game is over.**

The following table summarizes the outcome (we restrict the length to $n \leq 7$ to avoid bankruptcy of players).

Number of Heads in a Row (n)	Total Winnings
1	\$2
2	\$4
3	\$8
4	\$16
5	\$32
6	\$64
7	\$128

A MARKET FOR COIN FLIPS (Cont'd)

Please answer the following question (depending on your role):

- **Bankers: if you are a banker (i.e., act as a bank in this game), how much would you need to be paid for sure to run the game? The person with the lowest amount will serve as the banker and play the game for real.**

- **Players: if you are a player (i.e., you get to potentially win in this game), how much would you be willing to pay to participate in the game? The person with the highest amount will play the game for real.**

MARKET FOR COIN FLIPS: ACTUARIAL VALUE

Let us compute the actuarial value of the coin-flip game X:

$$E[X] = (½)\ \$0 + (¼)\ \$2 + (1/8)\ \$4 + \ldots + (1/2^n)\ \$2^{n-1} + (1/2^n)\ \$2^n$$

$$= (n-1)\ (\$0.5) + \$1 \underset{n\,=\,7}{=} \$4.$$

The certainty equivalent CE(X) of the coin-flip lottery X *for a player*,(1) given a utility function u, satisfies therefore:

$$u(CE(X)) = (½)\ u(\$0) + (¼)\ u(\$2) + (1/8)\ u(\$4) + \ldots + (1/2^n)\ u(\$2^{n-1}) + (1/2^n)\ u(\$2^n) = EU(X),$$

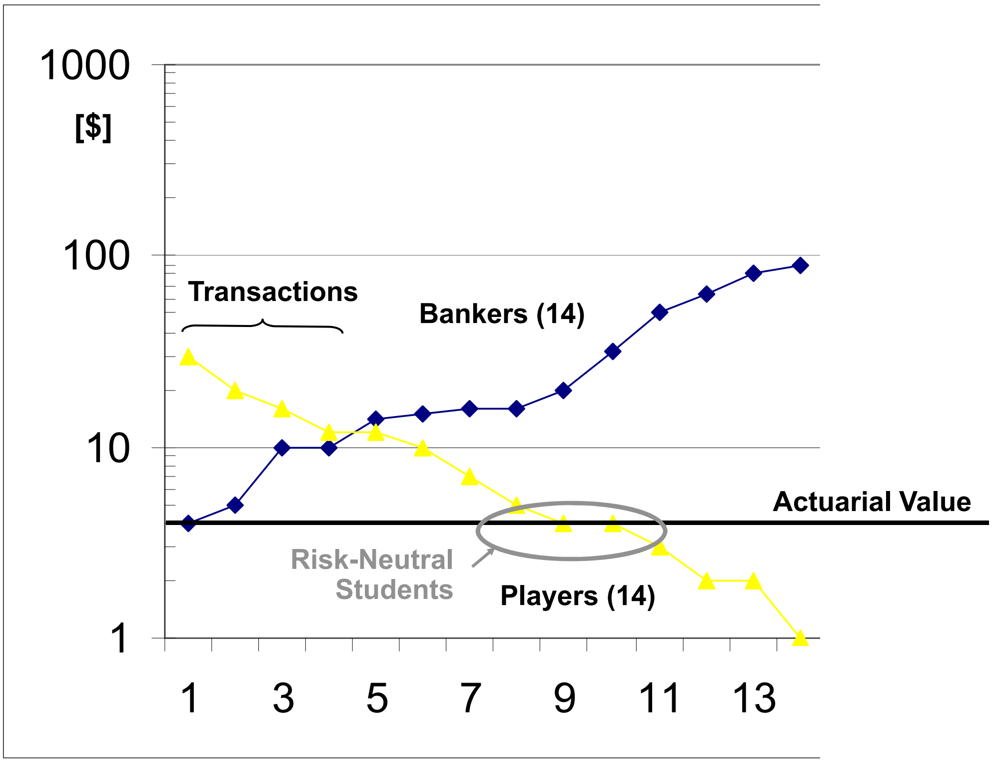
so that

$$CE = u^{-1}(EU(X))$$

You can read the utilities off your utility function constructed a couple of slides ago (for n = 7)

(1) For a *banker* one can compute the CE in a similar way, just by taking the outcomes in terms of losses instead of gains.

MARKET FOR COIN FLIPS: CLASS RESULTS



The plotted values correspond to the certainty equivalents of bankers and players respectively. Why are they not the same?

INDIVIDUALS ARE RISK-AVERSE IN GAINS AND RISK-SEEKING IN LOSSES (SENSITIVITY TO REFERENCE POINT)

We already did this experiment!

1. You have been given $200 and have a choice between the following two options

A: Win $150 with certainty

B: Win $300 with probability .5

Win $0 with probability .5

- **Do you prefer A or B ?**

2. You have been given $500 and have a choice between the following two options

C: Lose $150 with certainty

D: Lose $300 with probability .5

Lose $0 with probability .5

- **Do you prefer C or D?**

SENSITIVITY TO REFERENCE POINT: CLASS RESULTS

		Gamble C	Loss Averse Gamble D
Risk Averse	Gamble A	35	28
	Gamble B	7	8

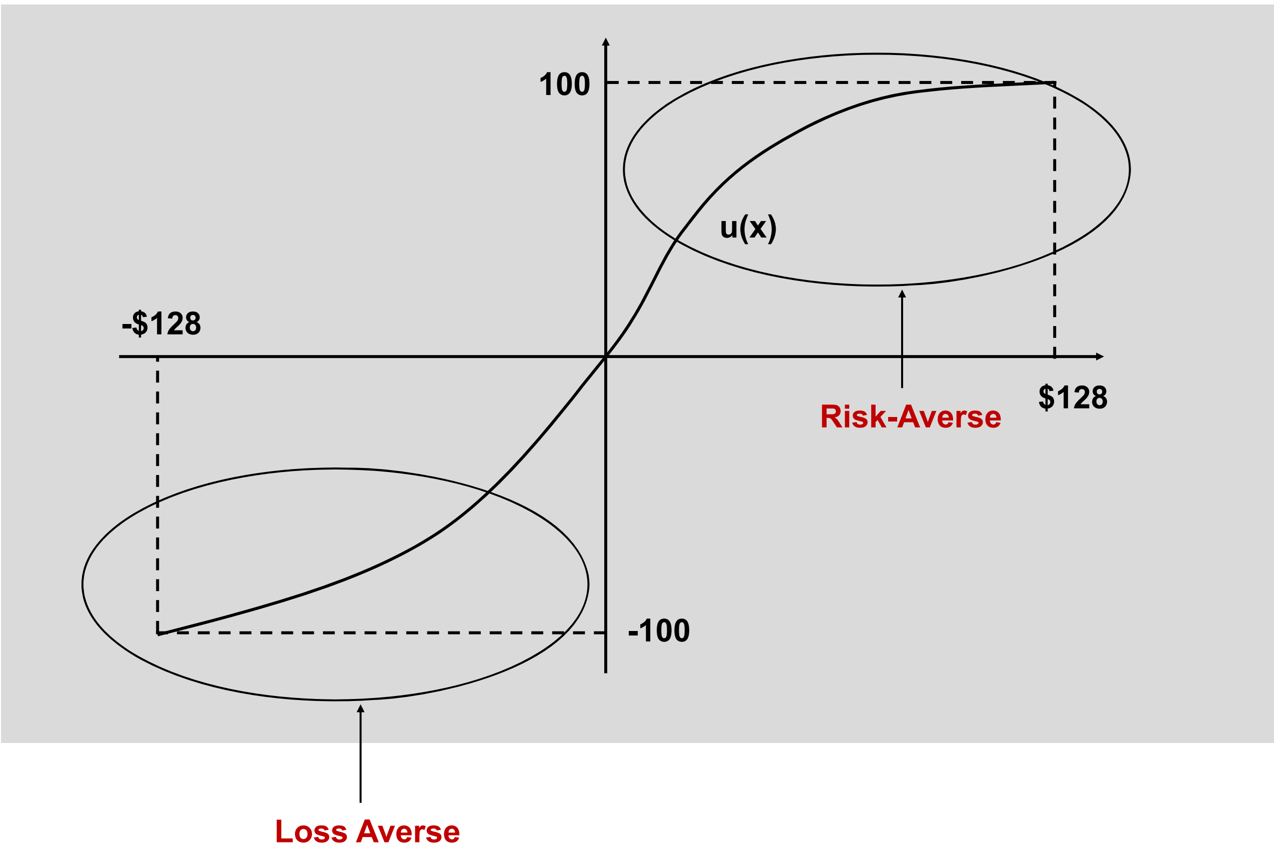
Rational choices (satisfying VNM axioms)

According to utility theory A = C and B = D; so if A is preferred to B then C should be preferred to D and vice versa.

The “modal choices” are (i.e., “most people prefer”) A and D to avoid losses.

Remark: This sensitivity to the framing of the lottery choice in terms of either gains or losses is sometimes also referred to as *reflection effect*: preferences tend reverse if the lottery is “reflected” from the domain of gains (with respect to the *status quo*) to the domain of losses. For details, see Kahneman, D., Tversky, A. (1979) “Prospect Theory: An Analysis of Decision Under Risk,” *Econometrica*, Vol. 47, No. 2, pp. 263—291.

“REAL” UTILITY FUNCTIONS OFTEN LOOK LIKE THIS



COMPARISON OF RISK AVERSION

Theorem (Pratt, 1964). *Assume that agents U and V have the same initial wealth w and suppose that their utility functions u and v are twice differentiable. Then the following statements are equivalent:*

(i) Agent U is more risk averse than agent V.

(ii) There is a strictly increasing concave function φ such that $u = \varphi \circ v$.

(iii) Agent U's absolute risk aversion is larger than agent V's absolute risk aversion, i.e., $\rho_A(u;w) \geq \rho_A(v;w)$ for all w.

(iv) The risk premium that agent U is willing to pay exceeds the risk premium that agent V is willing to pay, i.e., $\pi(u;w) \geq \pi(v;w)$ for all w.

Proof: See notes on "Risk and Uncertainty" posted on the course website.

WHEN IS ONE LOTTERY PREFERRED TO ANOTHER FOR A *CLASS* OF UTILITY FUNCTIONS?

$$Eu(\tilde{x}) \leq Eu(\tilde{y}) \quad \forall u \in \mathcal{U}. \qquad (**)$$

Definition. *If* **(**)** *holds, the risk $\tilde{y}$ is said to* stochastically dominate *$\tilde{x}$ with respect to $\mathcal{U}$, denoted by $\tilde{x} \preceq_{\mathcal{U}} \tilde{y}$. Necessary and sufficient conditions on $\tilde{x}$ and $\tilde{y}$ for* **(**)** *to hold are called a* stochastic dominance order (representation) relative to $\mathcal{U}$.

Class of Utility Functions (= Class of Agents)

Answer: Construct a stochastic dominance order

- **First-order stochastic dominance: all agents with increasing utility**
- **Second-order stochastic dominance: all agents with increasing concave utility**

STOCHASTIC DOMINANCE: DEFINITIONS

Let $X = [p_1, x_1; p_2, x_2; \ldots; p_n, x_n]$ and $Y = [q_1, y_1; q_2, y_2; \ldots; q_m, y_m]$ be two given ***discrete* random variables**, each with a finite number of realizations.

Without any loss of generality we can assume that $m = n$ and that $x_i = y_i$ for all i in $\{1, \ldots, n\}$, and that $x_1 < x_2 < \ldots < x_n$. This situation can always be achieved by extending the discrete variables X and Y to all events in the union of $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ assigning zero probabilities if necessary and subsequent relabeling.

Definition. Let $m = n$ and $x_i = y_i$ for all i in $\{1, \ldots, n\}$. **Y first-order stochastically dominates X** if

$$\sum_{i=1}^{k} q_i \leq \sum_{i=1}^{k} p_i$$

for all k in $\{1, \ldots, n\}$.

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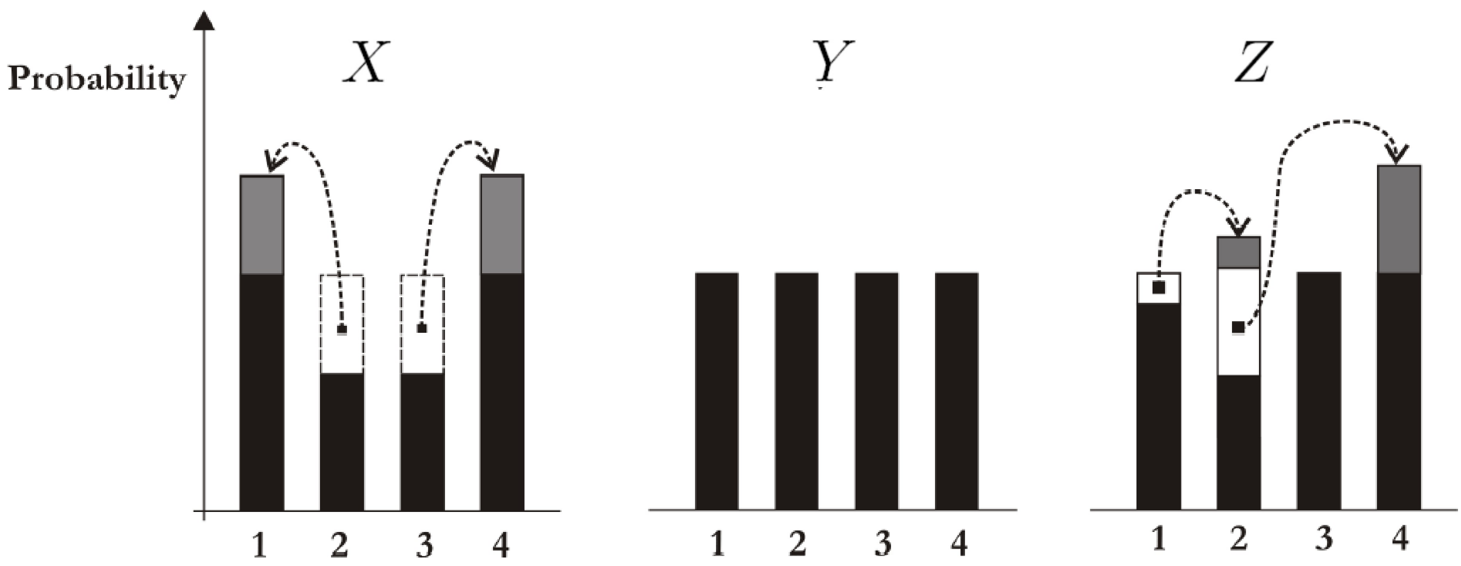
FIRST- AND SECOND-ORDER STOCHASTIC DOMINANCE

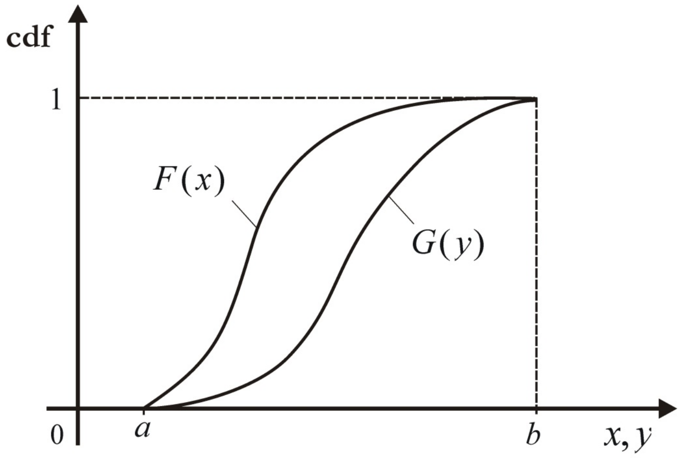


SOSD-Shift | **Original Distribution** | **FOSD-Shift**

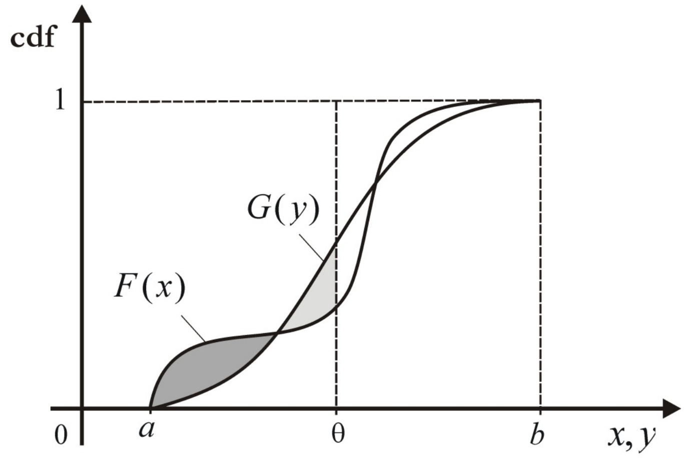
$\boxed{X \preceq_{\mathrm{SOSD}} Y}$ $\boxed{Y \preceq_{\mathrm{FOSD}} Z}$

FIRST- AND SECOND-ORDER STOCHASTIC DOMINANCE
General Case

Let F, G be two cumulative distribution functions (measures) for random variables X and Y, respectively, distributed on the set [a,b]. When does Y FOSD/SOSD-dominate X ?



FOSD



SOSD

$$X \preceq_{\text{FOSD}} Y \quad \Leftrightarrow \quad G(z) \leq F(z) \;\; \forall\, z \in [a,b]$$

$$X \preceq_{\text{SOSD}} Y \quad \Leftrightarrow \quad \int_a^z G(y)dy \leq \int_a^z F(x)dx \;\; \forall\, z \in [a,b]$$

AGENDA

Elements of Probability

Choice Under Uncertainty

Expected Utility Theory

Risk Aversion and Decision Biases

Key Concepts to Remember

KEY CONCEPTS TO REMEMBER

- Risk and Uncertainty
- Objective and Subjective Probability
- Discrete Random Variable
- Lottery
- Von Neumann-Morgenstern Axioms → Expected Utility Representation
- Allais Paradox
- Expected Utility Maximization
- Risk Aversion (Absolute & Relative)
- Sensitivity to Reference Point (Reflection Effect)
- Stochastic Dominance (First-Order & Second-Order)