MGT 621 – MICROECONOMICS

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2. Demand Theory

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AGENDA

SOME SPECIAL UTILITY FUNCTIONS

Let $\alpha_i, \rho > 0$ **and** $k : \Re \to \Re$ **be an increasing function.**

1. **Cobb-Douglas:**

$$u(x) = k\left(\prod_{i=1}^{L} x_i^{\alpha_i}\right)$$

Preferences are strongly monotonic and strictly convex on $\Re_{++}^{L}$

2. **Constant-Elasticity-of-Substitution (CES)**

$$u(x) = k\left(\left[\sum_{i=1}^{L} \alpha_i x_i^{\rho}\right]^{1/\rho}\right)$$

CES preferences are strongly monotonic and strictly convex on $\Re_{++}^{L}$**. Cobb-Douglas is a special case for** $\rho = 1$ **.** (1)

3. **Fixed-Coefficient (Leontief)**

$$u(x) = k\left(\min_{i \in \{1,\ldots,L\}} \{\alpha_i x_i\}\right)$$

Preferences are continuous, monotonic (not strongly), convex (but not strictly)

(1) The "elasticity of substitution" (introduced by John Hicks) between good 1 and 2 is $E_{12} = -\, d\ln(x_1/x_2)\, /\, d\ln(u_{x1}/u_{x2})$ ($= E_{21}$).

AGENDA

Some Special Utility Functions

Wealth Effects

Price Effects

Demand Aggregation

Standard Welfare Measures

Welfare Changes

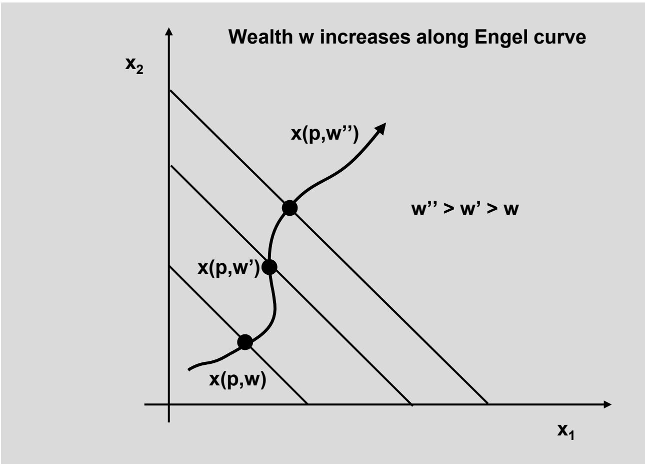
Key Concepts to Remember

COMPARATIVE STATICS: WEALTH EFFECT

Definition. Let x(p,w) be a consumer's Walrasian demand function.

For any fixed price p, the function f(w) = x(p,w) is called a wealth expansion function; its graph is a **wealth expansion path** (also known as **Engel curve**).

The derivative of x(p,w) with respect to wealth, $f'(w) = D_w x(p,w) = \left[\frac{\partial x_i(p,w)}{\partial w}\right]_{i=1}^{L}$,
is called the **wealth** (or **income**) **effect** (on demand).



Good i is called **normal** if $\frac{\partial x_i(p,w)}{\partial w} \geq 0$,
otherwise it is called **inferior** (at (p,w)).

Demand is called **normal** if all goods are normal at any (p,w).

INCOME ELASTICITY

Income elasticity of demand for good i:

- **differentiate Walrasian demand x_i with respect to w, then multiply by (w/x_i):**

$$e_i = (w/x_i)\ \partial x_i/\partial w$$

AVERAGE INCOME ELASTICITY

$\Sigma \{ p_i x_i \} = w$

Differentiate with respect to w

$\Sigma \{ p_i \partial x_i/\partial w \} = 1$

Multiply each term by $(x_i w / x_i w)$ and regroup

$$\Sigma \underbrace{[(p_i x_i)/w]}_{\text{Fraction of income spent on good i}} \underbrace{[(\partial x_i/\partial w)(w/x_i)]}_{\text{Income elasticity of demand for good i}} = 1$$

Weighted average of income elasticities equals 1. Weights are fractions of income spent on various goods

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PRICE ELASTICITY

(Own-)Price elasticity of demand for good i

- **differentiate Walrasian demand for good i with respect to p_i, then multiply by (p_i/x_i):**

$$\varepsilon_{ii} = (p_i/x_i)\ (\partial x_i / \partial p_i)$$

Cross-Price elasticity of demand for good i with respect to changes in the price of good j

- **differentiate Walrasian demand for good i with respect to p_j, then multiply by (p_j/x_i):**

$$\varepsilon_{ij} = (p_j/x_i)\ (\partial x_i / \partial p_j)$$

PRICE / INCOME ELASTICITIES FOR DIFFERENT COMMODITIES

Commodity	Price Elasticity	Income Elasticity
Electricity	- 1.2	+ 0.2
Beef (Meat)	- 0.9	+ 0.4
Women's hats	- 3.0	
Sugar	- 0.3	
Corn	- 0.5	
Potatoes	- 0.3	
Movies	- 3.7	
Flour		- 0.4
Restaurant Meals		+ 1.5
Margarine		- 0.2
Butter		+ 0.4
Furniture		+ 1.5
Milk		+ 0.1

ELASTICITIES FOR DIFFERENT COMMODITIES (Cont'd)
Cross-Price Elasticities

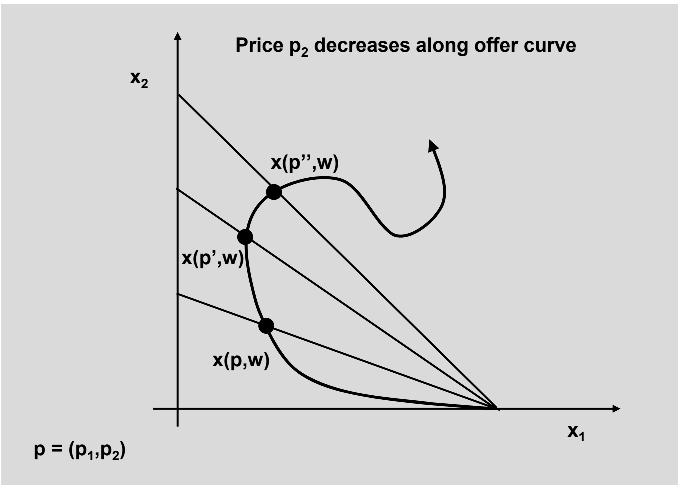
Commodity	With Respect to Price of	Cross-Price Elasticity
Electricity	Natural Gas	+ 0.2
Natural Gas	Fuel Oil	+ 0.4
Beef	Pork	+ 0.3
Pork	Beef	+ 0.1
Margarine	Butter	+ 0.8
Butter	Margarine	+0.7
Gasoline	Automobiles	Negative
Solar Panels	Electricity	Positive
Software	Computers	Negative
Hotel Rooms	Airline travel	Negative

COMPARATIVE STATICS: PRICE EFFECT

Definition. Let x(p,w) be a consumer's Walrasian demand function.

For any fixed wealth w, the graph of the function g(p) = x(p,w) is called the consumer's **offer curve**.

The derivative of x(p,w) with respect to price, $Dg(p) = D_p x(p,w) = \left[\frac{\partial x_i(p,w)}{\partial p_j}\right]_{i,j=1}^{L}$, is called the **price effect** on demand.



Good i is called a **Giffen good (at (p,w))** if $\frac{\partial x_i(p,w)}{\partial p_i} > 0$, otherwise it is called a non-Giffen good (= standard case).

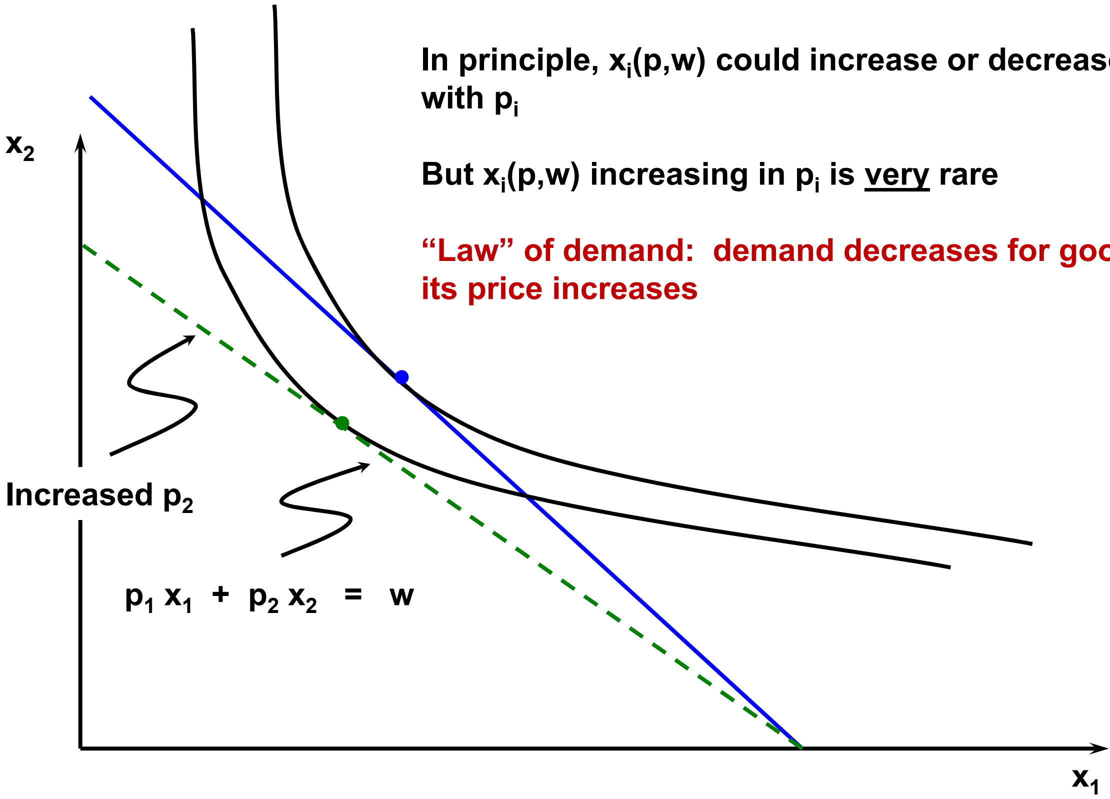
Demand exhibits **own-price effects** (of price of good i on demand of good i), and **cross-price effects** (of price of good i on demand of good j).

COMPARATIVE STATICS WITH RESPECT TO PRICES

In principle, $x_i(p,w)$ could increase or decrease with p_i

But $x_i(p,w)$ increasing in p_i is <u>very</u> rare

"Law" of demand: demand decreases for good if its price increases

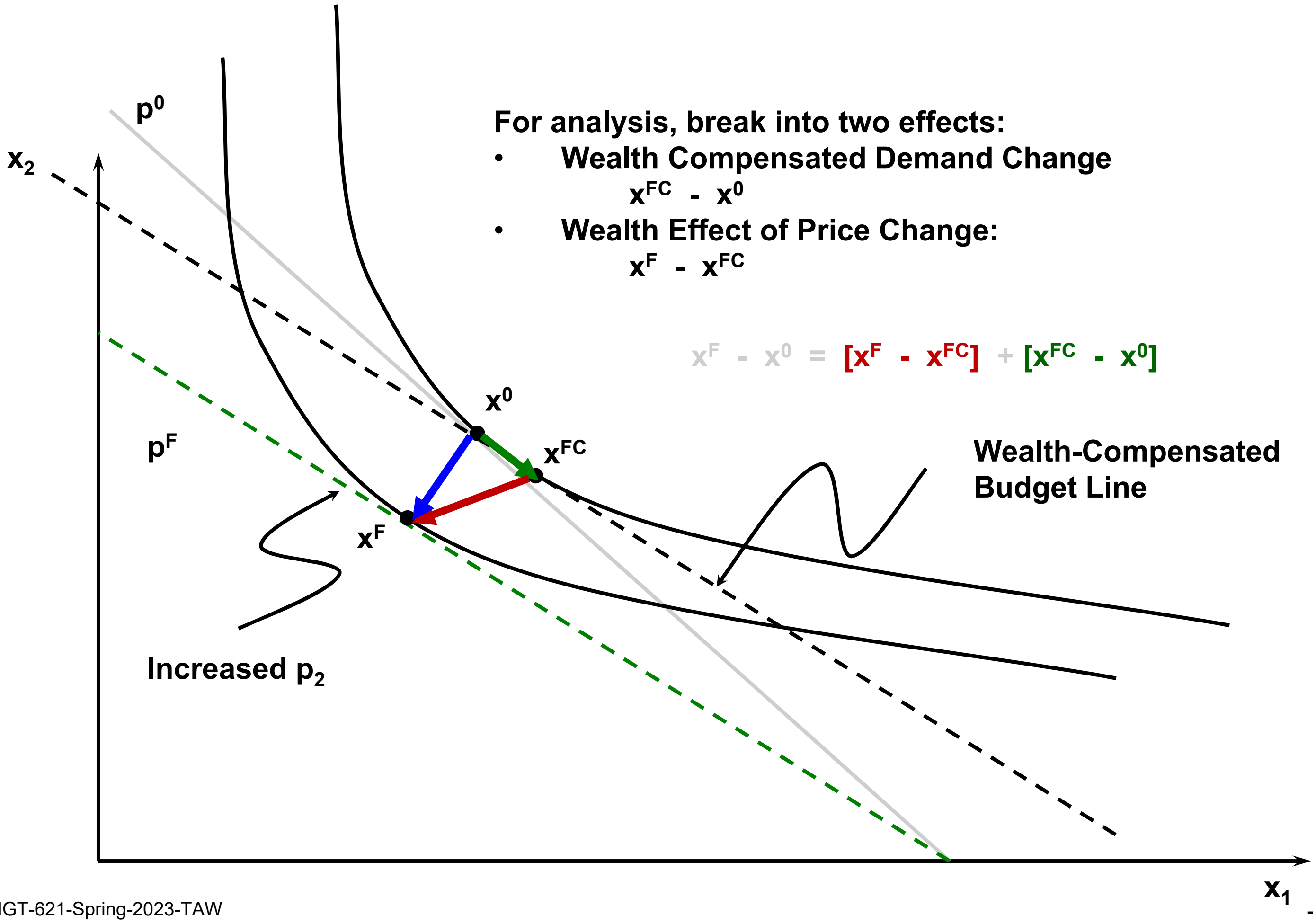


RESPONSE TO A PRICE INCREASE

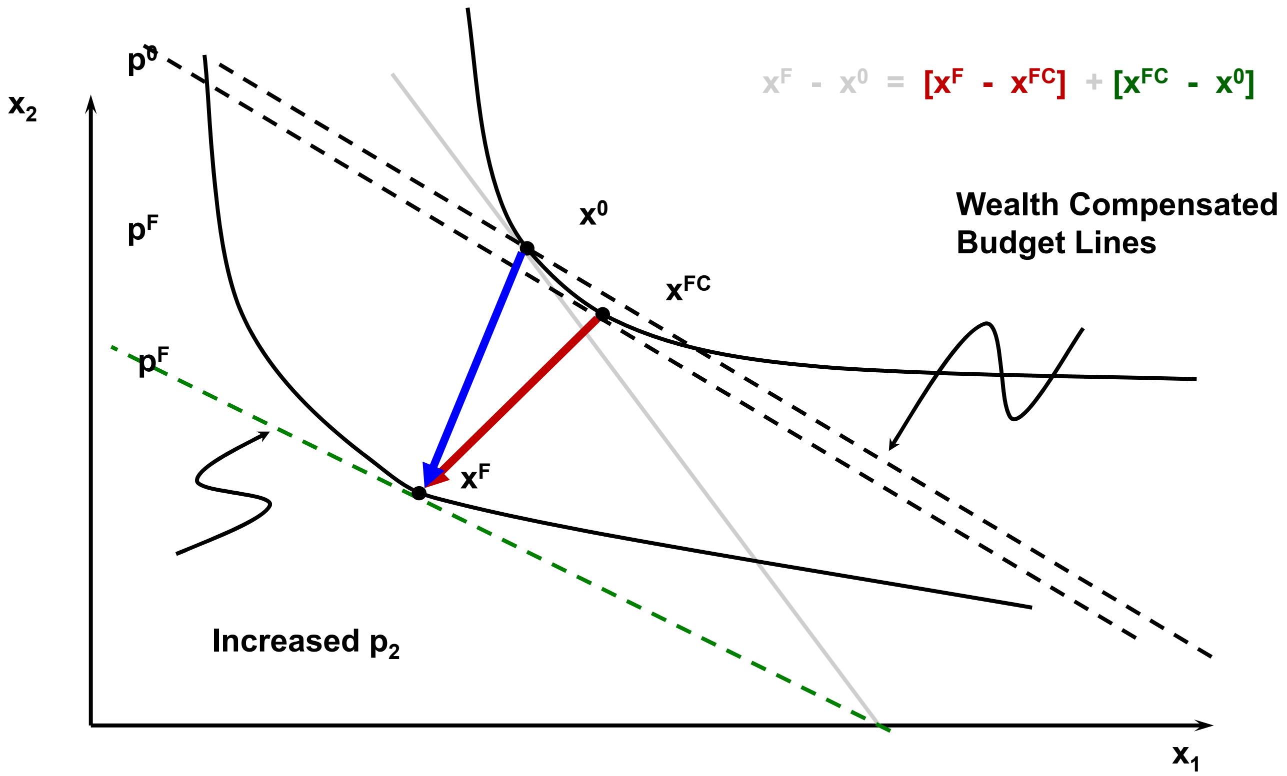
For analysis, break into two effects:

- Wealth Compensated Demand Change
  $x^{FC} - x^0$
- Wealth Effect of Price Change:
  $x^F - x^{FC}$

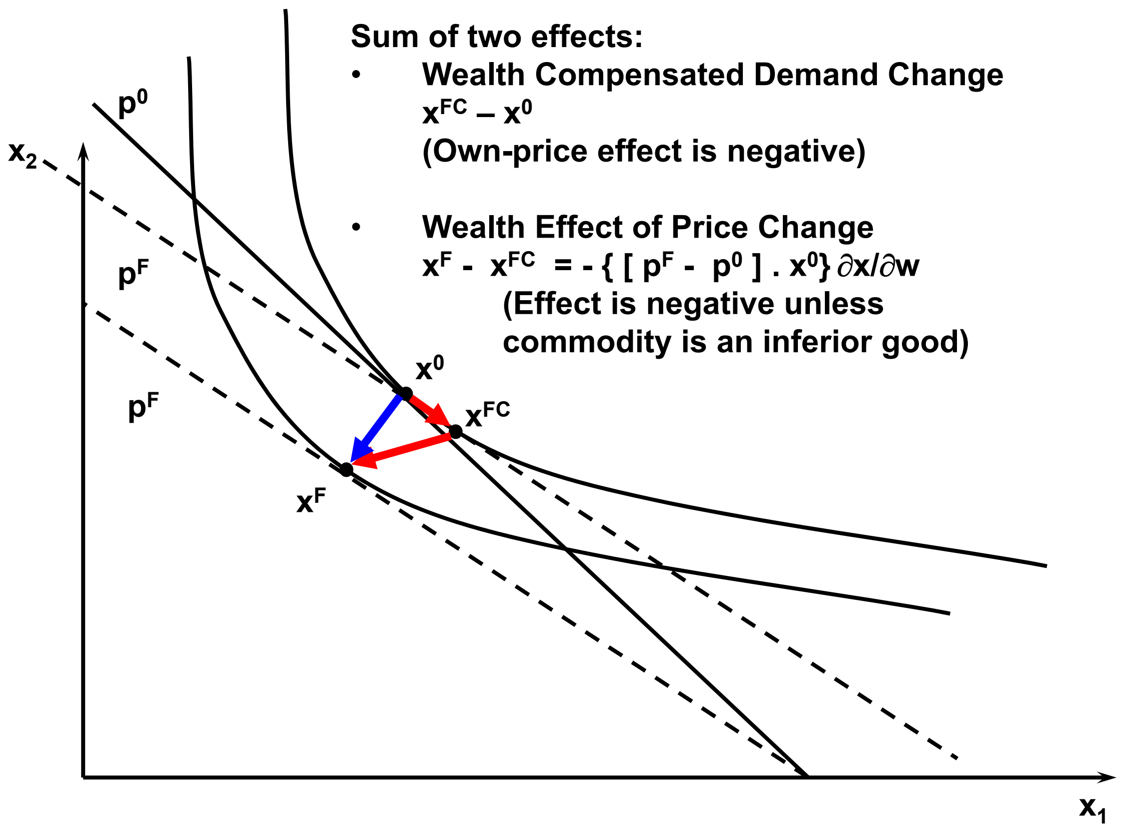
$$x^F - x^0 = [x^F - x^{FC}] + [x^{FC} - x^0]$$



RESPONSE TO A PRICE INCREASE (Cont'd)



RESPONSE TO A PRICE INCREASE (Cont'd)



Sum of two effects:

- Wealth Compensated Demand Change
  $x^{FC} - x^0$
  (Own-price effect is negative)
- Wealth Effect of Price Change
  $x^F - x^{FC} = -\{[p^F - p^0] \cdot x^0\} \partial x/\partial w$
  (Effect is negative unless commodity is an inferior good)

INDIRECT UTILITY AND EXPENDITURE FUNCTION

Definition. A consumer's indirect utility function v(p,w) corresponds to her maximized utility over her Walrasian budget set B(p,w),

$$v(p,w) = \max_{x \in B(p,w)} u(x)$$

Hence, if x(p,w) is the consumer's Walrasian demand function, then $v(p,w) = u(x(p,w))$. Note that v(p,w) is strictly increasing in w, as long as the preferences are locally nonsatiated; hence, v(p,w) for a fixed p, has an inverse with respect to w.

Now, given any utility level U, we can define the expenditure function e(p,U) in terms of the consumer's indirect utility implicitly, by setting,

$$v(p, e(p,U)) = U$$

i.e., the expenditure function defines the minimum expenditure necessary to achieve a given utility level U. In other words,

$$e(p,U) = \min_{x \in \{\hat{x} \in \Re^L_+ : u(\hat{x}) \ge U\}} p \cdot x$$

HICKSIAN DEMAND FUNCTION

Definition. The Hicksian demand function h(p,U) for a given price p and utility level U is given by the Walrasian demand function evaluated at the price p and the minimum wealth necessary to achieve U, i.e.,

$$h(p,U) = x(p, e(p,U)) \in \arg \min_{x \in \{\hat{x} \in \Re^L_+ : u(\hat{x}) \ge U\}} p \cdot x \qquad (*)$$

From (*) we can conclude that

$$\frac{\partial h_j(p,U)}{\partial p_i} = \frac{\partial x_j(p,e(p,U))}{\partial p_i} + \frac{\partial x_j(p,e(p,U))}{\partial w} \underbrace{\frac{\partial e(p,U)}{\partial p_i}}_{h_i(p,U)}$$

Roy's Identity

(follows directly from application of envelope theorem)

whence

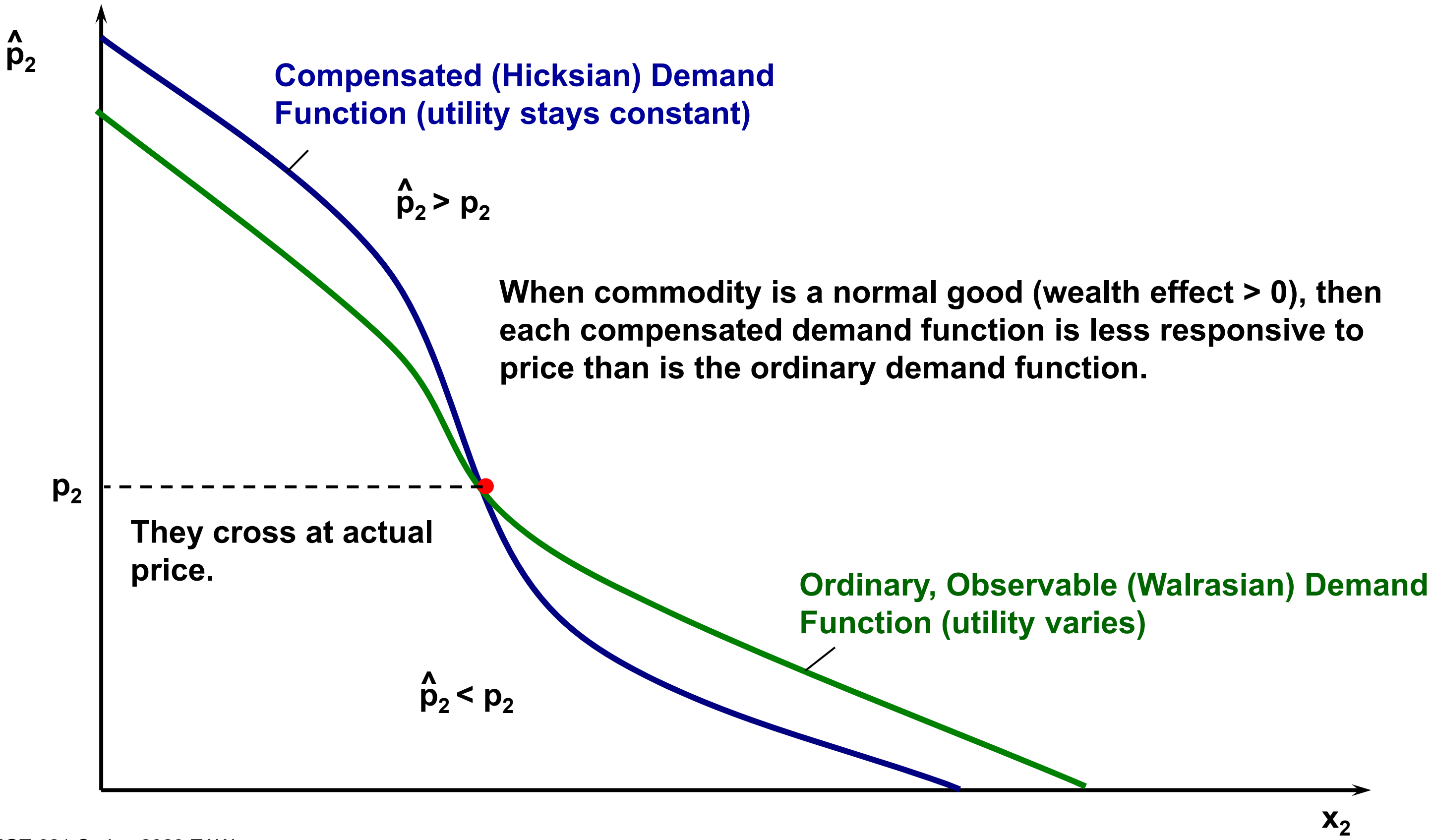
$$\frac{\partial x_j(p,e(p,U))}{\partial p_i} = \frac{\partial h_j(p,U)}{\partial p_i} - \frac{\partial x_j(p,e(p,U))}{\partial w} h_i(p,U)$$

Slutsky Equation

Price Effect | **Wealth-Compensated Demand Change** | **Wealth Effect**

WALRASIAN VS. COMPENSATED (HICKSIAN) DEMAND FUNCTION

Consider a change from p_2 to $\hat{p}_2$ (for a normal good)



PROOF OF ROY'S IDENTITY

Differentiate expenditure function with respect to one price component,

$$\frac{\partial e(p,U)}{\partial p_i} = \frac{\partial}{\partial p_i}\left[\min_{x \in \{\hat{x} \in \Re_+^L : u(\hat{x}) \geq U\}} p \cdot x\right] = \frac{\partial}{\partial p_i}\left[p \cdot x - \lambda(U - u(x))\right]\Big|_{x = x(p,e(p,U))} = x_i(p, e(p,U)) = h_i(p,U)$$

Envelope Theorem

This is Roy's identity. **QED**

We conclude that $$\frac{\partial h_j(p,U)}{\partial p_i} = \frac{\partial^2 e(p,U)}{\partial p_i \partial p_j}$$

The matrix $S(p,U) = \left[\frac{\partial h_j(p,U)}{\partial p_i}\right]_{i,j=1}^{L} = \left[\frac{\partial^2 e(p,U)}{\partial p_i \partial p_j}\right]_{i,j=1}^{L}$ **is called Slutsky matrix.**

PROPERTIES OF THE SLUTSKY MATRIX

Proposition. **The Slutsky matrix S(p,U) is symmetric, negative semidefinite, and satisfies S(p,U) p = 0.**

Proof.

(i) S is symmetric as long as expenditure function is twice continuously differentiable (theorem by Cauchy [sometimes attributed to H.A. Schwarz]).

(ii) The negative semi-definiteness of S (i.e., the fact that $D_p h(p,U) \le 0$) follows from the "law of compensated demand", which states that

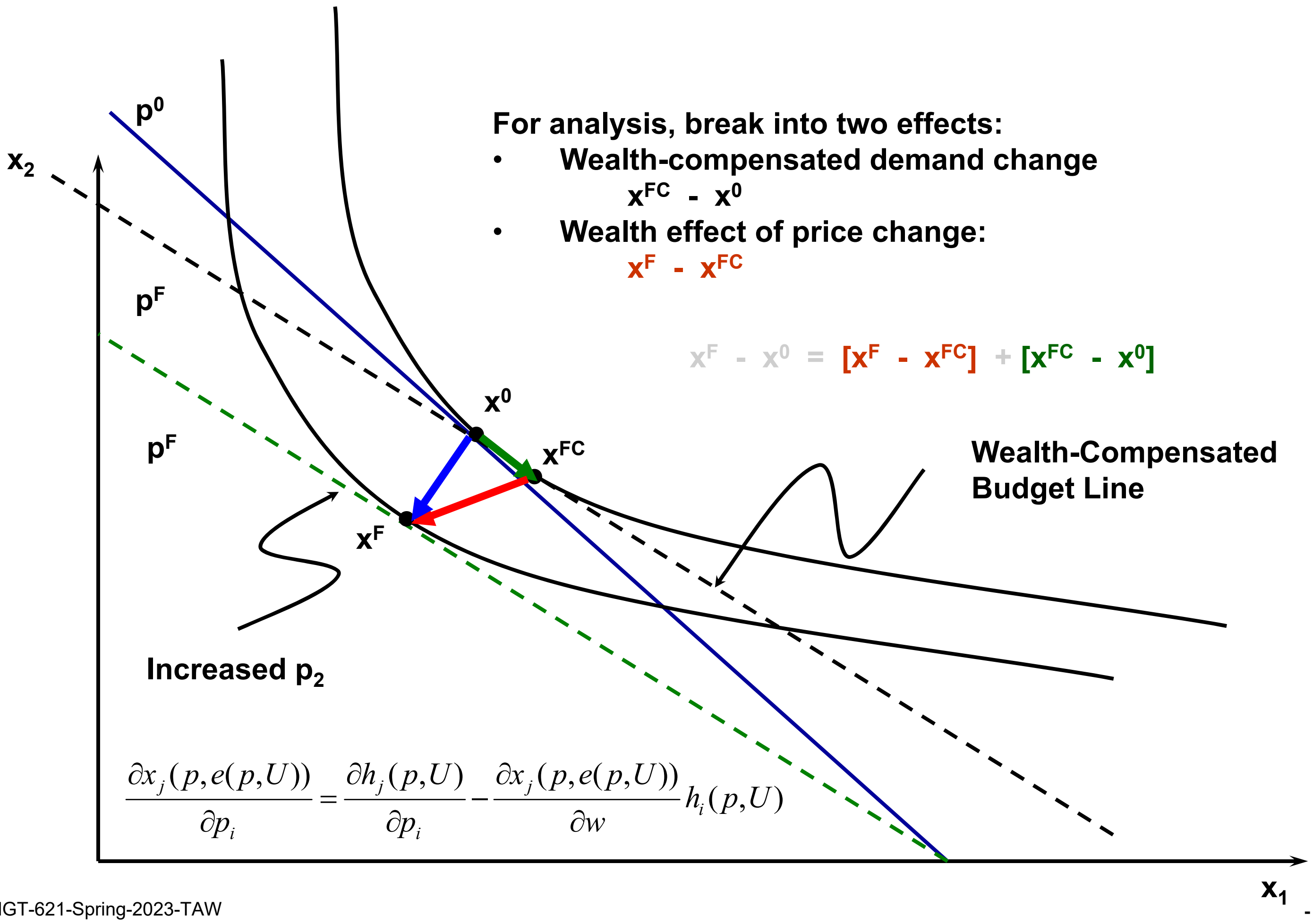
$$(p' - p)(h(p',U) - h(p,U)) \le 0 \qquad \text{(\#)}$$

Relation (#) holds, because $p' \cdot h(p',U) \le p' \cdot h(p,U)$ **and** $p \cdot h(p,U) \le p \cdot h(p',U)$ **.**

(iii) Note that Hicksian demand h(p,U) is homogeneous of degree zero in p (prove this as an exercise!), so that

$$\left.\frac{\partial h(\alpha\, p,U)}{\partial \alpha}\right|_{\alpha=1} = D_p h(p,U)p = S(p,U)p = 0 \qquad \textbf{QED}$$

RESPONSE TO A PRICE INCREASE



For analysis, break into two effects:

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  x^{FC} - x^0
- **Wealth effect of price change:**
  x^F - x^{FC}

$x^F - x^0 = [x^F - x^{FC}] + [x^{FC} - x^0]$

$$\frac{\partial x_j(p,e(p,U))}{\partial p_i} = \frac{\partial h_j(p,U)}{\partial p_i} - \frac{\partial x_j(p,e(p,U))}{\partial w} h_i(p,U)$$

INTERPRETATION OF THE SLUTSKY EQUATION

Decompose change of demand: $x^F - x^0 = [x^{FC} - x^0] + [x^F - x^{FC}]$

For very small price changes Δp, obtain:

$$[x^{FC} - x^0] = S(p,w) \cdot \Delta p$$
$$[x^F - x^{FC}] = -\partial x/\partial w \; \{x^0 \cdot \Delta p\}$$

$$\Delta x = S(p,w) \, . \, \Delta p \; - \partial x/\partial w \; \{x \cdot \Delta p\}$$

This is the Slutsky Equation. Most often it is written in terms of partial derivatives of x_i with respect to p_j (for small change $x = x^0$):

$$\partial x_i/\partial p_j = S_{ij} - \partial x_i/\partial w \; x_j$$

S_{ij} is (i,j)-th element of Slutsky substitution matrix, derivative of wealth compensated demand x^{FC}_i with respect to p_j. Referred to as "substitution effect" of a price change.

$-\partial x_i/\partial w \; x_j$ is referred to as the "income effect" of a price change.

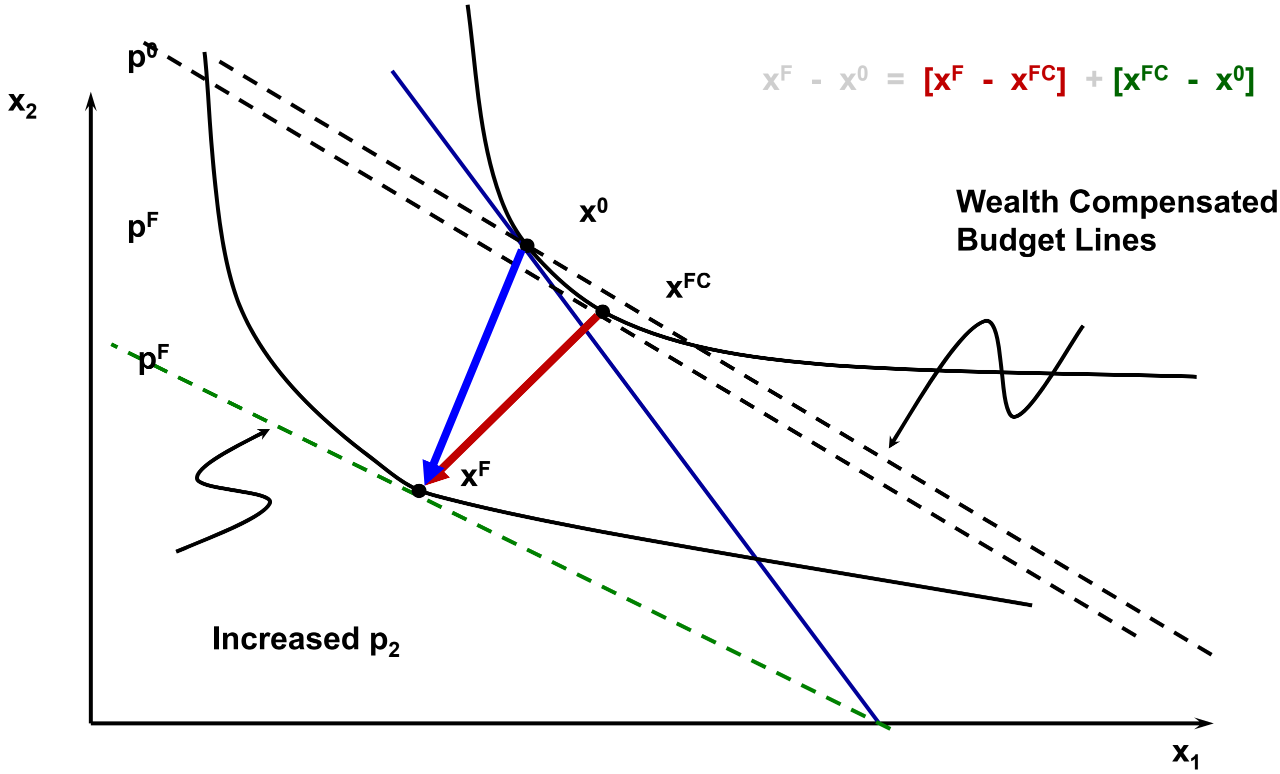
INTERPRETATION OF THE SLUTSKY EQUATION (Cont'd)

$$D_p x(p,w) = \underbrace{S(p,w)}_{\text{"Substitution Effect"}} \underbrace{- \frac{\partial x(p,w)}{\partial w} x(p,w)}_{\text{"Income Effect"}}$$

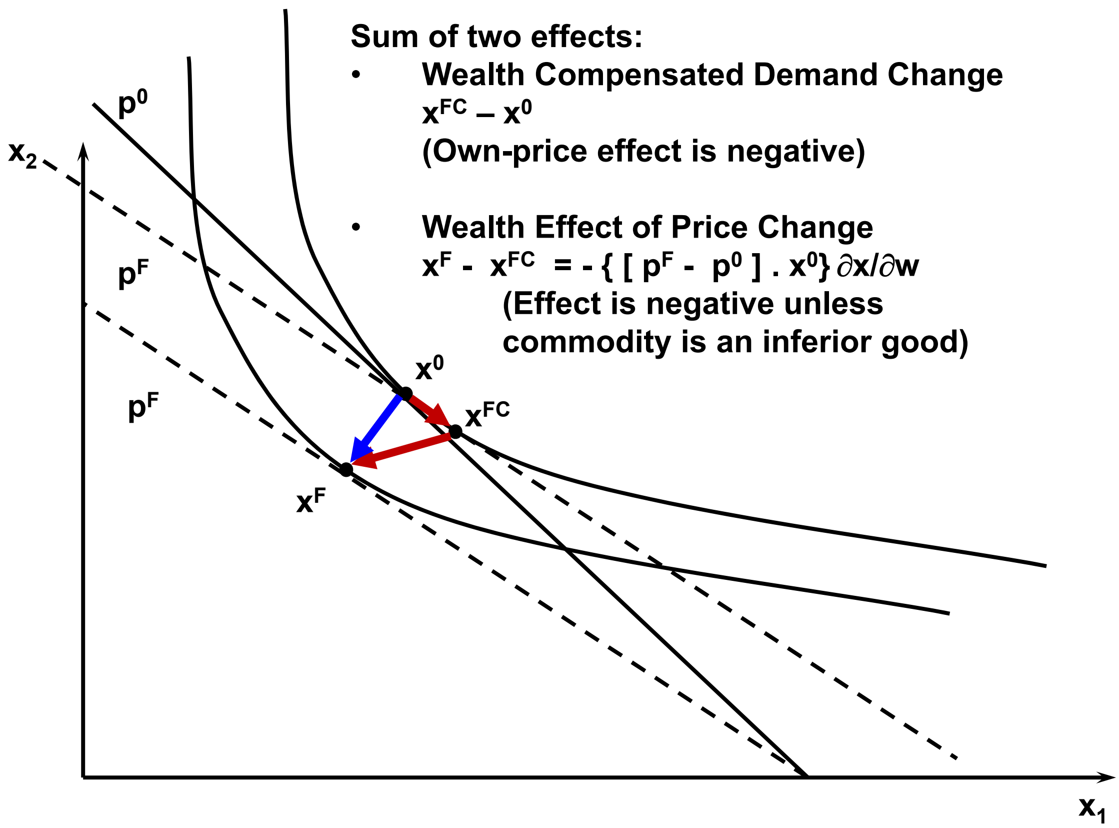
Note that (compared to earlier slides)

$$w = e(p,U)$$
$$x(p,w) = h(p,U)$$

RESPONSE TO A PRICE INCREASE (Cont'd)



RESPONSE TO A PRICE INCREASE (Cont'd)



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Some Special Utility Functions

Wealth Effects

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Key Concepts to Remember

MARKET DEMAND FUNCTION

Demand aggregated over (finitely) many individuals: Market Demand Function

If following properties hold for each individual demand function:

- Continuity
- Homogeneity of degree zero
- Walras' Law

Then they also hold for the market demand function

MARKET DEMAND FUNCTIONS (Cont'd)

Is it possible to find aggregate demand function D(p,w) (for n individuals), such that

$$D(p,w)=\sum_{k=1}^{n} x^k(p,w^k) \quad \textbf{for} \quad w=\sum_{k=1}^{n} w^k \quad ?$$

In general, if everyone faces the *same* price vector p, then aggregate demand can be written as a function of p, but NOT necessarily also as a function of aggregate income w, unless

$$\sum_{k=1}^{n}\frac{\partial x^k}{\partial w^k}dw^k=0$$

for any small wealth change dw that leaves aggregate wealth the same, i.e., for which

$$dw=(dw^1,\ldots,dw^n)$$

$$\sum_{k=1}^{n} dw^k=0$$

In other words, all the $\partial x^k/\partial w^k$ **have to be the same across all consumers.**

Wealth effects must compensate each other in the aggregate, no matter how the wealth is redistributed among the individuals!

MARKET DEMAND FUNCTIONS (Cont'd)

Proposition. A (necessary and) sufficient condition for demand aggregation to be possible is for preferences to be such that each consumer k's indirect utility v^k is quasilinear ("of the Gorman form"), i.e.,

$$v^k(p,w^k)=a^k(p)+b(p)w^k$$

Proof: (sufficiency only)

By the definition of indirect utility it is $v^k(p,e^k(p,u))=u$.

Thus,

$$v_p^k(p,w^k)+v_{w^k}^k(p,w^k)e_p^k(p,u^k(x^k(p,w^k)))=v_p^k(p,w^k)+v_{w^k}^k(p,w)h^k(p,u^k(x^k(p,w^k)))$$

$$=\underbrace{v_p^k(p,w^k)}_{a_p^k(p)+b'(p)w^k}+\underbrace{v_{w^k}^k(p,w^k)}_{b(p)}x^k(p,w^k)$$

$$=0 \qquad (=\partial u/\partial p)$$

And therefore, $\dfrac{\partial x^k(p,w^k)}{\partial w^k}=-\dfrac{b'(p)}{b(p)}$ **is the same for any consumer k, no matter what his or her wealth level w^k.**

QED

MARKET DEMAND FUNCTIONS (Cont'd)

A market demand function is useful for making statements about consumer response to changes in price and/or aggregate income.

Example.

Sometimes a market demand function is useful to explain other aggregate effects, such as the "bandwaggon effect," under which the demand for a good depends on the collective *expectation* about how many consumers will purchase the product.

A LITTLE DETOUR: NETWORK EXTERNALITIES

Externalities exist when the action of one agent directly affects the environment of another agent; *network externalities* are externalities between participants of a common network

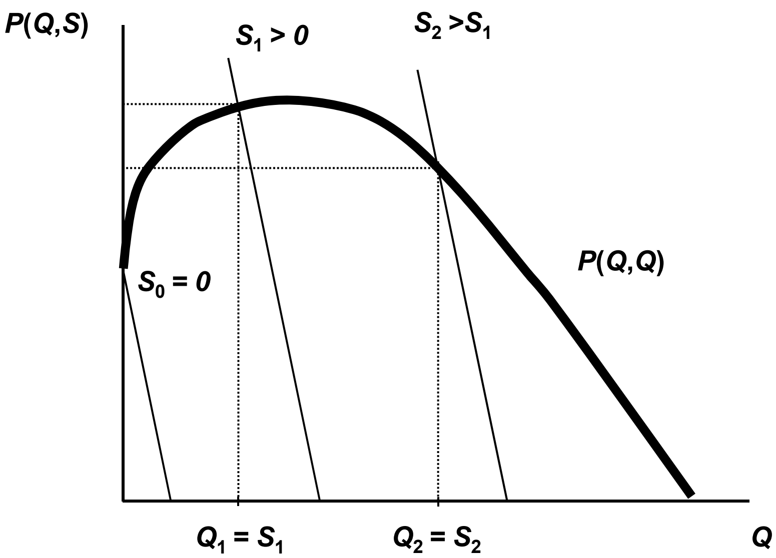
"How much would you pay for the first fax machine?"

Complementarity

- ***Direct* (e.g., in 2-way networks, "exchange transactions")**
- ***Indirect (e.g., Microsoft Word)***
- **Necessary conditions:**
  - **Compatibility (= ability to connect, usually to some hardware)**
  - **Interoperability (= ability to exchange and make use of information)**

Aggregate demand depends on the expected demand.

GENERATING FULFILLED-EXPECTATIONS DEMAND CURVE
Demand in the Presence of Network Externalities



Bandwaggon Effect

DEMAND CURVE SHIFTS DUE TO NETWORK EXTERNALITIES
Fulfilled-Expectations Demand

Quantitative Approach

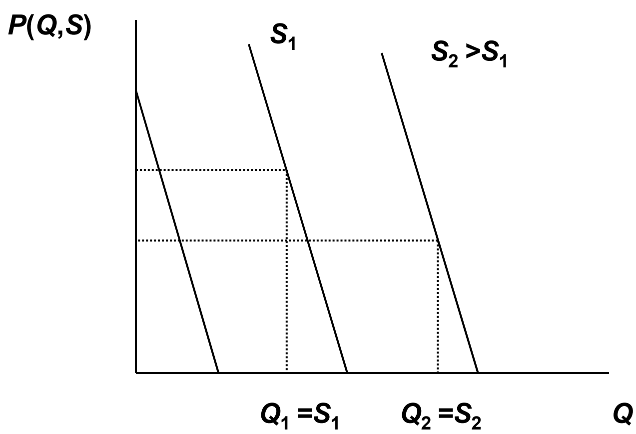
P(Q,S) = P(Q,0) + f(S)

P Willingness-to-pay

f(S) Externality function, f(0) = 0, f ”< 0 < f ’

Q Output

S Expected demand / installed base



Source: N. Economides

NETWORK EXPANSION PATH CAN HAVE SEVERAL FULFILLED EXPECTATIONS EQUILIBRIA

'Chicken and Egg' Paradox

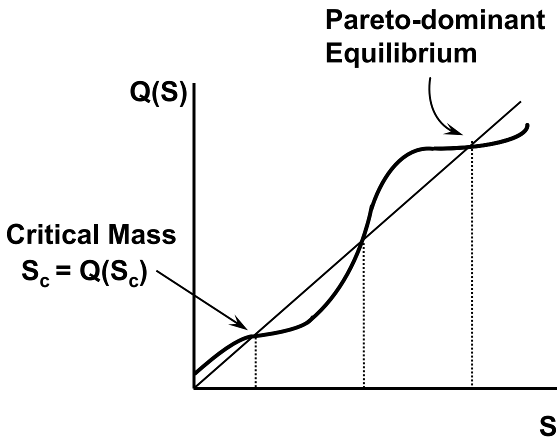
If perceived installed base is too small, customers may not be willing to purchase the product.

For certain products, small networks are not observed: "*Discontinuous Network Expansion*" (at least seemingly)

Existence of *minimum* feasible network ("critical mass") does not depend on the market structure (i.e., a demand-side phenomenon)

Equilibrium network sizes resulting from perfect competition / oligopoly / monopoly generally different (e.g., Economides/Himmelberg 1994)

Network Expansion Path



WHAT IS THE CRITICAL MASS?
Let's compute it!

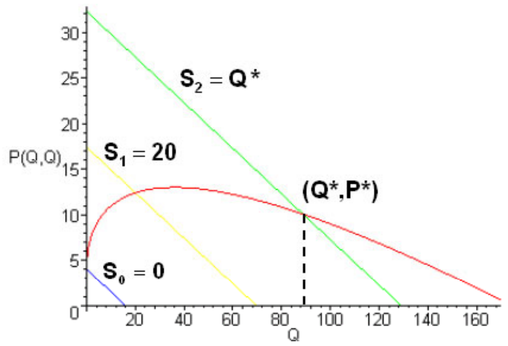
Example(1)

Willingness to Pay:	$P(Q,S) = 4 - Q/4 + 3\sqrt{S}$
Profit in Fulfilled-Expectations Equilibrium (S=Q):	$\Pi(Q) = Q \cdot P(Q,Q) - 2Q = \left(4 - Q/4 + 3\sqrt{Q}\right)Q - 2Q$
Maximization yields:	$Q^* = S^* = 88.8$, $P^* = P(Q^*,S^*) = 10.1$
Network Expansion Path:	$P^* = P(Q,S) = 4 - Q/4 + 3\sqrt{S} \Rightarrow S(Q) = (P^* - 4 + Q/4)^2$
Critical Mass:	$Q_c = \min\{Q' : S(Q') = Q' \geq 0\} \Rightarrow Q_c = 6.6$

Note that $Q_c/Q^* = 6.6/88.8 = 7.4\%$ is significant. Thus, in order to have a chance to achieve the optimum, the firm has to instill the belief that in equilibrium more than the critical mass of users (i.e., more than 6.6 Million) will eventually adopt.

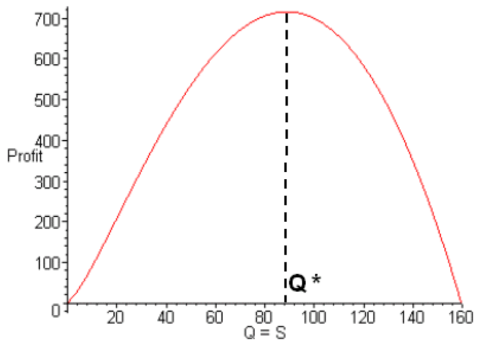
(1) Quantities measured in millions of units. Number of potential customers is 174.5 million (there willingness to pay in the fulfilled-expectations equilibrium will be zero, i.e., p(174.5,174.5) = 0)

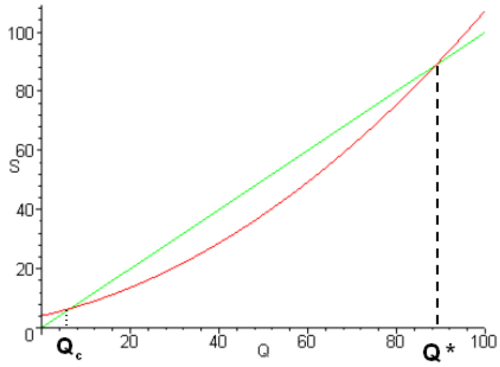
WHAT IS THE CRITICAL MASS?
Let's compute it! (Cont'd)



$$P(Q,S) = 4 - Q/4 + 3\sqrt{S}$$

$$\Pi(Q) = Q \cdot P(Q,Q) - 2Q$$





BUSINESS IMPLICATIONS FOR SELLERS OF NETWORK GOODS

- **Penetration pricing** (initially possibly < 0) to reduce adoption costs for the consumer. Pulling one consumer over is likely to induce further consumers ("herding") to adopt, also due to the effect of network externalities

- **Growth** is a strategic imperative
  - Production-side economies can help: e.g., lower marginal cost lowers optimal (monopoly) price, which in turn lowers critical mass
  - Demand-side economies are most important for achieving market dominance

- **Strategic pre-announcements** to reduce uncertainty. Market uncertainty can prevent the consumers from exploiting beneficial network externalities since consumers fear being stranded with a new technology(1)

- **Tradeoff among current and future benefits** through lock-in. Difficult tradeoff and frequent cause of business failure:
  - Myopia (too high prices) vs. overestimating future benefits (too low prices)

(1) Cf. Farrel, J., and Saloner, G., Installed Base and Compatibility: Innovation, Product Preannouncements, and Predation, AER 76(5):940—955, 1986.

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Key Concepts to Remember

MEASURING WELFARE CHANGES

Consider a simple example of valuing a nonmarket good (e.g., a national park).

- Assume that there are N standard market goods and one nonmarket good.
- A consumer has preferences represented by a smooth increasing utility function u(x,q), where x denotes the consumption in the market goods and q the consumption of the nonmarket good
- The consumer's income (wealth) is y > 0

Given any q, the consumer's indirect utility function is

$$v(p,q,y) = \max_{x \in \{\hat{x} \in \Re_+^N : p \cdot \hat{x} \le y\}} u(x,q)$$

where p is the price vector for the market goods.

Question. How much is an exogenous change of q from q^0 to $q^1 > q^0$ worth to the consumer?

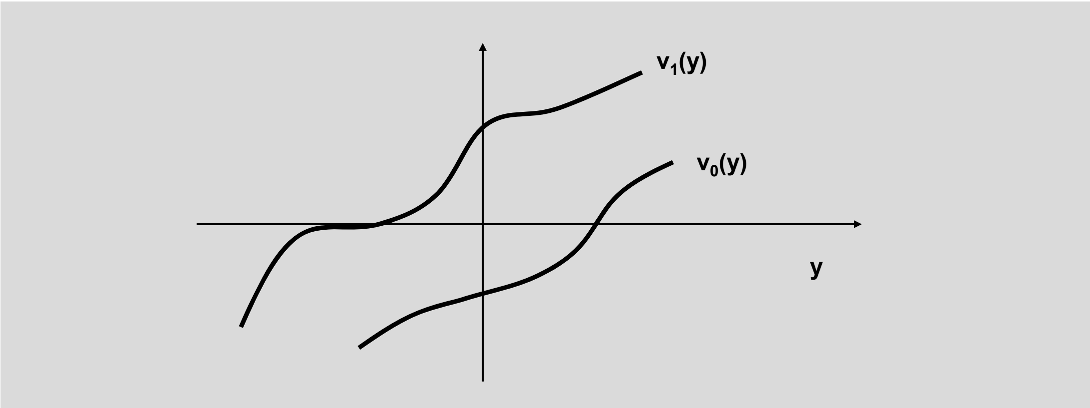
MEASURING WELFARE CHANGES (Cont'd)

One can interpret q^0 and q^1 as two "states" of the economy, and the consumer has some value for the change of the state (assume that $q^1 > q^0$, without loss of generality).

Let

$$v_i(y) = v(p, q^i, y)$$

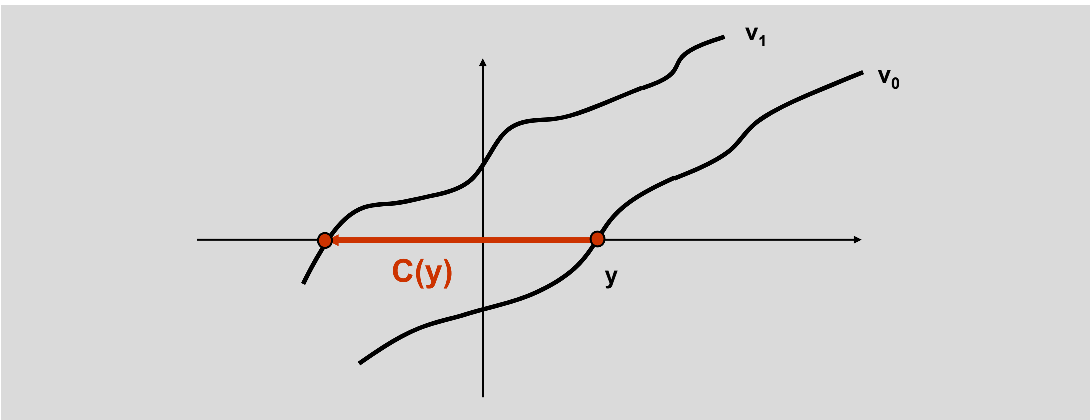
denote the consumer's indirect utility as a function of his income y for i in {0,1}.(1)



From standard demand theory we know that $v_i(y)$ is strictly increasing in y.

(1) We suppress the dependence on the constant price vector for simplicity. More generally, $v_i(y)$ can denote the consumer's (indirect) utility function in state i of the economy as a function of his or her wealth. Thus, the analysis here can also be applied when the state of the economy is defined by different commodity price vectors (corresponding to the treatment in standard economics textbooks such as MWG).

MEASURING WELFARE CHANGES (Cont'd)



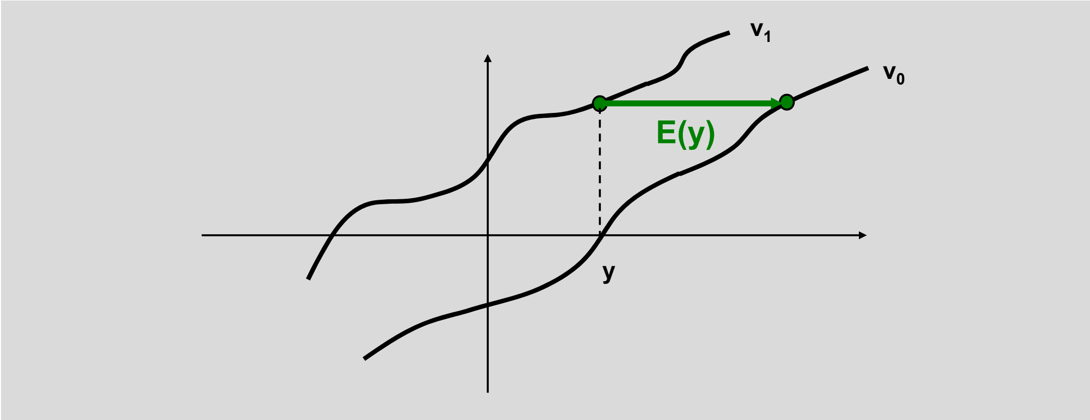
Question I. **How much would a consumer of income y be willing to pay for the transition from q^0 to q^1?**

Answer I. **A income change of (– C(y)) would "compensate" the consumer for having q^1 instead of q^0. [*The word compensate means here to bring the consumer back to the original utility level before the change.*]**

$$v_1(y - C(y)) = v_0(y)$$

C(y) is the Compensating Variation

MEASURING WELFARE CHANGES (Cont'd)



Question II. How much would a consumer of income y be willing to accept for the transition from q^1 to q^0?

Answer II. A income change of +E(y) would make the consumer feel "equivalent" between having q^0 (at income y+E(y)) and having q^1 (at income y).

$$v_1(y) = v_0(y + E(y))$$

E(y) is the Equivalent Variation

COMPENSATING AND EQUIVALENT VARIATIONS

Definition. Let $v_i(y)$ be a consumer's increasing (indirect) utility function for an economy in state $i \in \{0,1\}$ as a function of income y.

(i) The compensating variation C(y) is defined as the consumer's maximum *willingness to pay* to transition from state 0 to state 1, i.e.,

$$C(y) = \sup\{c \in \Re : v_0(y) \le v_1(y - c)\}$$

(ii) The equivalent variation E(y) is defined as the consumer's minimum *willingness to accept* to transition from state 1 to state 0, i.e.,

$$E(y) = \inf\{e \in \Re : v_1(y) \le v_0(y + e)\}$$

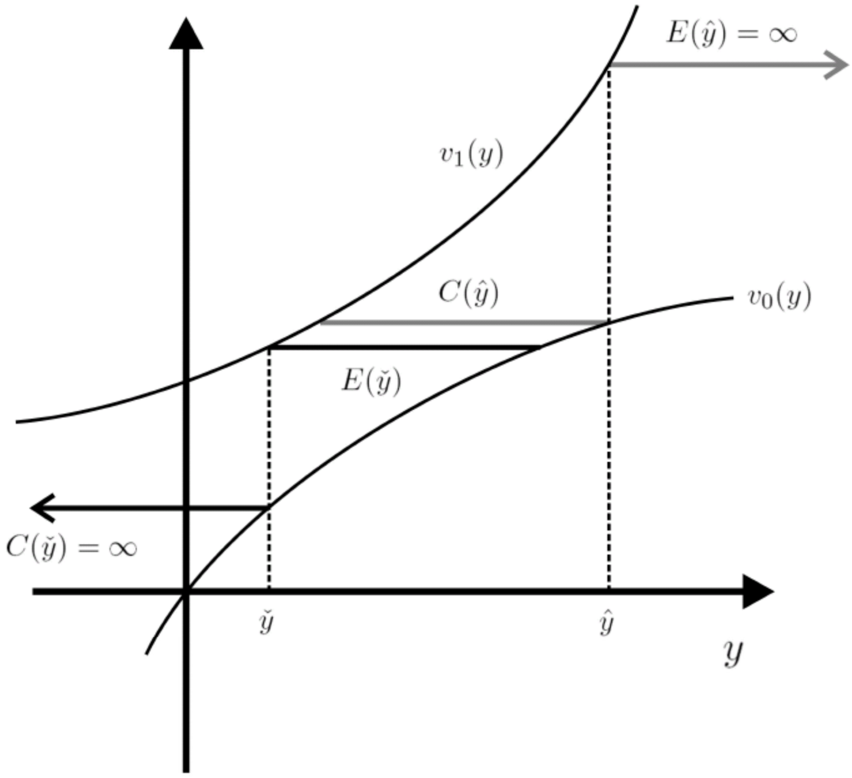
Remark. If C(y) and E(y) are bounded, we have that

$$v_0(y) = v_1(y - C(y)) \quad \text{and} \quad v_0(y + E(y)) = v_1(y),$$

corresponding to the standard definition of these two welfare measures.

COMPENSATING AND EQUIVALENT VARIATIONS

They can be very different!



WHAT IS THE RELATION BETWEEN C(y) AND E(y)?

The answer is simple. Since both v_0 and v_1 are invertible functions, we obtain from the definition of C and E that

$$C(y) = y - v_1^{-1}(v_0(y)) \qquad \text{and} \qquad E(y) = v_0^{-1}(v_1(y)) - y$$

This immediately implies that C and E are independent of the particular utility representation of the consumer's preferences (why?).

As a result, we could choose the utility representation such that $v_0(y) = y$, so that $v_1(y) = E(y) + y$ (from the definition of E(y)). Thus, from the definition of C(y) we know that we simply need to form the inverse of v_1 to find C(y), so that

$$C(y) = y - w_{01}(y) = E(w_{01}(y)),$$

where the compensated income $w_{01}(y)$ is such that $y = w_{01}(y) + E(w_{01}(y))$.

Similarly, one can show that

$$E(y) = w_{10}(y) - y = C(w_{10}(y)),$$

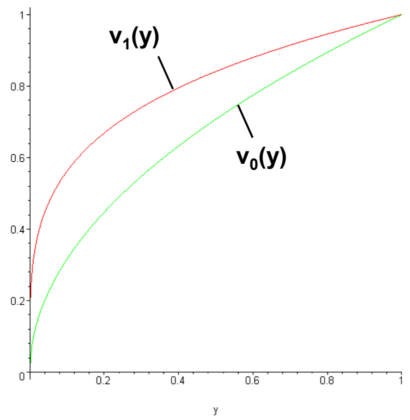
where the compensated income $w_{10}(y)$ is such that $y = w_{10}(y) - C(w_{10}(y))$

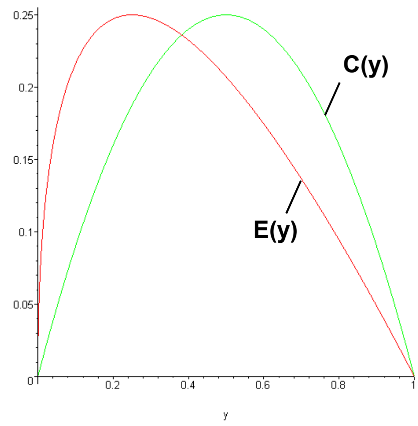
EXAMPLE: COMPUTATION OF C(y) AND E(y)

Consider a consumer with indirect utility functions $v_0(y) = y^{\alpha}$ and $v_1(y) = y^{\beta}$, where $\alpha = ½$ and $\beta = ¼$, and the income y lies in [0,1].(1)

$$C(y) = y - v_1^{-1}(v_0(y)) = y - (\sqrt{y})^4 = (1-y)y$$

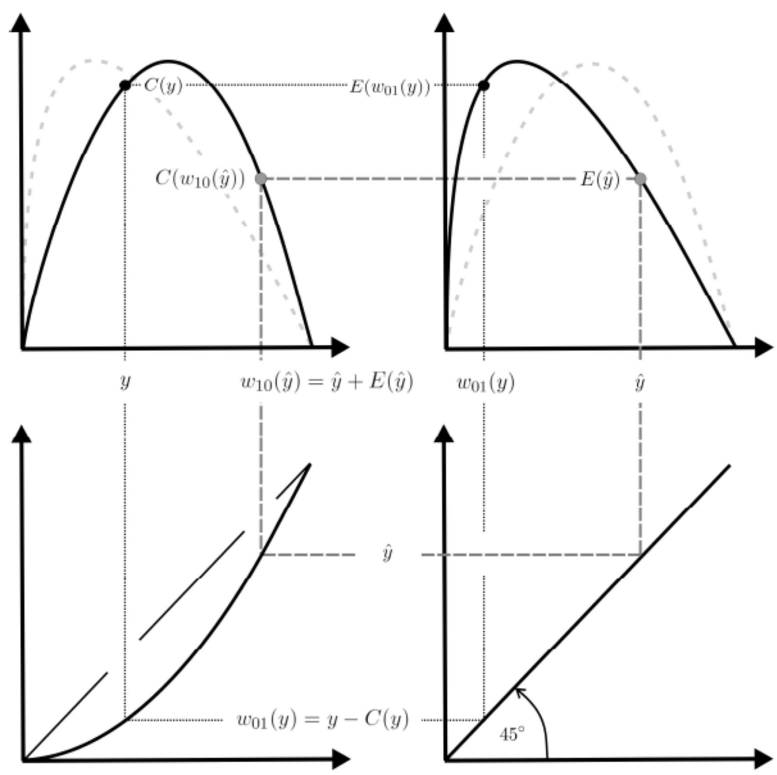
$$E(y) = v_0^{-1}(v_1(y)) - y = \left(y^{1/4}\right)^2 - y = \sqrt{y} - y$$





(1) These indirect utility functions can be obtained after solving the utility maximization problem for appropriate Cobb-Douglas utilities.

EXAMPLE (Cont'd)



$$C(y) = E(w_{01}(y))$$
$$E(\hat{y}) = C(w_{10}(\hat{y}))$$
$$w_{10}(\hat{y}) = w_{01}^{-1}(\hat{y}) = \hat{y} + E(\hat{y})$$
$$w_{01}(y) = y - C(y)$$

EXAMPLE: TRANSFER OF A NONMARKET GOOD

Assume that there are two consumers, the first has welfare measures $C(y), E(y)$, while the second has the welfare measures $\hat{C}(y), \hat{E}(y)$. For simplicity, we assume that both start with the same income level y. The first consumer holds one unit of a nonmarket good, while the second consumer possesses none.

Questions. **(i) At what transfers t will there be a transaction of the nonmarket good?**

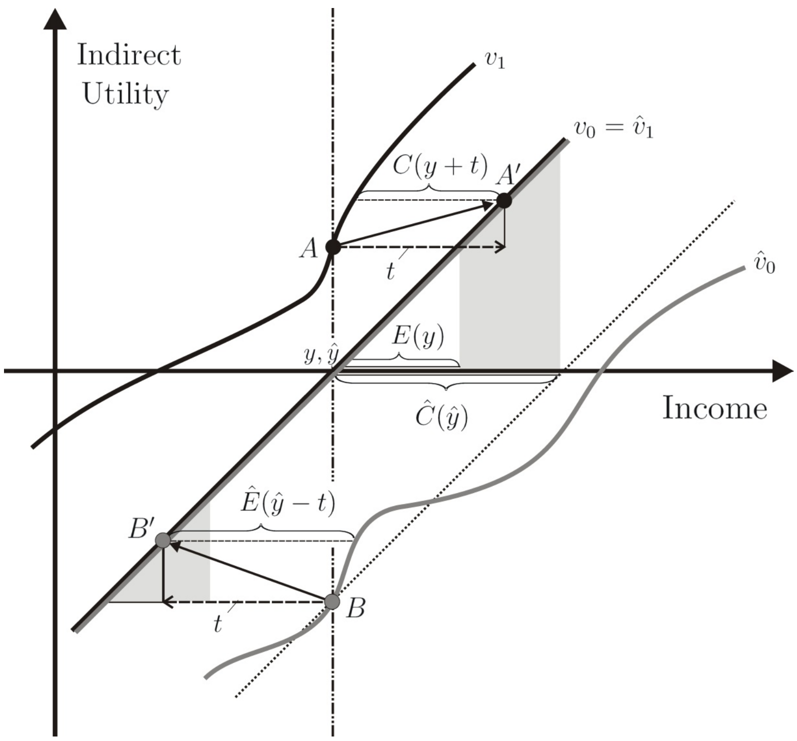
(ii) Is it possible that after the first transfer takes place, another such transfer occurs moving the good back to the first consumer?

Answers.

(i) A necessary and sufficient condition for a transfer is that $\hat{C}(y) \geq E(y)$

(ii) A necessary and sufficient condition for a second transfer (after the good had been exchanged under (i) at price t) is that $C(y+t) \geq \hat{E}(y-t)$. This can never happen if the first transaction realized gains from trade!

TRANSFER OF A NONMARKET GOOD (Cont'd)



The first transfer leads to a Pareto improvement! A second transfer cannot generate another (strict) Pareto improvement.

AGENDA

Some Special Utility Functions

Wealth Effects

Price Effects

Demand Aggregation

Standard Welfare Measures

Welfare Changes

Key Concepts to Remember

HOW TO COMPUTE E(y) AND C(y) FOR PRICE CHANGES?

Price change from p to $\hat{p}$

$$C(y) = e(p, v(p, y)) - e(\hat{p}, v(p, y))$$
$$= y - e(\hat{p}, v(p, y))$$

Compensating Variation

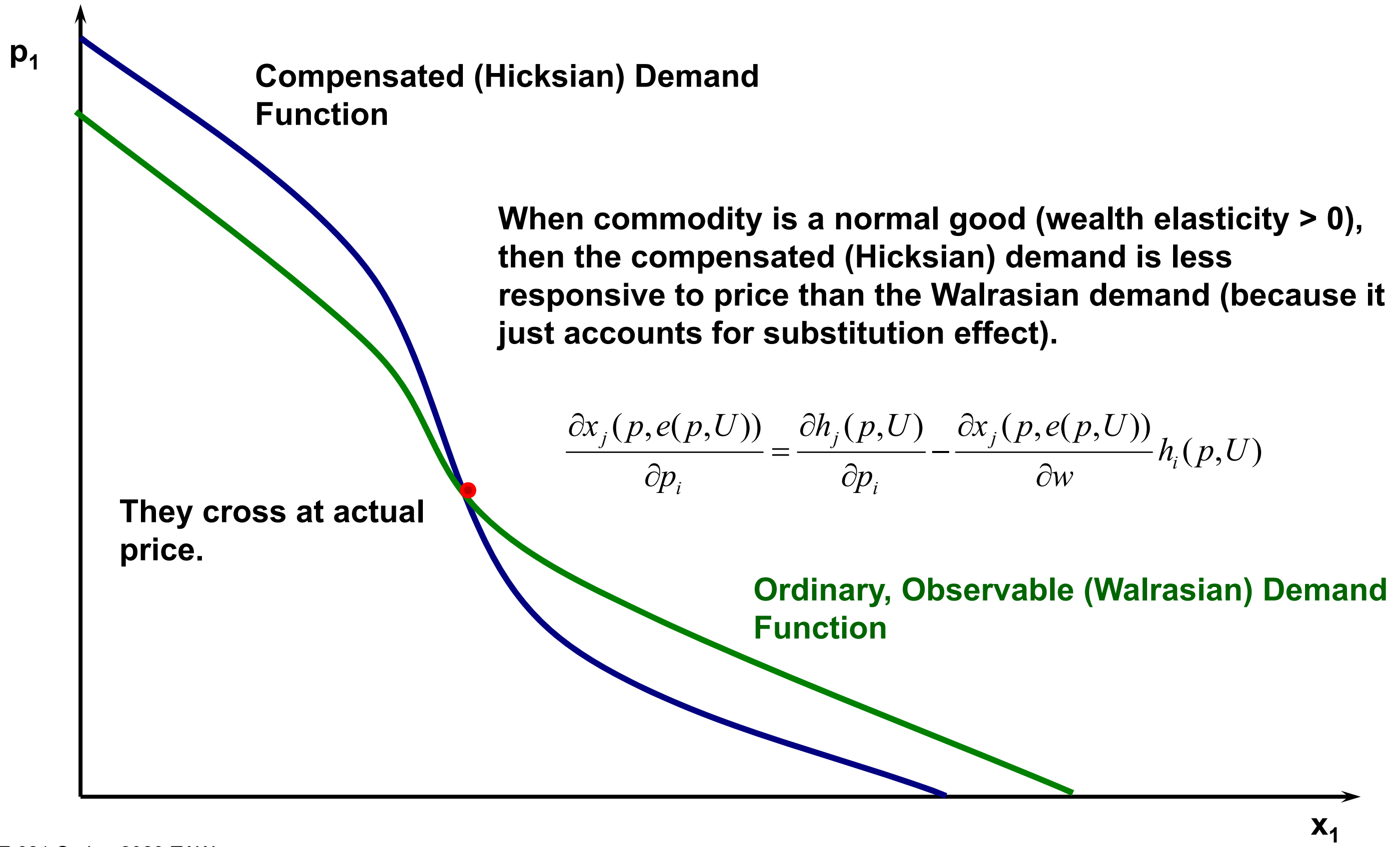
Expenditure at initial price minus expenditure at final price, evaluated at initial utility level

$$E(y) = e(p, v(\hat{p}, y)) - e(\hat{p}, v(\hat{p}, y))$$
$$= e(p, v(\hat{p}, y)) - y$$

Equivalent Variation

Expenditure at initial price minus expenditure at final price, evaluated at final utility level

WALRASIAN DEMAND VS. COMPENSATED (HICKSIAN) DEMAND



When commodity is a normal good (wealth elasticity > 0), then the compensated (Hicksian) demand is less responsive to price than the Walrasian demand (because it just accounts for substitution effect).

$$\frac{\partial x_j(p,e(p,U))}{\partial p_i}=\frac{\partial h_j(p,U)}{\partial p_i}-\frac{\partial x_j(p,e(p,U))}{\partial w}h_i(p,U)$$

COMPENSATED DEMAND FUNCTION

Slope of the actual demand function, or Walrasian demand function is $\partial x_i/\partial p_i$

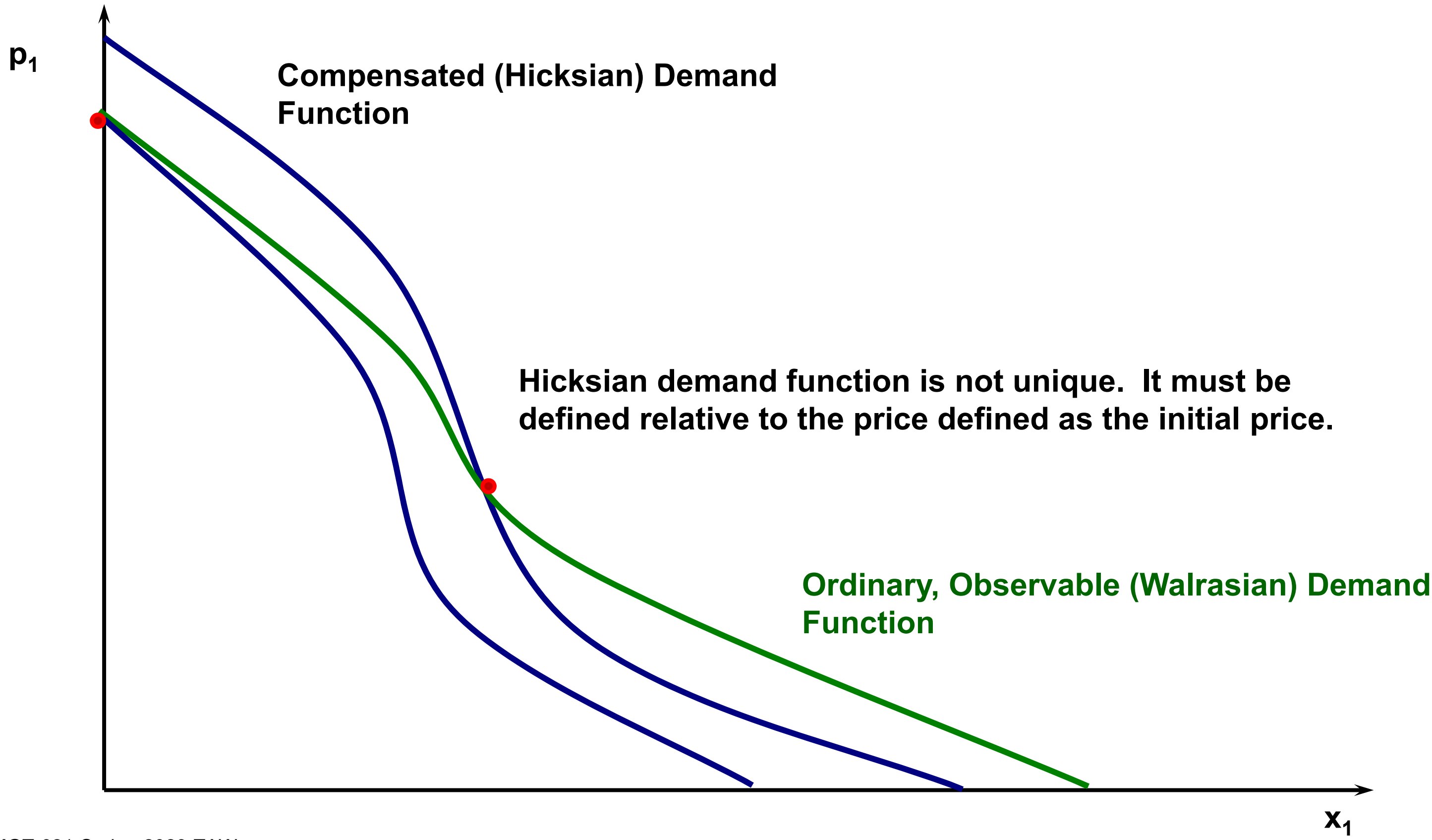
Can use Slutsky equation to construct the "compensated demand function", also called the "Hicksian demand function"

Construct compensated demand function around some specific combination of p_i and the resulting x_i but with the slope S_{ii}

$$S_{ii} = \partial x_i/\partial p_i + \partial x_i/\partial w \ x_i$$

This is the slope of the artificial demand function, constructed as if at the same time the price is increasing, consumer is given exactly enough additional wealth to keep utility constant.

COMPENSATED (HICKSIAN) DEMAND FUNCTIONS FROM TWO DIFFERENT INITIAL PRICES



SIGNIFICANCE

Hicksian demand curve is used to create two conceptually correct measurements of welfare impacts of a price change (called "compensating variation" and "equivalent variation")

- **Compensating Variation: Negative of dollar amount to compensate consumer for facing price change, so that utility remains unchanged.**
- **Equivalent Variation: Dollar amount consumer would accept in place of a price change, so utility change would be the same as it would be with the price change.**

Compensating Variation and Equivalent variation are both positive for price decrease and negative for a price increase.

- **A less conceptually correct measure, the change in consumer's surplus, will be approximately equal to compensating variation and to equivalent variation. Consumer's surplus is somewhat easier to calculate**

CALCULATION OF WELFARE MEASURES FOR PRICE CHANGE

Compensating Variation: Negative of dollar amount to compensate consumer for facing price change, so that utility remains unchanged.

- **Integrate along Hicksian demand curve, crossing through the original price and quantity**

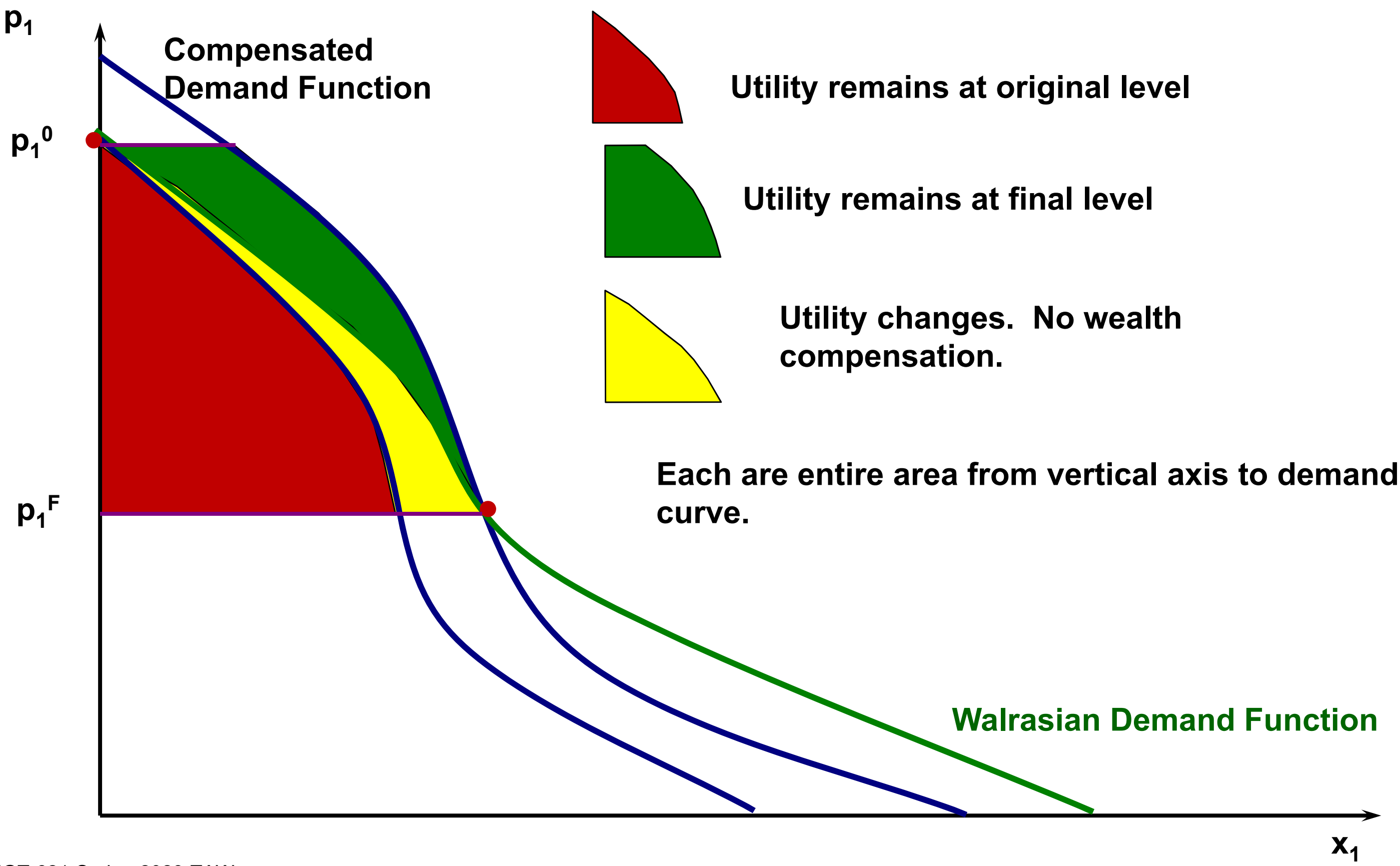
Equivalent Variation: Dollar amount consumer would accept in place of a price change, so utility change would be the same as it would be with the price change.

- **Integrate along Hicksian demand curve, crossing through the final price and quantity**

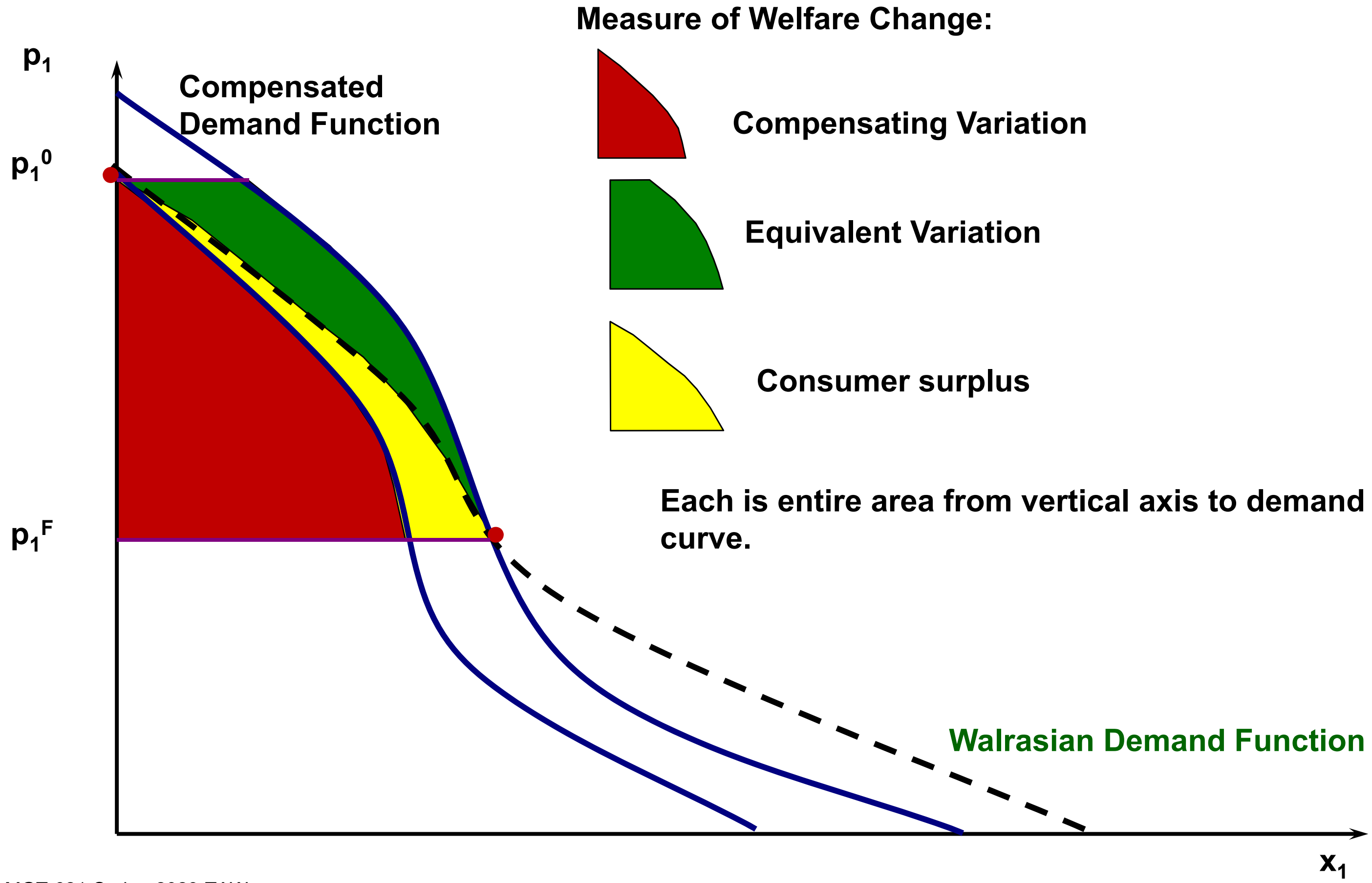
Consumer Surplus: Integrate along ordinary (Walrasian) demand curve.

WELFARE IMPACTS OF PRICE REDUCTION

Nature of Compensation:

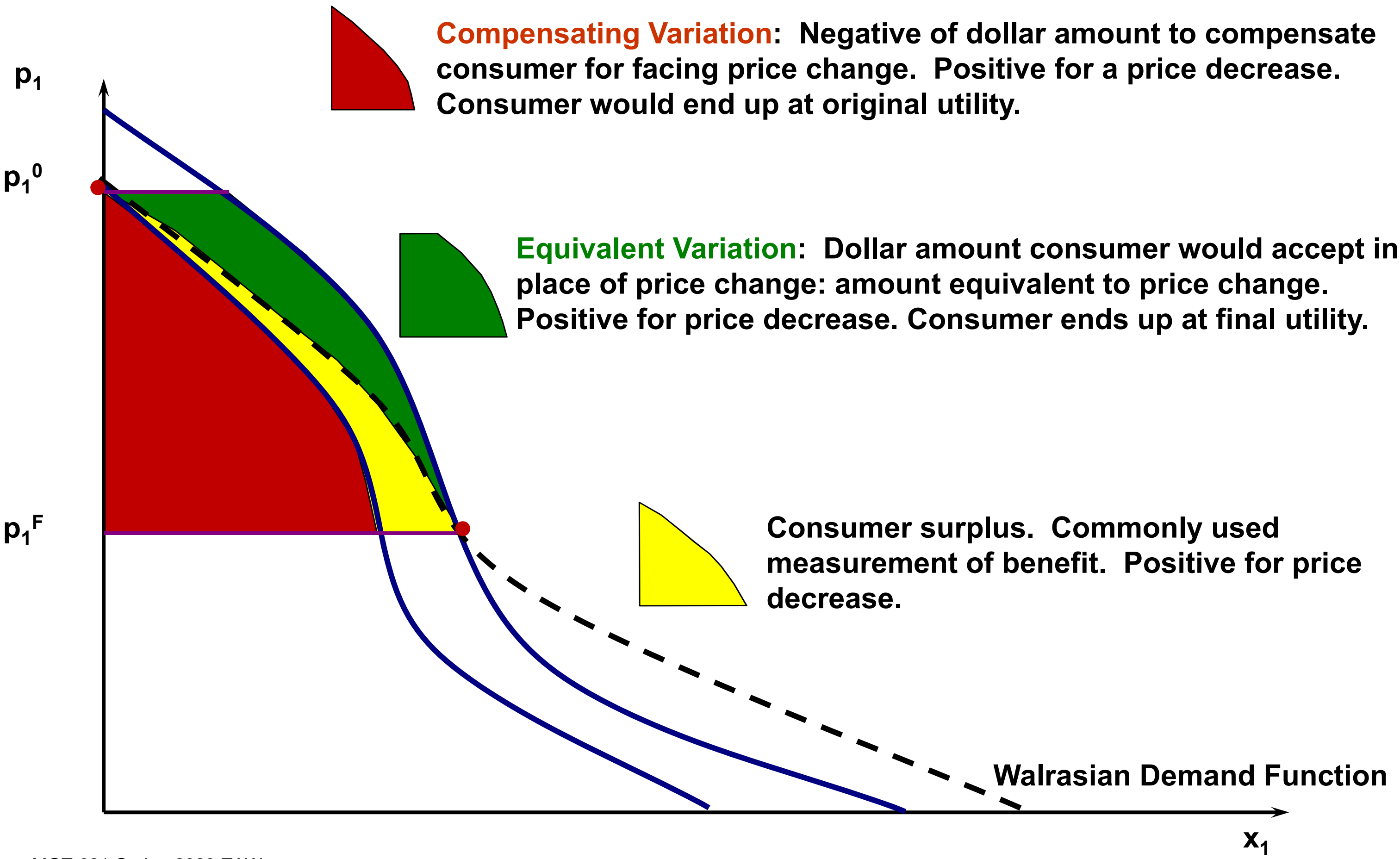


WELFARE IMPACTS OF PRICE REDUCTION (Cont'd)



Each is entire area from vertical axis to demand curve.

WELFARE IMPACTS OF PRICE REDUCTION (Cont'd)



Compensating Variation: Negative of dollar amount to compensate consumer for facing price change. Positive for a price decrease. Consumer would end up at original utility.

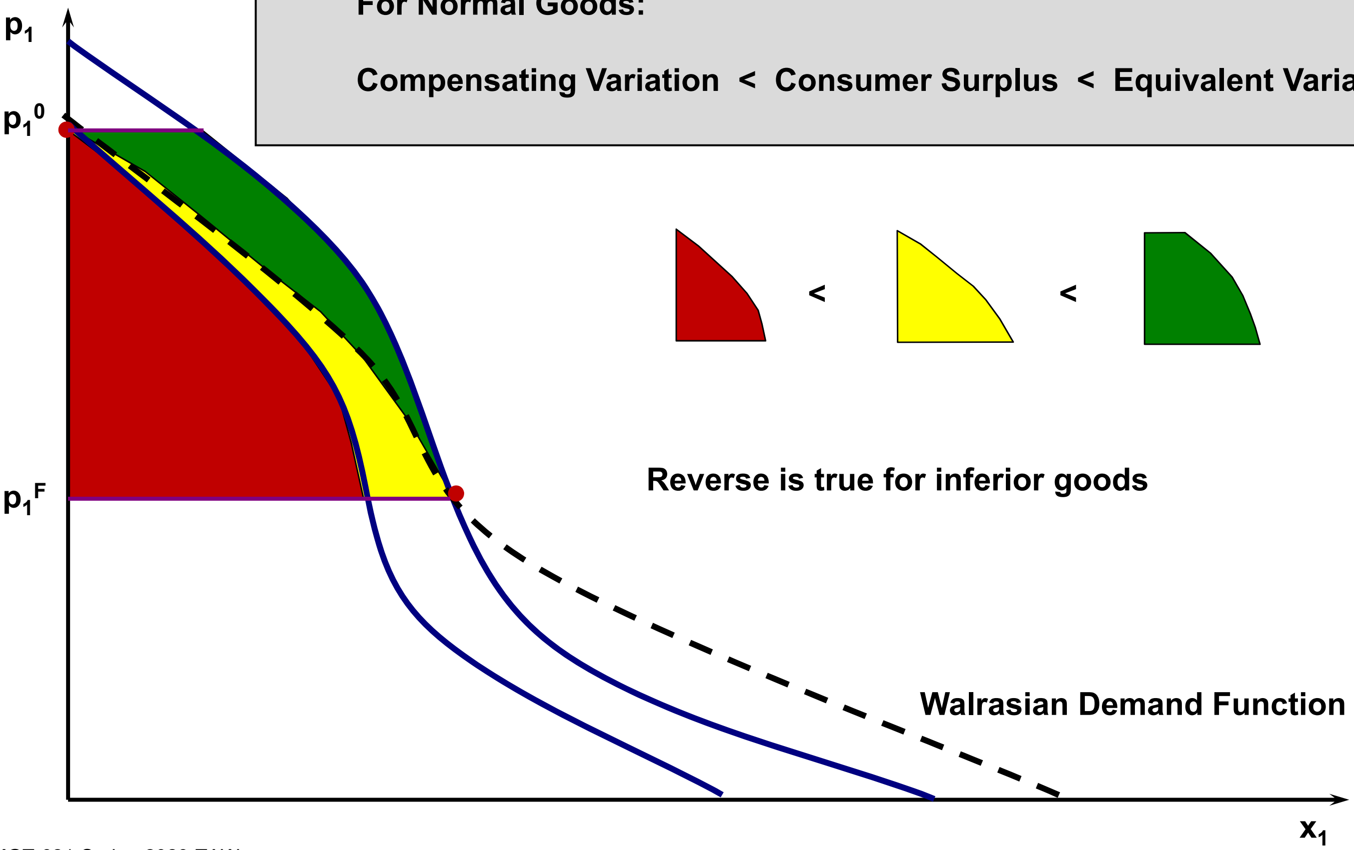
Equivalent Variation: Dollar amount consumer would accept in place of price change: amount equivalent to price change. Positive for price decrease. Consumer ends up at final utility.

Consumer surplus. Commonly used measurement of benefit. Positive for price decrease.

WELFARE IMPACTS OF PRICE REDUCTION (Cont'd)

For Normal Goods:

Compensating Variation < Consumer Surplus < Equivalent Variation



Reverse is true for inferior goods

EXAMPLE: CONSTANT ELASTICITY OF DEMAND
Compute Hicksian Compensated Demand Function

$$x(p,w) = \begin{bmatrix} \alpha(w/p_1) \\ (1-\alpha)(w/p_2) \end{bmatrix}$$

Walrasian Demand Function **corresponding to a Cobb-Douglas utility function** $u(x) = Kx_1^{\alpha}x_2^{1-\alpha}$ **in a two-good economy, where** $\alpha \in (0,1), K > 0$ **.**

$$U = K(x_1(p,w))^{\alpha}(x_2(p,w))^{1-\alpha} = \frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}{p_1^{\alpha}p_2^{1-\alpha}}Kw \Rightarrow e(p,U) = \frac{p_1^{\alpha}p_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\frac{U}{K}$$

Hicksian Compensated Demand Function

$$h_1(p,U) = x_1(p,e(p,U)) = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha}\left(\frac{U}{K}\right)$$

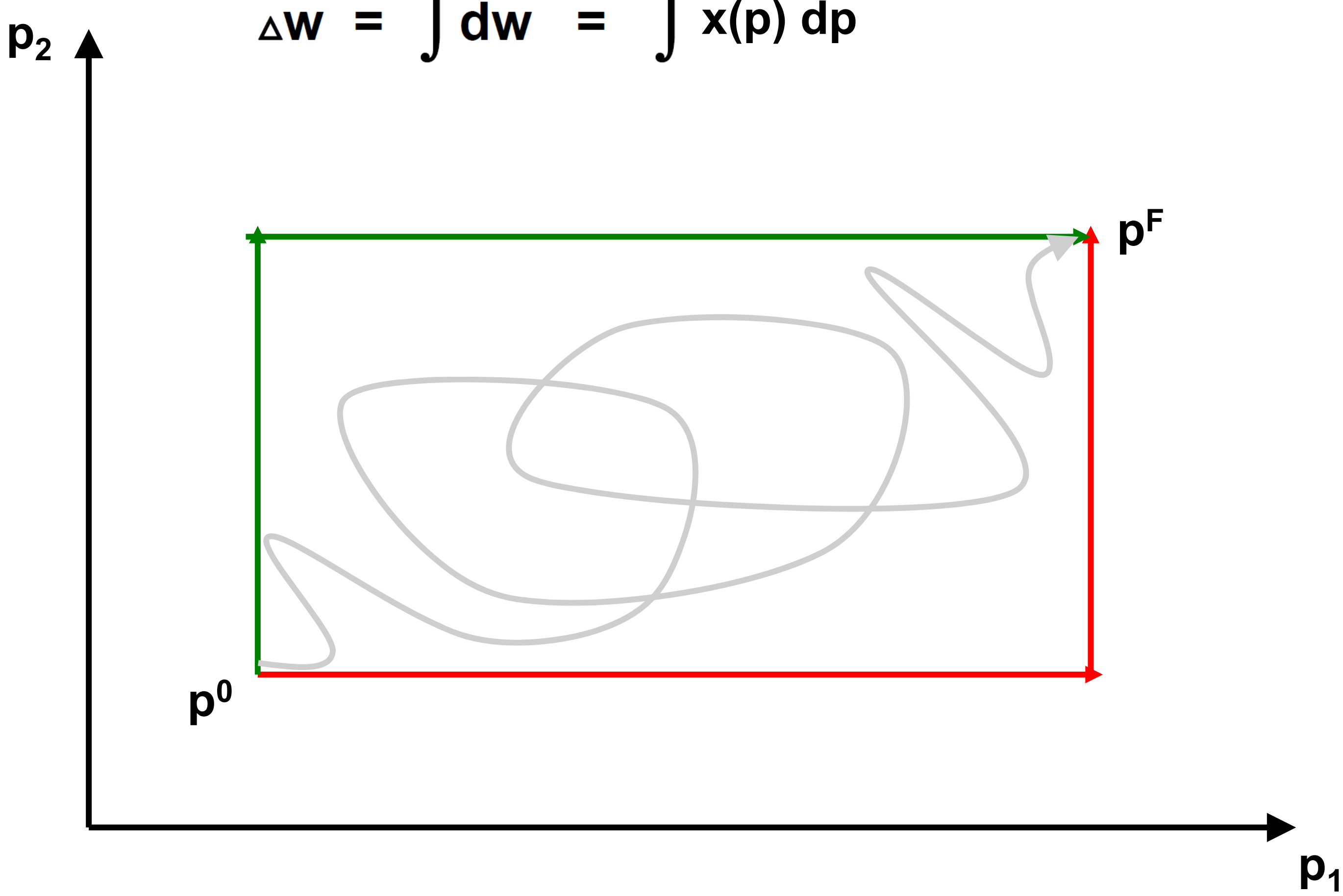
$$h_2(p,U) = x_2(p,e(p,U)) = \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}\left(\frac{U}{K}\right)$$

INTEGRABILITY

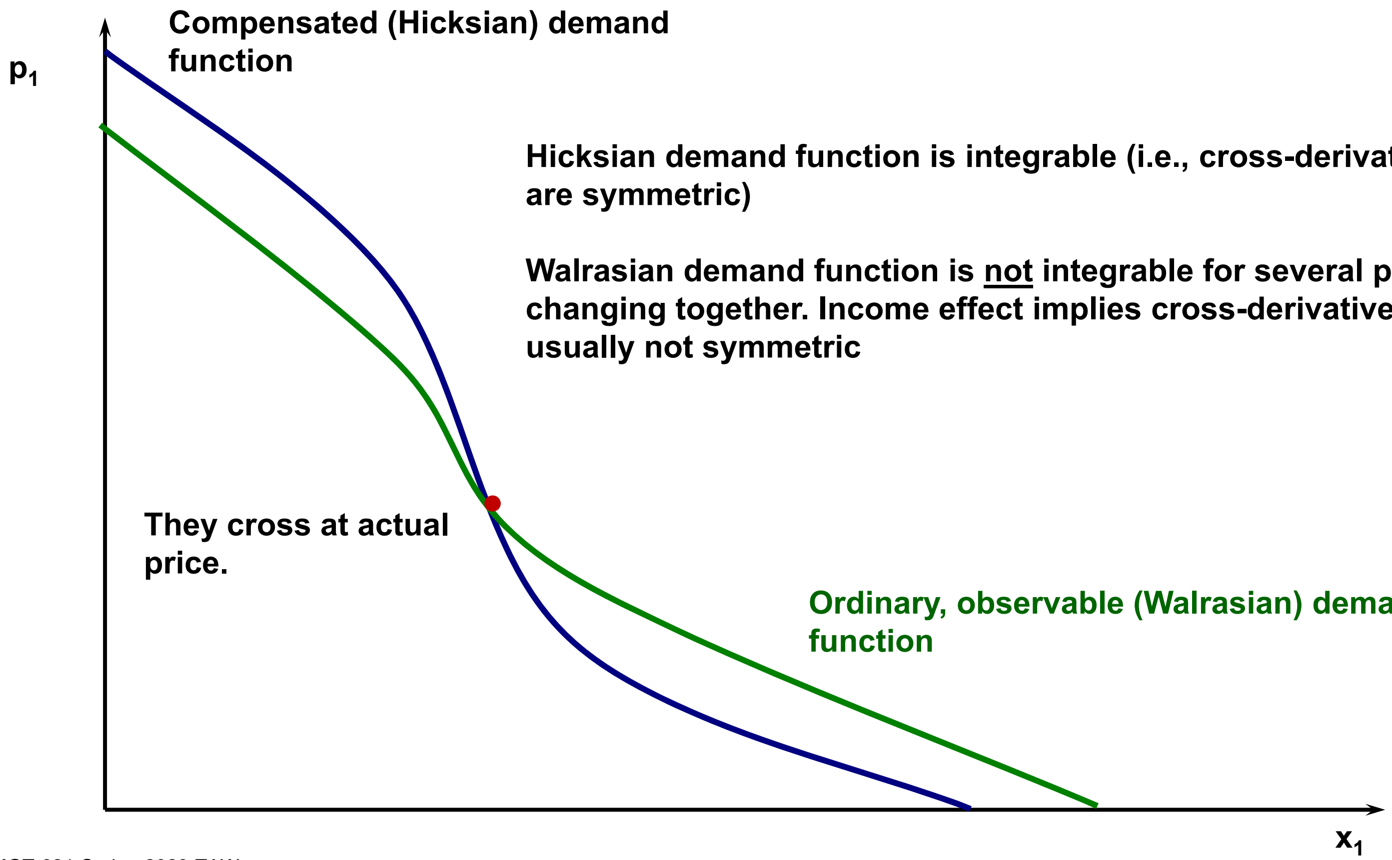
- To calculate welfare effects we integrate x(p)
- Integration is along *some* path of p from p^0 to p^F
- For the welfare measure to be unique, the integral must not depend on which particular path of p (from p^0 to p^F) was chosen for the integration
- Integrability conditions provide us with criterion for whether the measure is unique, that is, whether the integral is path-dependent
- Note that if only one price changes, all continuous functions are integrable
- For multiple prices changing, the demand function is integrable if and only if all cross-derivatives are symmetric, so

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial x_j}{\partial p_i} \qquad \text{(for all } i, j\text{)}$$

$$\Delta w = \int dw = \int x(p)\, dp$$



WALRASIAN DEMAND VS. COMPENSATED (HICKSIAN) DEMAND



Hicksian demand function is integrable (i.e., cross-derivatives are symmetric)

Walrasian demand function is not integrable for several prices changing together. Income effect implies cross-derivatives usually not symmetric

SUMMARY OF WELFARE MEASURES

Equivalent variation and compensating variation

- conceptually precise measurements
- based on integrable demand function

Consumer surplus

- not based on conceptually precise concept (e.g., not based on integrable demand functions)
- very easy to measure
- measure most often seen in calculations

All three measures tend to be similar UNLESS income effect is large

Welfare measures for state changes in the economy are routinely used in benefit-cost calculations

AGENDA

Some Special Utility Functions

Wealth Effects

Price Effects

Demand Aggregation

Standard Welfare Measures

Welfare Changes

Key Concepts to Remember

KEY CONCEPTS TO REMEMBER

- **Special Utility Functions (Cobb-Douglas, CES, Leontief)**
- **Envelope Theorem**
- **Monotone Comparative Statics**
- **Expenditure Minimization Problem**
- **Hicksian Demand**
- **Law of Compensated Demand**
- **Slutsky Compensation (Wealth Compensation)**
- **Indirect Utility (and Gorman Form)**
- **Roy's Identity**
- **Slutsky Equation**
- **Income Effect & Substitution Effect of a Price Change**
- **Aggregate Demand**
- **Bandwaggon Effect & Network Externalities & Fulfilled-Expectations Demand & Critical Mass**
- **Compensating Variation / Equivalent Variation / Consumer Surplus**