MGT 621 – MICROECONOMICS

Thomas A. Weber

1. Theory of Choice

Autumn 2023

École Polytechnique Fédérale de Lausanne
College of Management of Technology

AGENDA

Administrivia & Course Overview

Preferences and Utility Representation

Some Properties

Utility Representation (Cont'd)

Demand Theory: Basics

A Little Refresher on Constrained Optimization

Key Concepts to Remember

INFRASTRUCTURE

My Coordinates

- Room ODY 3.01 (OES: Operations, Economics and Strategy) / Zoom
- Phone: +41 21 693 0141
- Email: thomas.weber@epfl.ch

Grading Assistance

- Jun Han
- Room: ODY 4.16 / Zoom
- Phone: +41 21 693 9054
- Email: jun.han@epfl.ch

Administrative Assistance

- Ms. Ilona Ball
- Room ODY 3.16 / Zoom
- Phone: +41 21 693 0039
- Email: ilona.ball@epfl.ch

INFRASTRUCTURE (Cont'd)

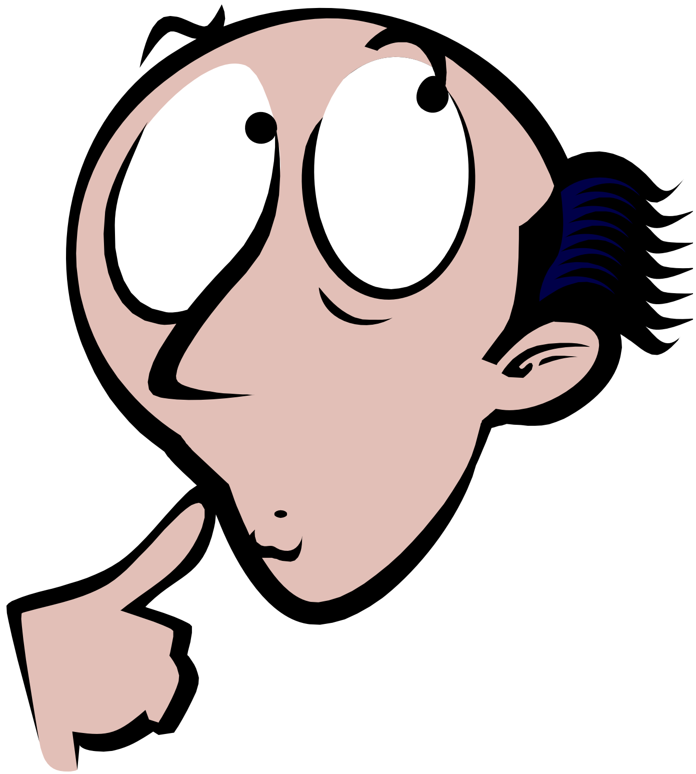
Course Material & Information

- Course website: http://econspace.net/MGT-621.html
  Access to content requires login
  Student ID: 621student
- Required Text:
  - [PR] Pindyck, R.S., Rubinfeld, D.L. (2012). *Microeconomics (8th Edition)*, Pearson/Prentice Hall, Upper Saddle River, NJ
  - All notes & additional readings will be posted
- Solid knowledge in calculus required
- Access to spreadsheet & math software (e.g., MS Excel, Matlab, Maple) may be useful for some homework and the course project
- Links to general information on course website

Honor Code(!)

ADMINISTRIVIA

Did we forget anything?



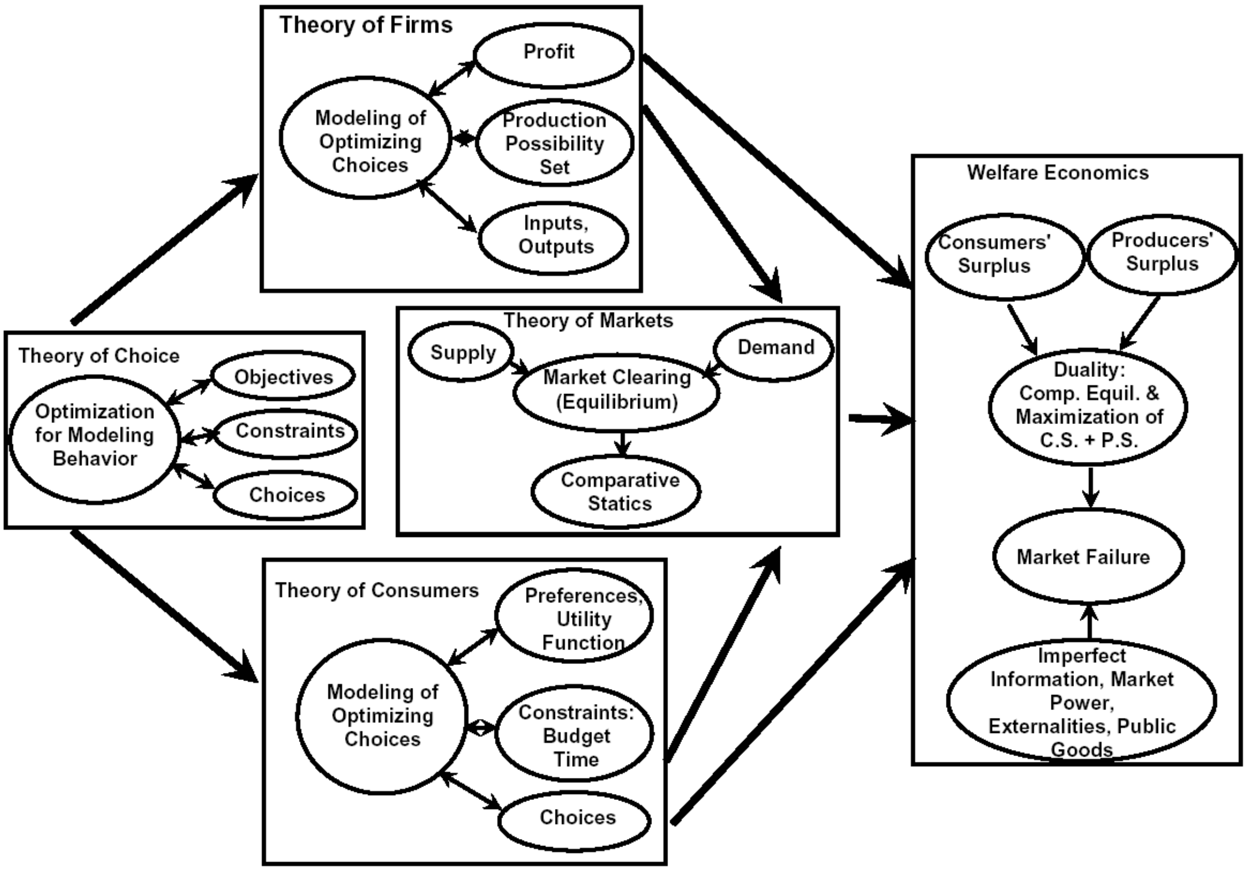
ASSESSMENT

- PROBLEM SETS (20%)
  - Reproductive & productive questions
  - Cooperation ok!!
  - Assignments need to be written up & turned in individually

- FINAL EXAM (40%)
  - Held on Monday, October 2, 2023; Room TBA; there is no makeup
  - *Any arrangements by September 25*
  - 3 hours (open book)
  - Covers everything discussed in the course

- COURSE PROJECT (40%)
  - Report due on October 30 (before 5 pm; by email to the instructor)

MGT 621 MICROECONOMICS – OVERVIEW



TOPICS IN THIS COURSE
Tentative List

I. <u>**Theory of Choice**</u>
- **Individual Decision Making**
- **Preferences and Utility Representation**
- **Consumer Choice (+ under uncertainty)**
- **Aggregate Demand**

II. <u>**Theory of the Firm**</u>
- **Production Sets**
- **Profit Maximization and Cost Minimization**
- **Aggregation**

III. **Market Equilibrium**

IV. **Market Failure**

TOPICS IN THIS COURSE (cont'd)
Tentative List

I. Theory of Choice

II. Theory of the Firm

III. Market Equilibrium
- Competitive Markets
- Profit Maximization and Cost Minimization
- Aggregation

IV. Market Failure
- Monopoly
- Externalities
- Public Goods
- Regulation & Taxation

AGENDA

Administrivia & Course Overview

Preferences and Utility Representation

Some Properties

Utility Representation (Cont'd)

Demand Theory: Basics

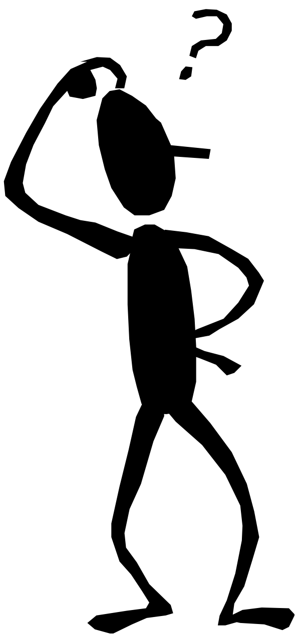
A Little Refresher on Constrained Optimization

Key Concepts to Remember

AN “ECONOMIC CHOICE PROBLEM”: BUYING A CAR

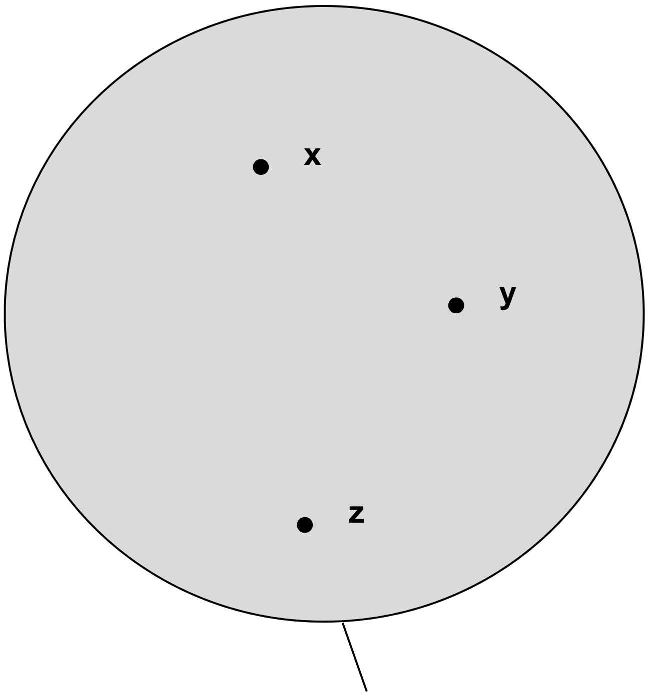
What are your goals? … alternatives? … selection criteria?

PREFERENCES



***Homo Economicus* : “Joe”**

(You? Me? Everybody else?)



Choice Set X = {x,y,z}

Contains all potentially feasible (mutually exclusive) alternatives.

PREFERENCES (Cont'd)

To decide which alternative to choose Joe needs to be able to rank them, i.e., he needs to have a preference ordering of all elements of his choice set X.

Definition. A preference relation on X is a binary relation "$\preceq$" that for any two elements x,y in X compares them so that

(i) $x \preceq y$ **: y is (weakly) preferred to x,**

or

(ii) $y \preceq x$ **: x is (weakly) preferred to y.**

If both (i) and (ii) hold, then we say that there is indifference between x and y, denoted by $x \sim y$ or, equivalently, $y \sim x$. If (i) but not (ii), then $x \prec y$, and we say that y is strictly preferred to x.

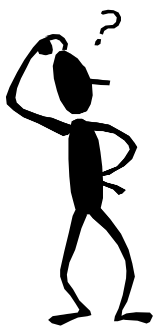
Only if Joe has a preference relation on X, is he able to establish a preference ordering of all elements of his choice set X

PREFERENCES (Cont'd)

Potential problem:

If Joe has a preference relation on X = {x,y,z}, he might have the following preference ordering between pairs of elements:

1. $x \prec y$
2. $y \prec z$
3. $z \prec x$



Example:
- **x: apple**
- **y: banana**
- **z: orange**

Problem?

Lack of transitivity!

(i.e., $x \prec y,\, y \prec z \;\Rightarrow\; x \prec z$ does not hold)

PREFERENCES (Cont'd)

Lack of transitivity can be generated through aggregation of preferences of individuals with transitive preference relations on X: Arrow's Impossibility Theorem

Example: Consider *three alternatives* $x, y, z \in X$ and three agents ($N = \{1,2,3\}$). Let the agents' preferences be as follows:

- **Agent 1:** $x \succ_1 y, y \succ_1 z$
- **Agent 2:** $z \succ_2 x, x \succ_2 y$
- **Agent 3:** $y \succ_3 z, z \succ_3 x$

If we use pairwise majority voting to aggregate the agents' preferences, then we obtain that socially $y \succ z, x \succ y, z \succ x$; in other words, social preferences would be intransitive

Arrow's Impossibility Theorem generalizes this result, and shows that dictatorship (or outside imposition) is required for a consistent aggregation of (at least 3 agents') preferences over (at least 3) independent alternatives.

MORE GENERALLY: QUASI-ORDER ON SETS

Consider a set X of alternatives (outcomes), which has at least two elements.

Definition: A quasi-order R on X is a binary relation that is complete, reflexive, and transitive, i.e.,

- **For all** $x, y \in X$ **:** xRy **or** yRx **(*Completeness*)**
- **For all** $x \in X$ **:** xRx **(*Reflexivity*)**
- **For all** $x, y, z \in X$ **:** $xRy, yRz \Rightarrow xRz$ **(*Transitivity*)**

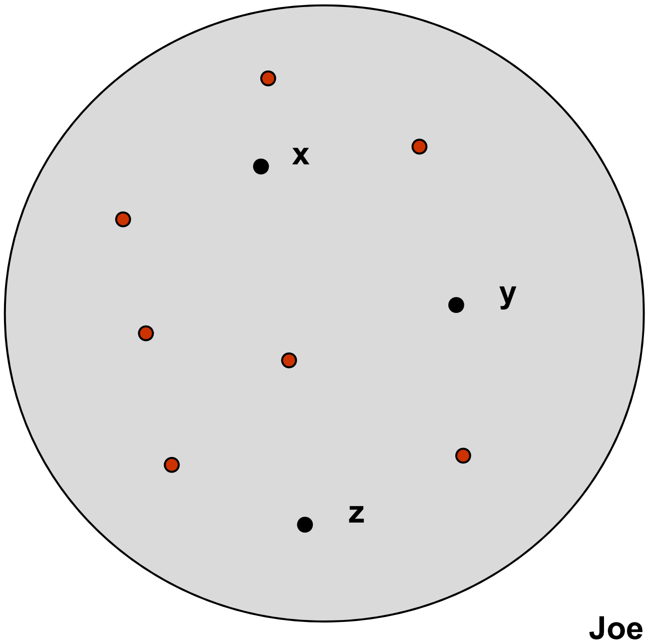
If xRy, we say that x is "weakly preferred" to y.

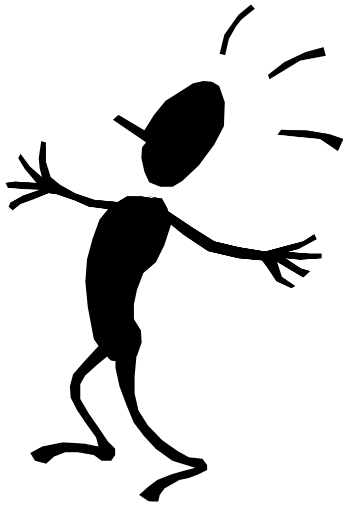
Definition: A strict partial order P is a binary relation that is irreflexive and transitive. For any $x, y \in X$ and quasi-order R on X, we define xPy as $(\text{not } yRx)$. If xPy, we say that x is "(strictly) preferred" to y .[(1)]

(1) In the same spirit we define xIy as $(xRy \text{ and } yRx)$ and say, if xIy, that x is "indifferent" to y.
Remark: For more on ordering alternatives and the representation of preferences, see Fishburn, P.C. (1970) *Utility Theory for Decision Making*," Wiley, New York, NY. Another useful reference is Kreps, D.M. (1988) *Notes on the Theory of Choice*, Westview Press, Boulder, CO.

RATIONAL PREFERENCE ORDER

Definition. A preference relation on X is rational if it is a quasi-order on X, i.e., if it is complete, reflexive, and transitive.





Joe can now make 'rational' choices …

… could they depend on the whole set X?

RATIONAL PREFERENCES – WHERE DO THEY COME FROM?

Mainstream economic theory does not try to explain preferences, but typically takes preferences as data, that is, as fixed for the economic agent.

Preferences in fact result from many forces, e.g.,

- **National culture**
- **Advertising**
- **Social institutions and norms**
- **Parental influence**
- **Education**
- **Religion**
- **Personal tastes**

Preferences can be rational – that is, complete & transitive – and still be the result of the various forces. And they can be rational and change over time (e.g., under the influence of advertising)

A CLASS EXPERIMENT

1. You have been given $200 and have a choice between the following two options

   A: Win $150 with certainty

   B: Win $300 with probability .5

   Win $0 with probability .5

- Do you prefer A or B?

2. You have been given $500 and have a choice between the following two options

   C: Lose $150 with certainty

   D: Lose $300 with probability .5

   Lose $0 with probability .5

- Do you prefer C or D?

RESULT: FRAMING GENERALLY DOES MATTER

			Loss Averse
		Gamble C	**Gamble D**
Risk Averse	**Gamble A**	35	28
	Gamble B	7	8

Rational choices

Since, A = C and B = D, a rational agent's choice should be such that if A is preferred to B then C is preferred to D and vice versa.

However, the "modal choices" are (i.e., "most people prefer") A and D to avoid losses.

Remark: The *reflection effect* refers to the observation that preferences tend to reverse when the lottery is "reflected" from the domain of gains (with respect to the *status quo*) to the domain of losses. For details, see Kahneman, D., Tversky, A. (1979) "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica*, Vol. 47, No. 2, pp. 263—291.

ANOTHER EXAMPLE: ELLSBERG PARADOX

An urn is known to contain 90 balls of which 30 are red and the other 60 black or yellow in unknown proportions. (Neither you nor the person with the urn knows the actual proportions.) One ball is to be drawn at random from the urn and your "reward" depends on the color of the ball drawn. You must choose between the following two bets, which have consequences as indicated.

	Red	Black	Yellow
a. Bet on red	$100	$0	$0
b. Bet on black	$ 0	$100	$0

Now under the same general conditions which bet would you choose in this second situation?

	Red	Black	Yellow
c. Bet on red and yellow	$100	$0	$100
d. Bet on black and yellow	$0	$100	$100

ELLSBERG PARADOX: CLASS RESULTS

	Gamble c	Gamble d
Gamble a	7	58
Gamble b	1	12

'consistent' choices

If you prefer a to b then you should prefer c to d because yellow ball is irrelevant for each pair of decisions.

The "modal choices" are (i.e., "most people prefer") a and d to avoid ambiguity (→ ambiguity aversion) … we will deal with choice under uncertainty later.

UTILITY REPRESENTATION OF PREFERENCES

Idea: Joe's rational preference relation on a nonempty choice set X could be represented by dots on the real line if there is a "**utility function**" u that maps every element x of X to a real number u(x), such that preferred elements get always mapped to larger real numbers.

Then instead of making a pairwise comparison between elements of X, Joe could 'simply' maximize his utility function u on X.

Definition. A function $u : X \to \mathbb{R}$ is a **utility function** that **represents the preference relation** $\preceq$ on X if for any x,y in X:

$$x \preceq y \quad \Leftrightarrow \quad u(x) \le u(y)$$

UTILITY REPRESENTATION (Cont'd)

For a utility representation of a preference relation to exist, the preference relation must necessarily be rational!

Proposition. If the function $u : X \to \mathbb{R}$ represents the preference relation $\preceq$ on X, then $\preceq$ is rational.

Proof (in 2 Steps)

1. Consider any x,y in X. Then, either $u(x) \le u(y)$ or $u(y) \le u(x)$
   Since u represents $\preceq$, it is therefore either $x \preceq y$ or $y \preceq x$, so that $\preceq$ is **complete**.
   It is also **reflexive** (trivial).
2. Consider any x,y,z in X, such that $x \preceq y$ and $y \preceq z$. Thus, $u(x) \le u(y) \le u(z)$, which implies that $x \preceq z$. Hence, the preference relation $\preceq$ is also **transitive**.

QED

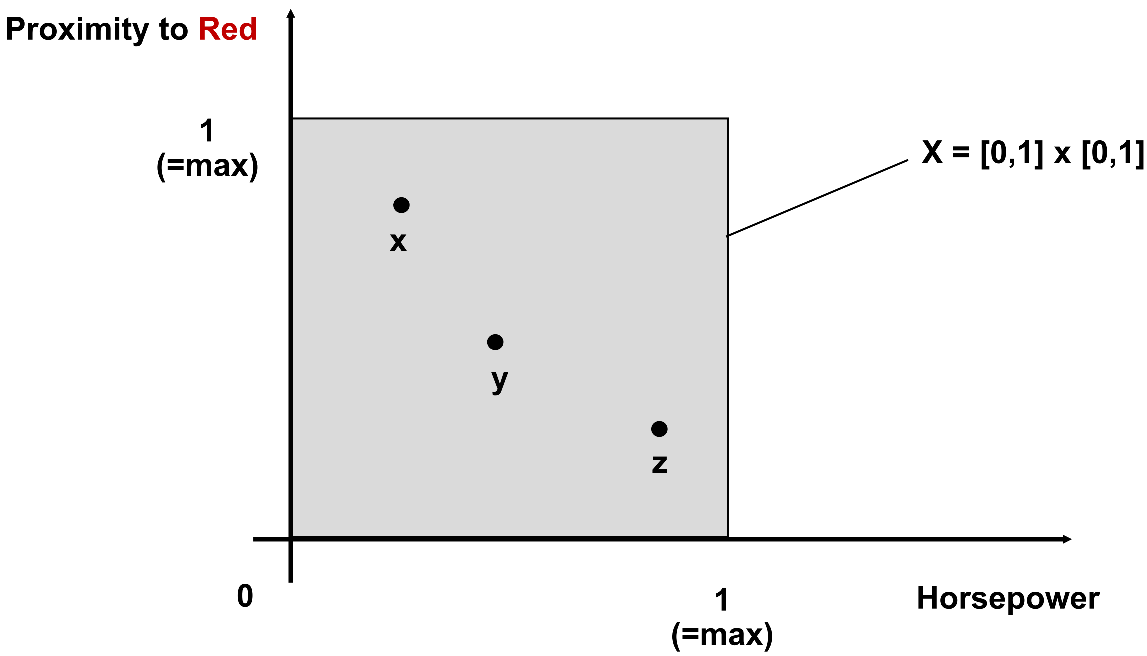
UTILITY REPRESENTATION MAY NOT EXIST!

Example: Preferences for a used car.

Joe would like to buy a Ford Mustang. He cares about two attributes: horsepower and color. Of two given models, he would always prefer the more powerful one. If they have the same power, then he would take the one that has a color closest to red.

$\longrightarrow$ **Lexicographic Preferences**

PREFERENCES FOR USED CAR (Cont'd)



Question. What is Joe's preference ordering?

Right, $x \prec y \prec z$ **!**

AGENDA

Administrivia & Course Overview

Preferences and Utility Representation

Some Properties

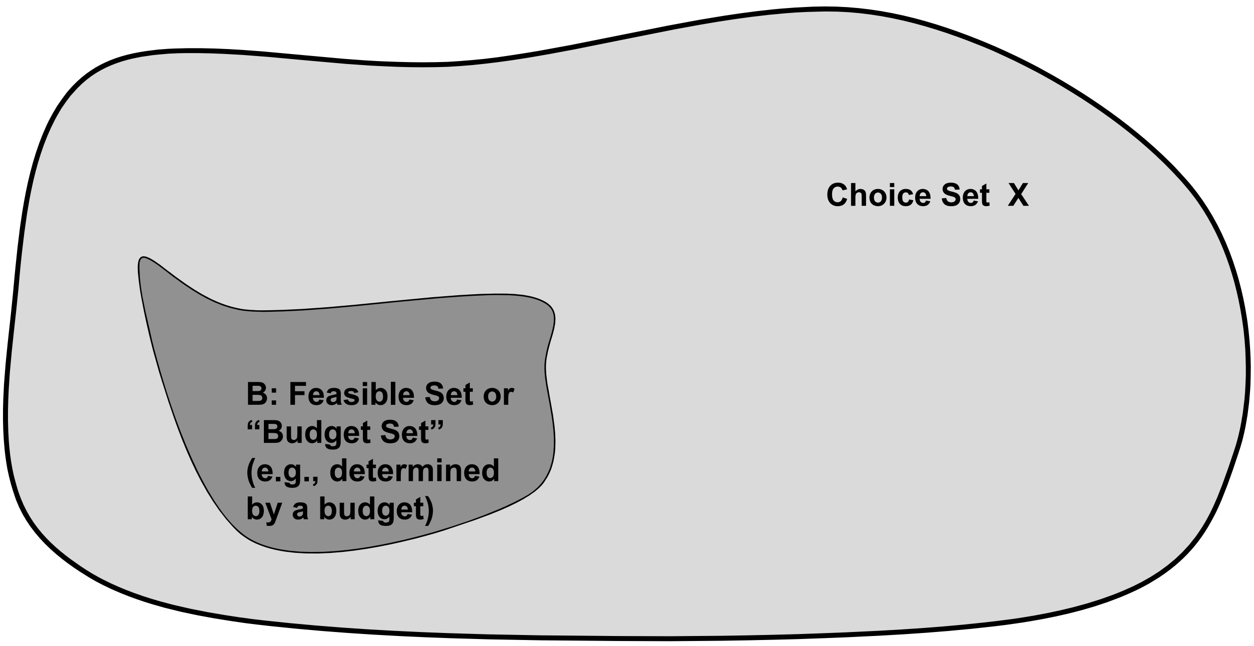
Utility Representation (Cont’d)

Demand Theory: Basics

A Little Refresher on Constrained Optimization

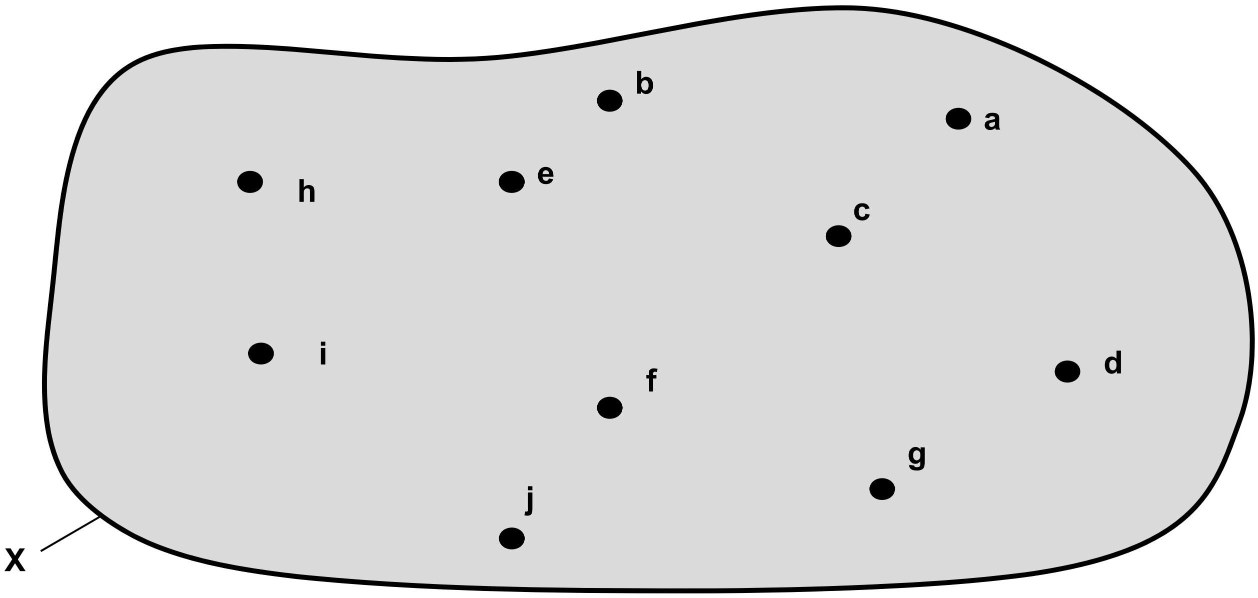
Key Concepts to Remember

ONLY A SUBSET OF THE CHOICE SET MAY BE FEASIBLE



$$\boxed{B \subset X}$$

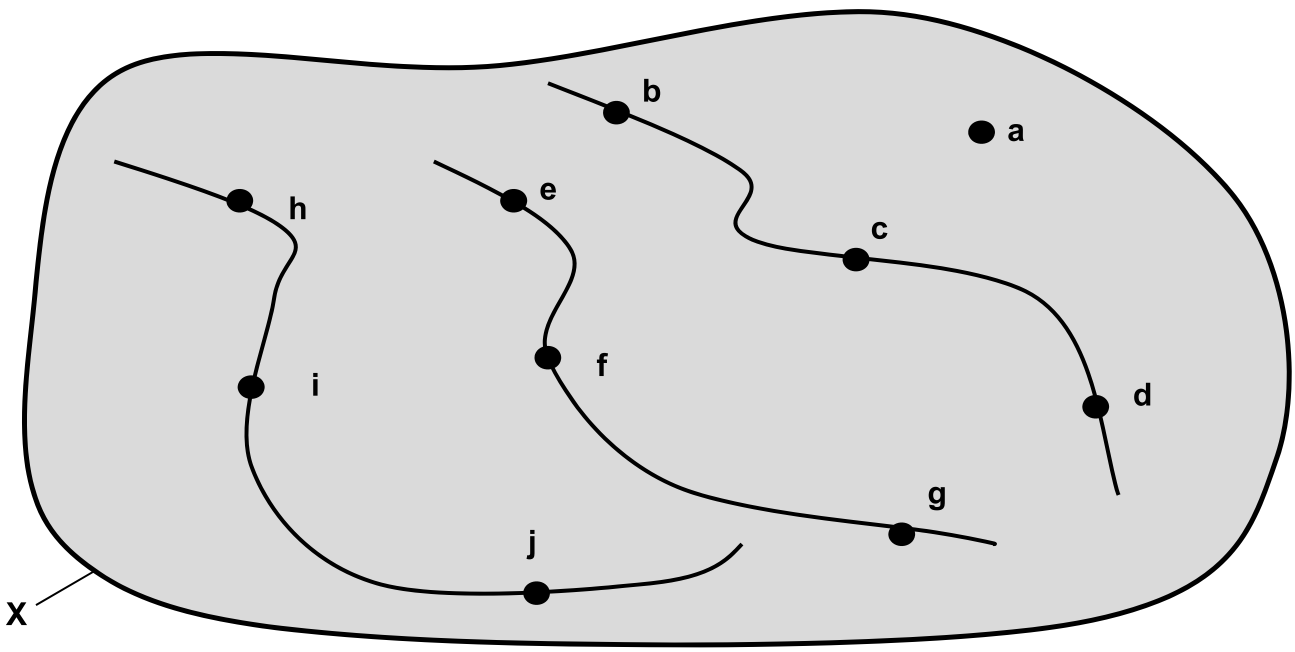
COMPLETE PREFERENCE RELATION …



$a \succeq c \succeq f \succeq j$

$b \sim c \sim d$ $e \sim f \sim g$ $h \sim i \sim j$

… ALLOWS TO DEFINE "EQUIVALENCE CLASSES" OR "INDIFFERENCE CURVES" WHEN THERE IS A UTILITY FUNCTION

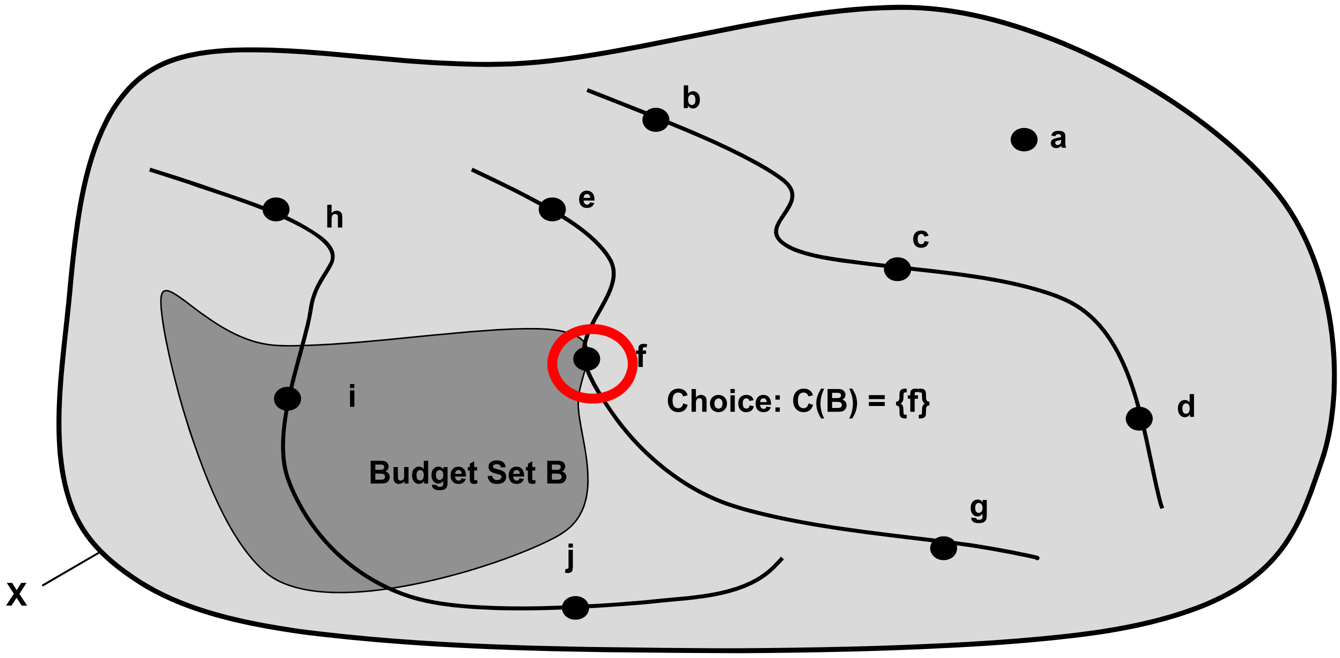


$a \succeq c \succeq f \succeq j$

$b \sim c \sim d$ $e \sim f \sim g$ $h \sim i \sim j$

Remark: It is not really necessary to be able to connect all the points of a choice set that an agent is indifferent about by "continuous" indifference curves. These "curves" might not look like curves at all, and do not really have to. What is important, is that different indifference curves do not intersect!

RATIONAL CHOICE: MOST PREFERRED ALTERNATIVE (HERE AT POINT f)



$a \succsim c \succsim f \succsim j$

$b \sim c \sim d$ $e \sim f \sim g$ $h \sim i \sim j$

Remark: A choice (set) is generally be set valued. For example, C(X\{a}) = {b,c,d}.

SOME PROPERTIES OF CHOICE

Independence of irrelevant alternatives

- **If we reduce the budget set, eliminating points that are not chosen, then the optimal point – the choice point – will not change**

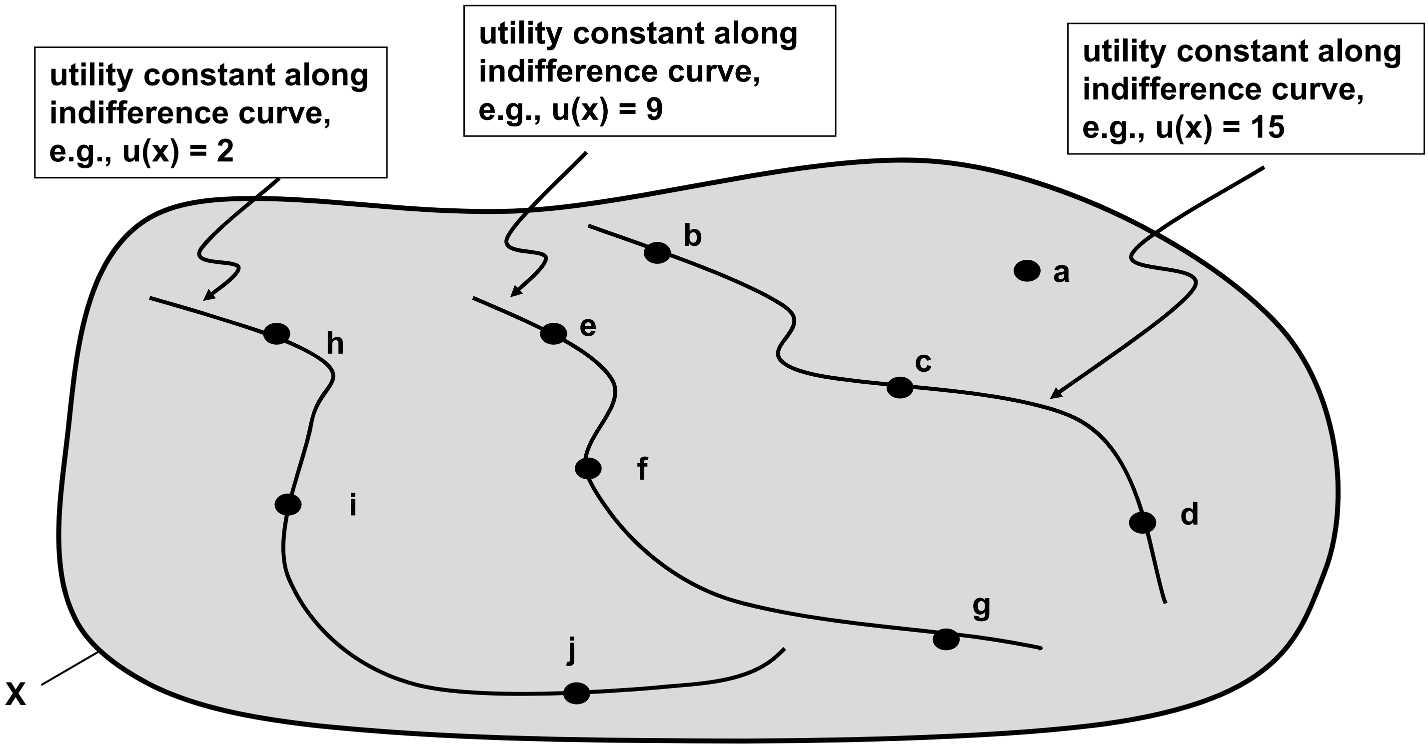
Intensity of preferences is irrelevant to choice

- **Saying that 'C is MUCH preferred to F' or that 'C is slightly preferred to F' has no relevance to what point will be chosen**

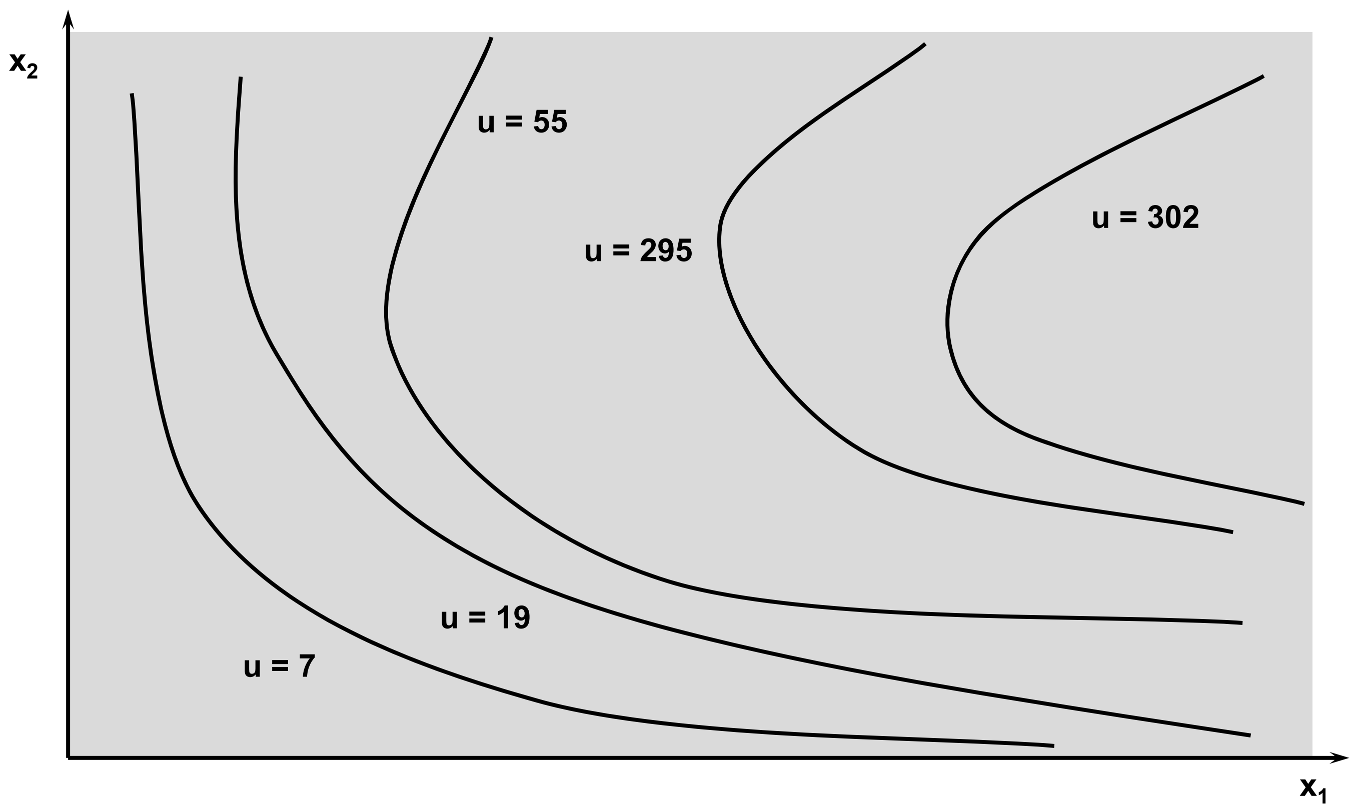
Choice is invariant with respect to changes that leave feasible set unchanged

- **Expansion or contraction of choice set (X) has no impact on choice if expansion or contraction does not impact feasible set**
- **Rescaling of problem parameters that leave the feasible set unchanged will not impact choice**

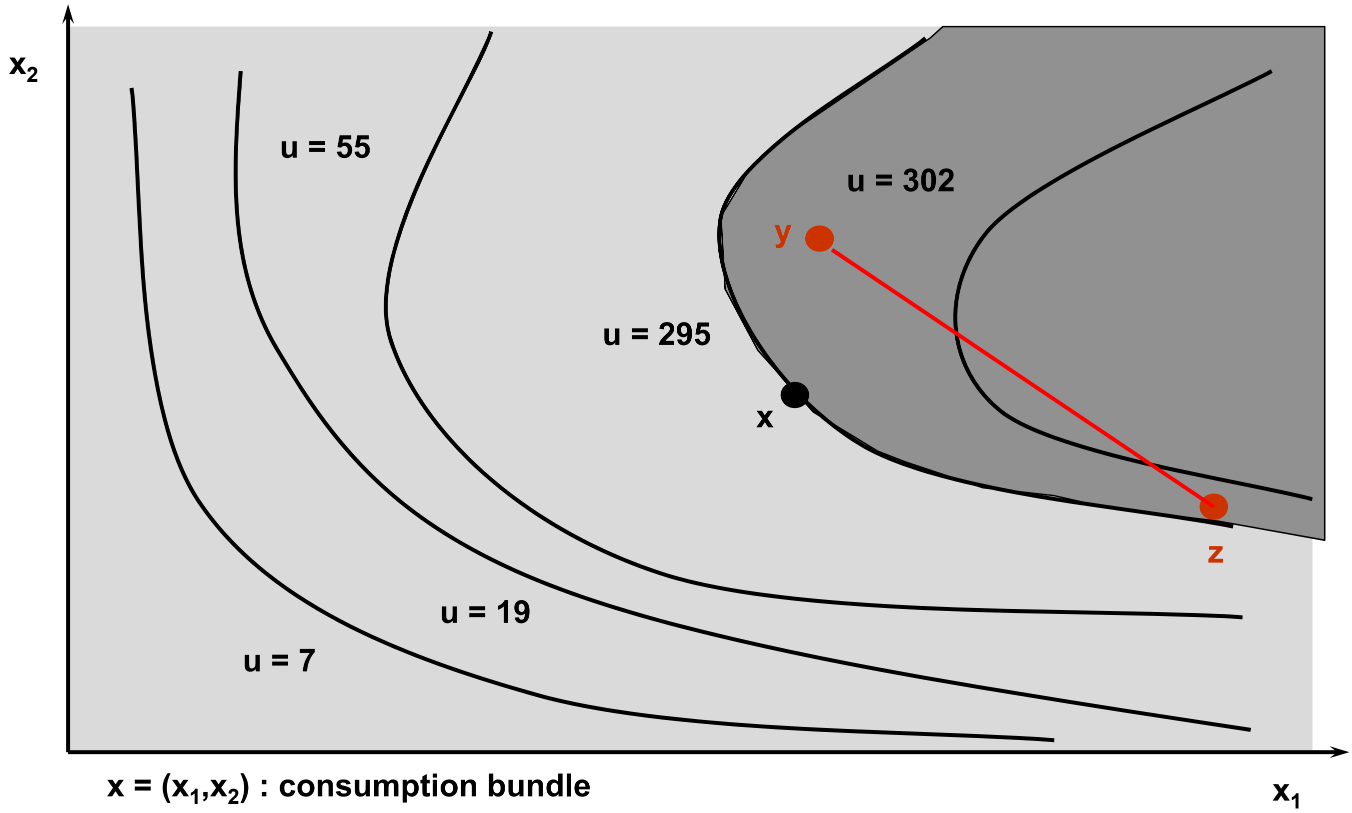
REPRESENTATION OF PREFERENCE RELATION BY UTILITY FUNCTION



EXAMPLE: CONSUMPTION SET WITH INDIFFERENCE CURVES



CONVEXITY OF PREFERENCES



AGENDA

Administrivia & Course Overview

Preferences and Utility Representation

Some Properties

Utility Representation (Cont'd)

Demand Theory: Basics

A Little Refresher on Constrained Optimization

Key Concepts to Remember

THEORY OF THE CONSUMER: PREFERENCES

All consumer preferences assumed to be rational

- **Complete**
- **Reflexive**
- **Transitive**

Preferences also assumed to be continuous

- **Preference order does not jump around discontinuously**

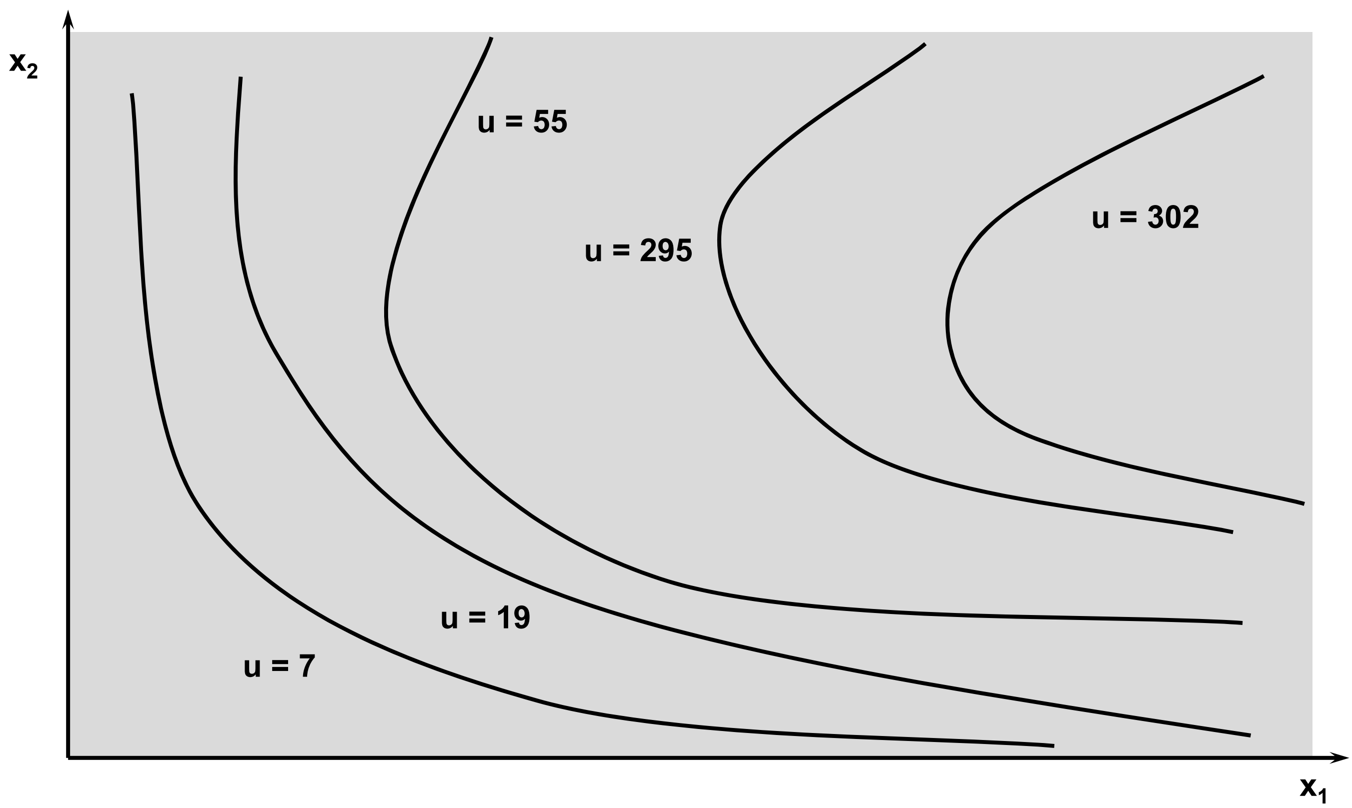
  $\{x: x \succeq y\}$ and $\{x: y \succeq x\}$ are both closed sets

- **Exclude situation: consumer prefers x(n) to y for sequence of x(n) converging to limit $x(\infty)$, but strictly prefers y to $x(\infty)$ (1)**

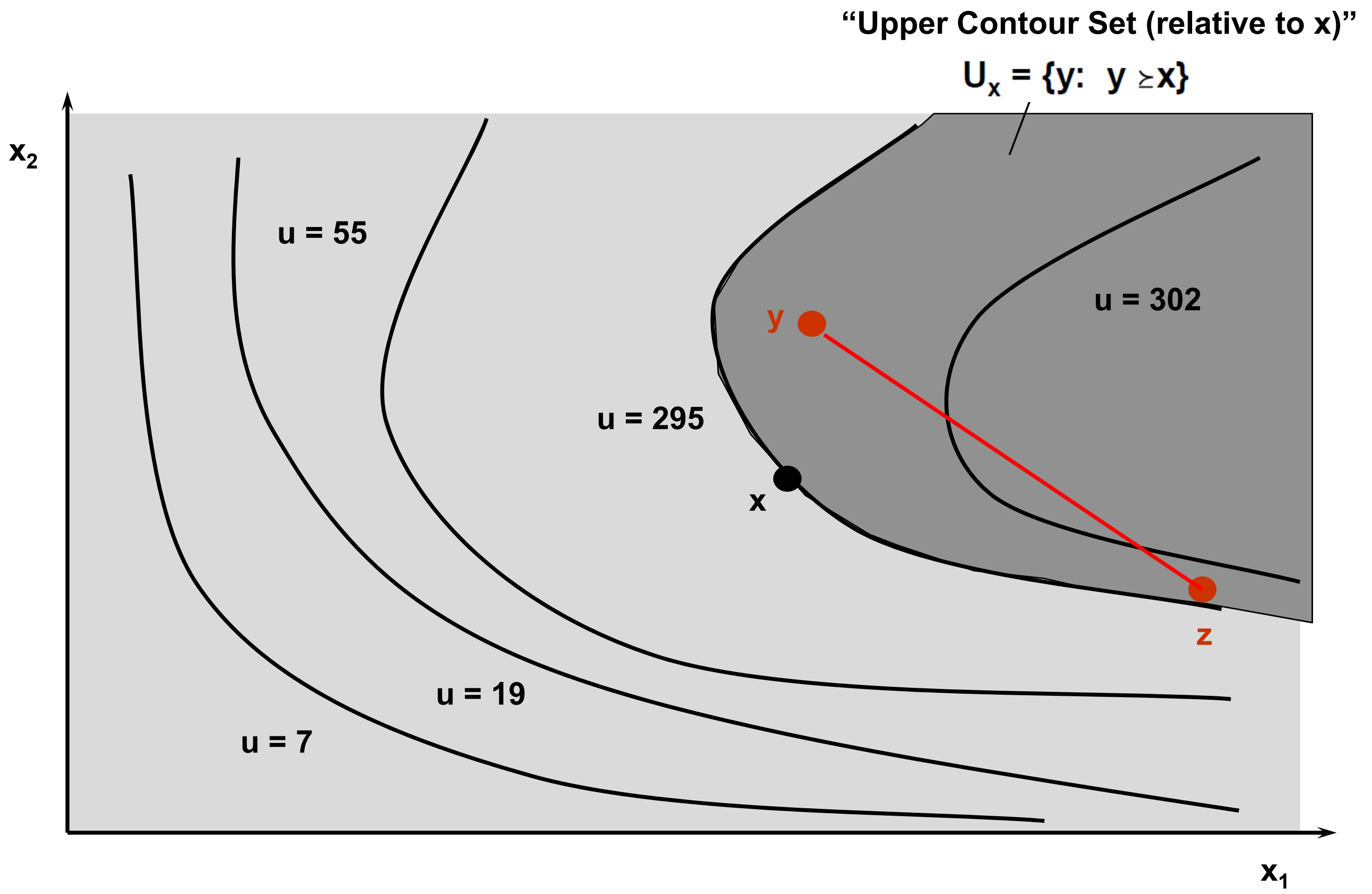
> **Theorem. If preferences are rational and continuous, then there exists a continuous utility function u(x) that describes preferences.**

(1) Example: Take lexicographic preferences on the square X = [0,1] x [0,1] (as in the first lecture) and let x(n) = (1/n,1/n), y=(0,1).

TYPICAL CONSUMPTION SET WITH INDIFFERENCE CURVES



CONVEXITY OF PREFERENCES



CONVEXITY

Definition. A rational preference relation $\preceq$ on X is **convex** if the upper contour set $U_x = \{y : x \preceq y\}$ is convex for any x in X, i.e.,

$$y, z \in U_x \quad \Rightarrow \quad \theta y + (1-\theta) z \in U_x \quad \forall \theta \in (0,1)$$

Proposition. A utility representation of a convex preference relation is quasi-concave (i.e., single-peaked).

ORDINAL VS. CARDINAL PROPERTIES

A utility representation u(x) for a given rational preference relation $\succeq$ on X is generally *not unique*.

The preference relation $\succeq$ fixes only ordering of elements of the choice X, and is therefore called ordinal.

Given the utility representation u(x) of $\succeq$ on X, the function $v(x) = \phi(u(x))$ is also a utility representation of $\succeq$ on X, as long as the (real-valued) transformation ϕ is increasing.

Each specific utility representation of $\succeq$ on X is called cardinal.

Thus, while the ordinal properties of utility functions are invariant with respect to increasing transformations, their cardinal properties are not!

UTILITY FUNCTIONS RELEVANT FOR CONSUMERS AND FIRMS

Theory of Consumers

- Need assumptions about preferences to ensure utility function exists.
- Normally only ordinal properties (which express ordering of options) of utility functions are important.
- For theories of consumer choice under uncertainty, cardinal properties are important. (Cardinal properties express how much better one option is than another.)

Theory of Firms

- Preferences assumed for firms – profit – can always be written as a function, a profit function.
- Profit function plays the same role as utility function.
- For theory of firm behavior under uncertainty, we can use utility function with cardinal properties.

AGENDA

Administrivia & Course Overview

Preferences and Utility Representation

Some Properties

Utility Representation (Cont'd)

Demand Theory: Basics

A Little Refresher on Constrained Optimization

Key Concepts to Remember

THEORY OF THE CONSUMER
Choice Set = Consumption Set

Consumer modeled as choosing among bundles of commodities ("market baskets")
Example: x_1 shirts, x_2 lbs of beef, x_3 gallons of gasoline

$x = (x_1, x_2, x_3)$ is vector of these quantities (L-dimensional vector if there are L commodities)

Choice set X (= "Consumption Set") contains all feasible (not necessarily affordable!) bundles $x \in X$.

Standard Assumptions

- We typically include x = 0
  - Sometimes x=0 may include necessary commodities for survival
  - Sometimes choice set may be discrete, e.g., when only integer amounts of consumption are possible

- We often assume that preferences are (locally) nonsatiated. This means that a bigger bundle (if available to the consumer) is always strictly preferred.

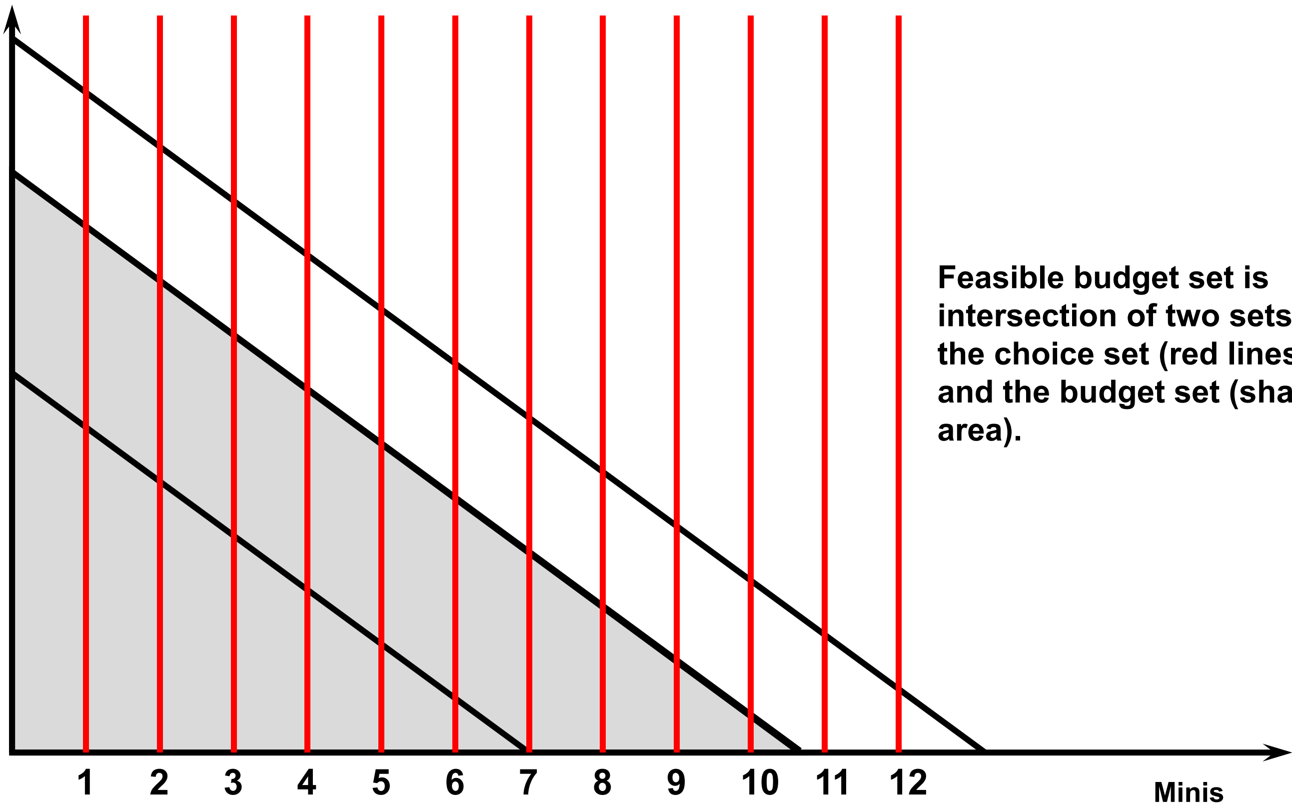
PROBLEM IN REALITY?

Indivisibilities … !

BUDGET SET FOR DISCRETE COMMODITIES

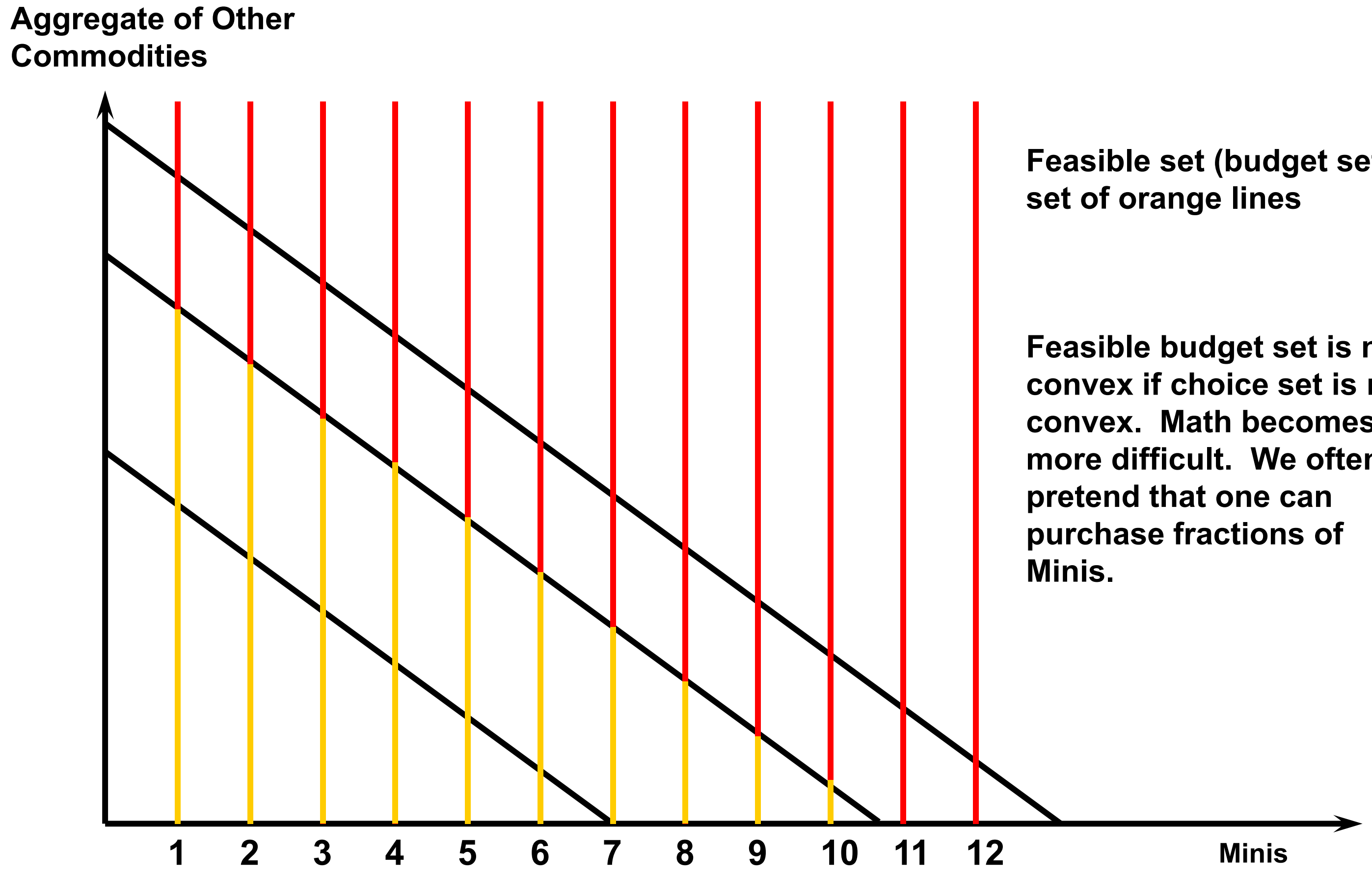
Aggregate of Other Commodities

Choice set consists of points on the red lines



Feasible budget set is intersection of two sets, the choice set (red lines) and the budget set (shaded area).

BUDGET SET FOR DISCRETE COMMODITIES (Cont'd)



Feasible set (budget set) is set of orange lines

Feasible budget set is not convex if choice set is not convex. Math becomes more difficult. We often pretend that one can purchase fractions of Minis.

PROBLEM IN REALITY?

THEORY OF THE CONSUMER
Budget Set

A consumer's choices are constrained to consumption bundles s/he can afford.

- **Commodities traded at prices $p_1, p_2, \ldots, p_L$**
- **Prices represented by an L-dimensional price vector $p = (p_1, p_2, \ldots, p_L)$**
- **Assume that consumer cannot influence prices**

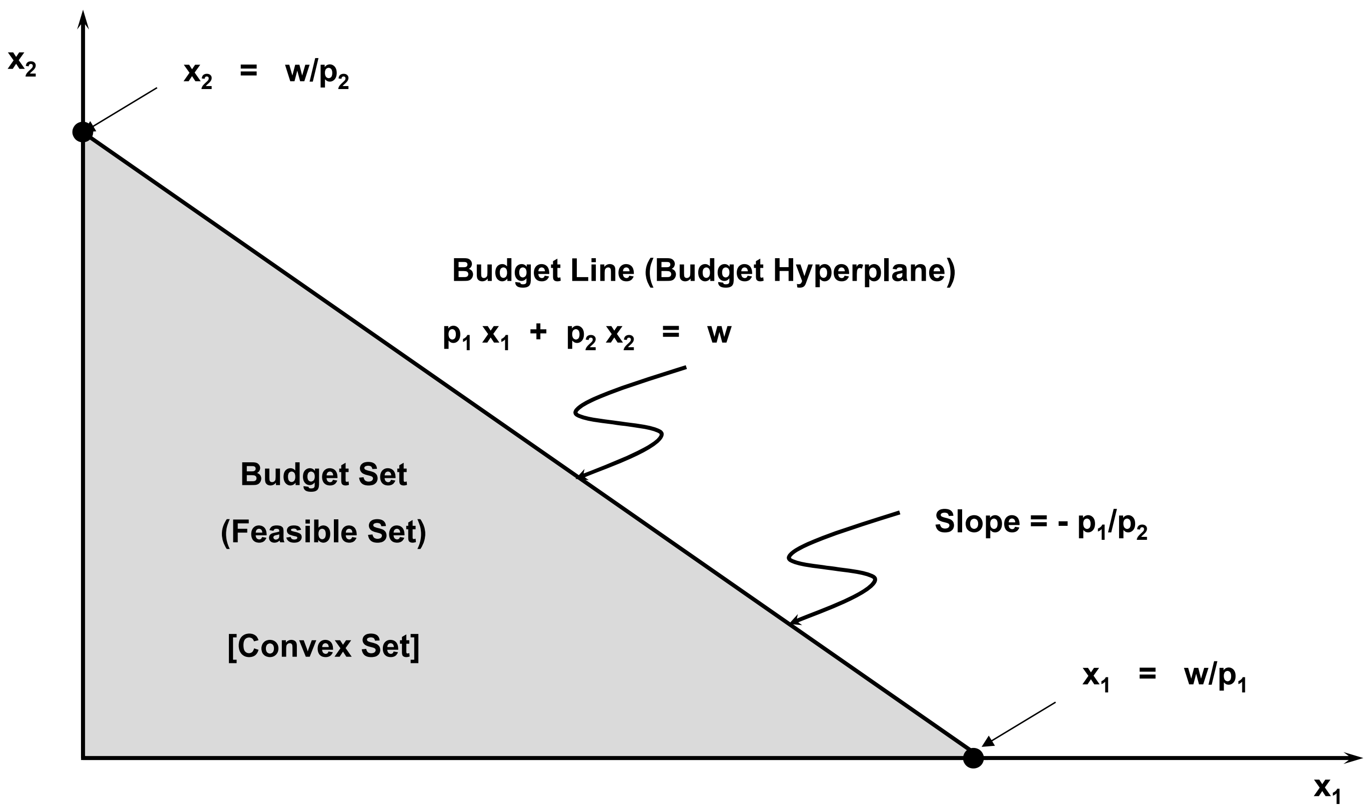
Consumption bundle is affordable if total cost does not exceed the consumer's nonnegative wealth (income), represented by w.

$$p \cdot x \leq w$$

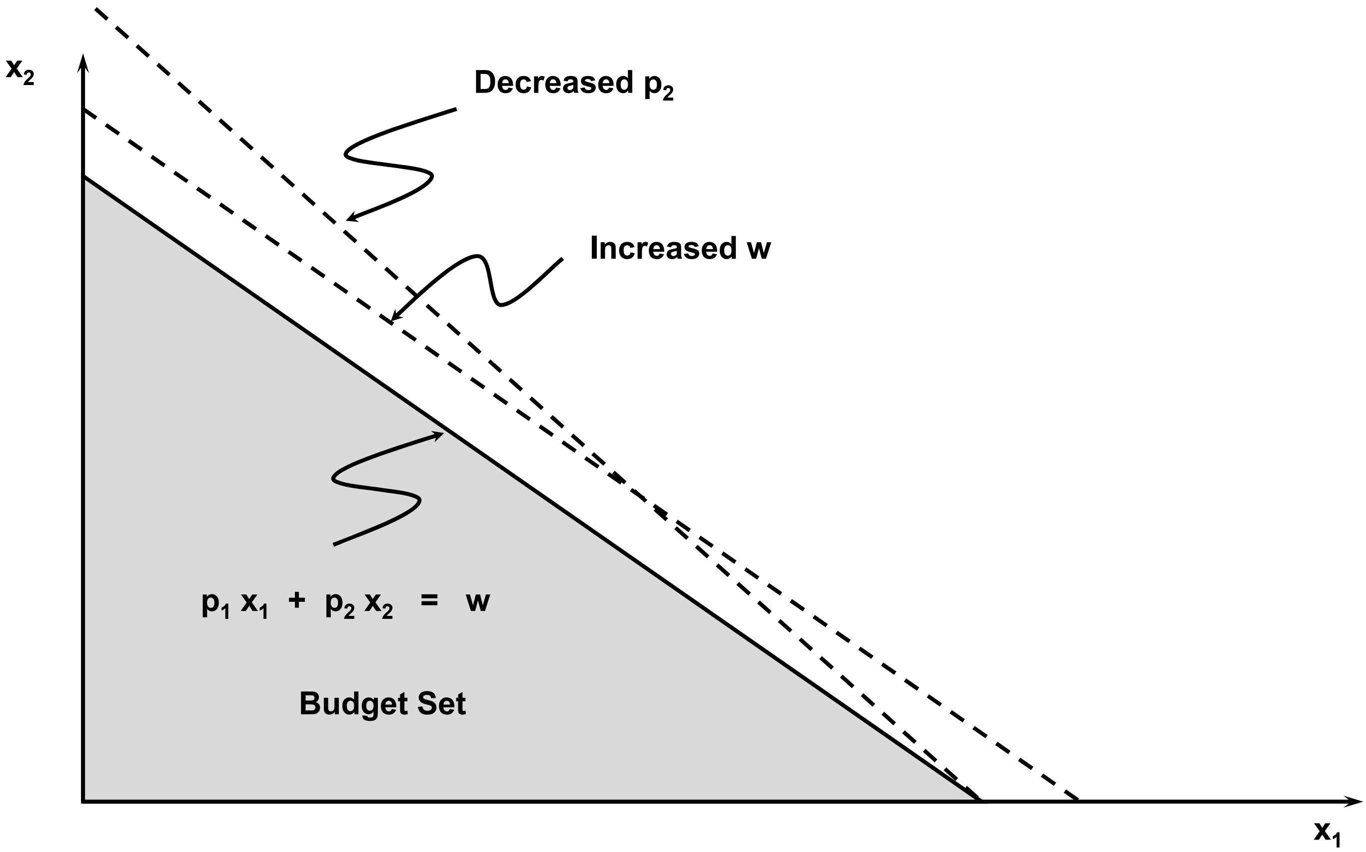
$$p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots + p_L x_L \leq w$$

Set of bundles x in X that satisfy this constraint are known as the budget set B(p,w).

BUDGET SET



BUDGET SET DEPENDS ON PRICES AND WEALTH



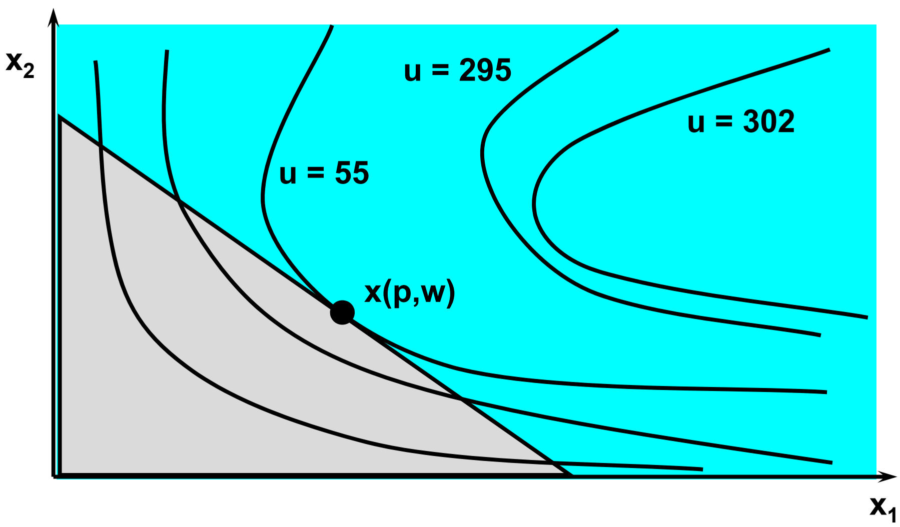
RATIONAL CHOICE = UTILITY MAXIMIZATION PROBLEM

Consumer chooses by maximizing utility over all alternatives in budget set, i.e., s/he solves

Maximize	$u(x)$
such that	$p \cdot x \leq w$

x(p,w) denotes the optimal choice, and is referred to as the demand function

Example



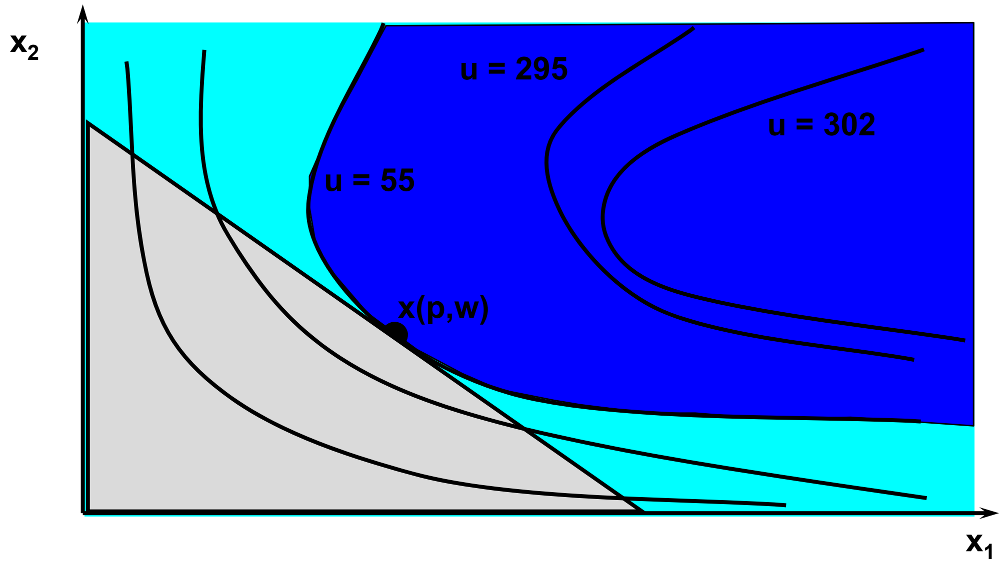
u(x(p,w)) = 55

UTILITY MAXIMIZATION PROBLEM (Cont'd)

x(p,w) denotes the optimal choice, or ("Walrasian") demand function.

- **No feasible point (x) has u(x) > 55**
- **No feasible point strictly preferred to x(p,w).**
- **Set of preferred points (= upper contour set) and feasible set have no common interior points.**

Example (Cont'd)



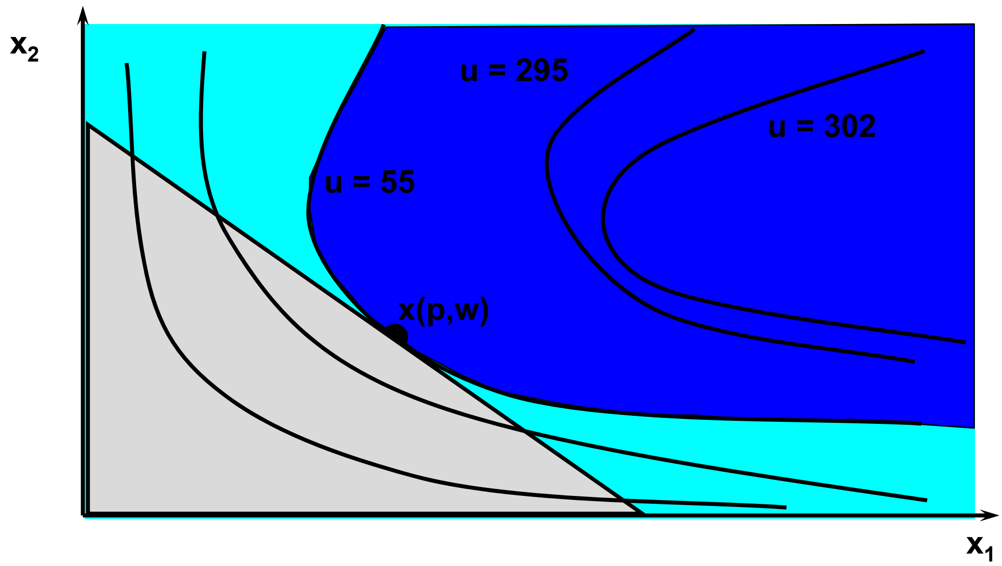
EQUIVALENT EXPENDITURE MINIMIZATION PROBLEM (Cont'd)

Equivalently, x(p,w) solves a second problem:

Minimize $p \cdot x$
such that $u(x) \geq 55$

That is, minimize expenditure, under constraint that $u(x) \geq 55$

Example (Cont'd)



PROPERTIES OF THE CONSUMER DEMAND FUNCTION

Homogeneity of degree zero in p,w:

$x(\alpha p, \alpha w) = x(p,w)$ for any $\alpha > 0$

Walras' Law:

$p \,.\, x(p,w) = w$

(holds if preference are locally nonsatiated)

Convexity:

If preferences are convex, then x(p,w) is a convex set

Uniqueness:

If preferences are strictly convex, then x(p,w) is a single point

AGENDA

Administrivia & Course Overview

Preferences and Utility Representation

Some Properties

Utility Representation (Cont'd)

Demand Theory: Basics

A Little Refresher on Constrained Optimization

Key Concepts to Remember

CONSTRAINED OPTIMIZATION

Let $f : X \to \Re$ be a real-valued function, where X is a nonempty, compact subset of $\Re^n$,[(1)] where n is a positive integer.

We would like to maximize f(x) on X, i.e., solve the problem

Objective Function (Maximand)

$$\max_{x \in X} f(x)$$

Constraint Set

(Constrained Optimization Problem)

Remark:

The utility maximization problem is of this form, with f(x) = u(x) and $X = \{x \in R_+^L : p \cdot x \le w\}$ for some price vector $p = (p_1, \ldots, p_L) >> 0$, wealth w>0, and number of commodities L>0.

(1) A subset of a finite-dimensional Euclidean space is *compact* if it is closed and bounded. A set is *closed* if the limit of any (sub-)sequence constructed from elements of the set also lies in the set. A solution to a constrained optimization problem on a compact set exists, provided the objective function f is continuous.

OPTIMALITY CONDITIONS WHEN CONSTRAINT NOT BINDING
One-Dimensional Case

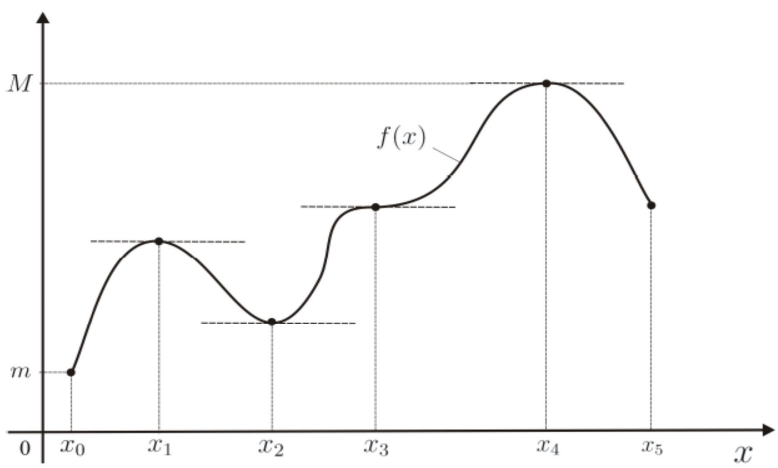
Fermat's Rule (= First-Order Necessary Optimality Condition ["FOC"])

$$x \text{ local extremum } \Rightarrow f'(x) = 0$$

Example:

$X = [x_0, x_5] \subset \Re$

f differentiable



Note that the FOC is satisfied for local maxima and local minima in the interior of X.

OPTIMALITY CONDITIONS (Cont'd)
One-Dimensional Case

In order to guarantee that one has arrived at an interior maximizer (= optimal x that maximizes the objective function and does not lie on the boundary of X), one can use additional optimality conditions: second-order optimality conditions

$$x \text{ local maximizer} \Rightarrow f''(x) \le 0$$
$$x \text{ local minimizer} \Rightarrow f''(x) \ge 0$$

Second-Order Necessary Optimality Condition

$$f''(x) < 0 = f'(x) \Rightarrow x \text{ local maximizer}$$
$$f''(x) > 0 = f'(x) \Rightarrow x \text{ local minimizer}$$

Second-Order Sufficient Optimality Condition ["SOC"]

Examples (Gap Between Necessary and Sufficient Second-Order Optimality Conditions):

(a) The function $f(x) = 1 - x^4$ **has a maximum at x = 0 which does not satisfy SOC.**

(b) The function $f(x) = 1 - x^3$ **has no extremum even though it satisfies** $f''(0) \le 0 = f'(0)$ **.**

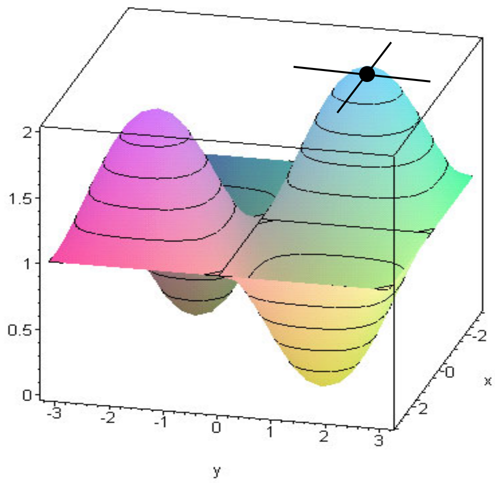
OPTIMALITY CONDITIONS WHEN CONSTRAINT NOT BINDING
Multidimensional Case

Fermat's Rule generalizes to the case of multiple dimensions (n>1)

$$x \text{ local extremum } \Rightarrow Df(x) = (\partial f(x)/\partial x_1, \ldots, \partial f(x)/\partial x_n) = 0$$

Example:

$$X = [-\pi, \pi]^2 \subset \Re^2$$
$$f(x) = 1 - \sin(x_1)\cos(x_2)$$



Again, the FOC is satisfied for local maxima and local minima in the interior of X.

CONSTRAINED OPTIMIZATION

Let $f : X \to \Re$ be a real-valued objective function, $X = \{x \in \Re^n : g(x) \le 0\}$ be a nonempty compact constraint set, where $g : X \to \Re^k$ is a vector-valued function.

The standard constrained optimization problem is then often written in the form

$$\max_{x \in \Re^n} f(x), \quad \text{s.t.} \quad g(x) \le 0$$

Idea:

Relax this problem by introducing *k* additional variables ("Lagrange Multipliers"), one for each constraint component.

Then find critical points (= points that satisfy FOC) of "Lagrangian" L (= relaxed objective function), where

$$L(x;\lambda) = f(x) - \lambda \cdot g(x)$$

CONSTRAINED OPTIMIZATION: INTUITION

Interpret Dg(x) as a vector perpendicular to frontier (determined g(x) = 0), pointing in direction of increasing g(x).

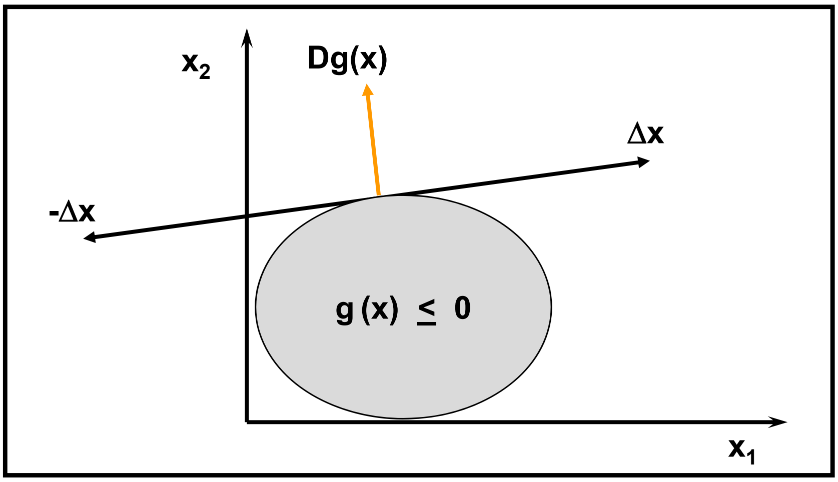
Choose Δx that is tangent to frontier. For tiny movements along Δx or along $-\Delta x$, the function g(x) does not change in value. Thus,

$$\Delta x_1\, \partial g/\partial x_1 + \cdots + \Delta x_n\, \partial g/\partial x_n = 0 \qquad \text{or} \qquad \Delta x \cdot Dg(x) = 0$$

Any two vectors whose inner product is zero must be perpendicular to each other. Thus, Δx and Dg(x) are perpendicular to each other.

Now take Δx = Dg(x) (assumed nonzero). Then, $\Delta g = \Delta x \cdot Dg(x) > 0$, because all components of Δg are $(\partial g/\partial x_i)^2 > 0$.

Hence, g(x) is increasing in direction of Dg(x).



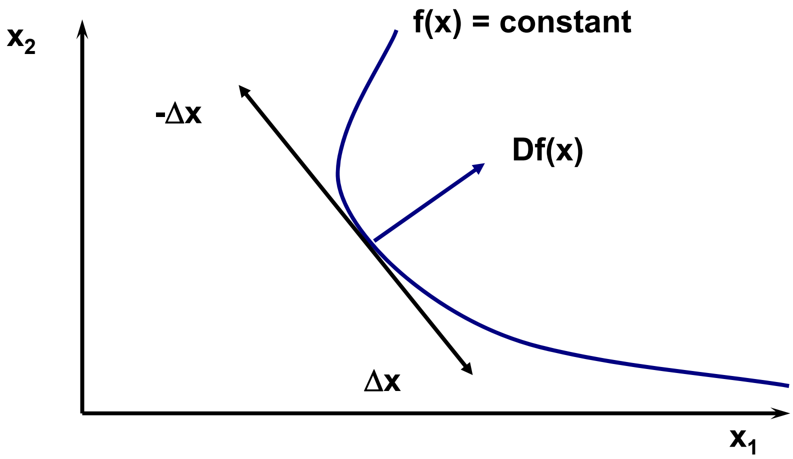
CONSTRAINED OPTIMIZATION: INTUITION (Cont'd)

Interpret Df(x) as a vector perpendicular to the level set {y : f(x) = f(y) }

This vector is pointed in the direction of increasing value of f(x).

Choose Δx that is tangent to the level set, f(x) = constant. For tiny movements along Δx or along $-\Delta x$, f(x) does not change in value. Thus $\Delta x_1 \, \partial f / \partial x_1 + \cdots + \Delta x_n \, \partial f / \partial x_n = 0$ or $\Delta x \cdot Df(x) = 0$.

Thus, Δx and Df(x) are perpendicular to each other.

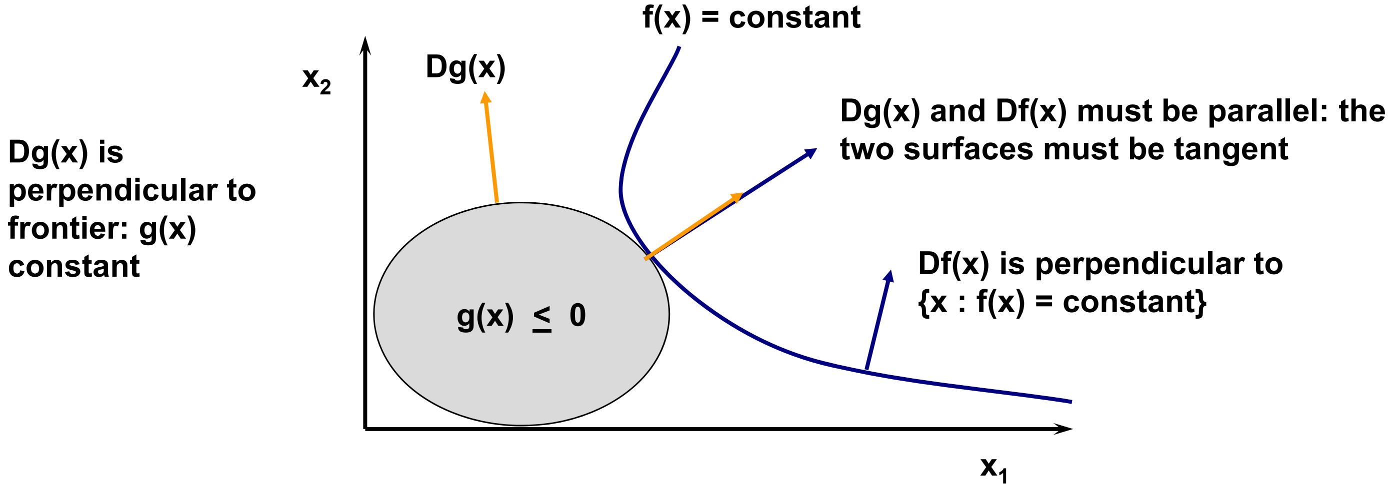


INTUITION OF FIRST-ORDER CONDITION

First-Order Necessary Optimality Condition for Constrained Optimization:

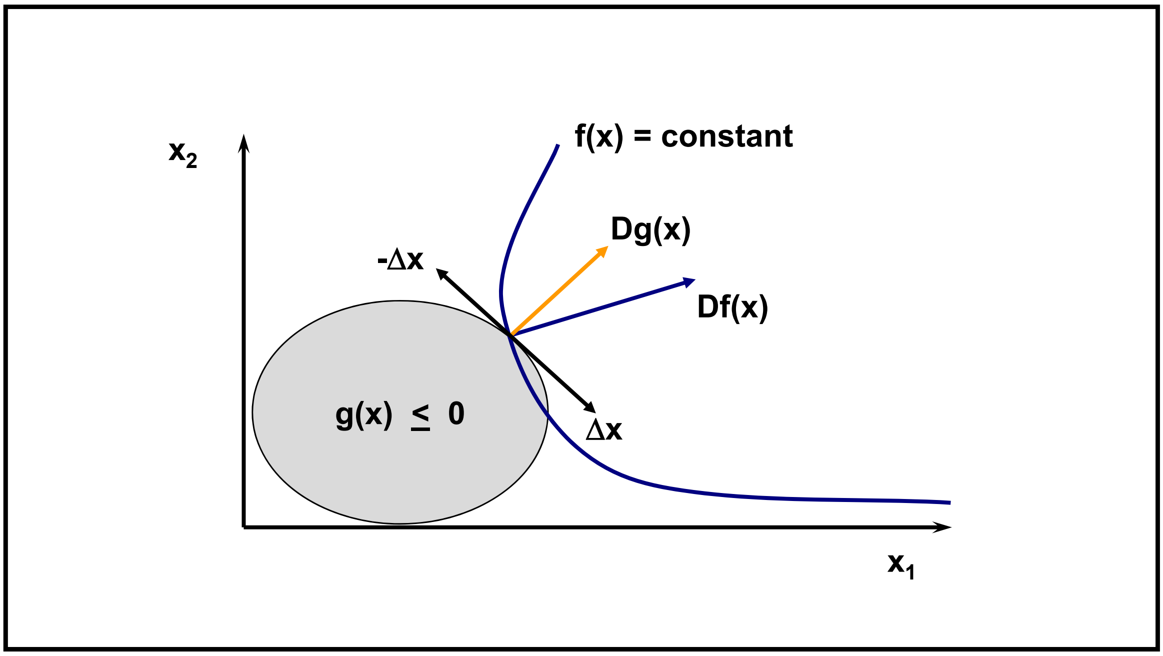
If the constraint g(x)=0 is binding, then a level set of f must be tangent to the constraint set at an extremal point. This implies that the gradient of f and the gradient of the constraint function g need to be parallel.

> **$Df(x) = \lambda \, Dg(x)$ for some $\lambda \geq 0$ (λ is a scalar)**
>
> **If constraint is binding (i.e., if $g(x) = 0$), then $\lambda > 0$**



INTUITION OF FIRST-ORDER CONDITION

If Dg(x) and Df(x) are not parallel, there are feasible points with greater f(x). They can be found by moving tiny distance in direction Δx or -Δx.



FIRST-ORDER NECESSARY OPTIMALITY CONDITION FOR CONSTRAINED OPTIMIZATION PROBLEM

Formal Statement

Let $L(x;\lambda) = f(x) - \lambda \cdot g(x)$ **be the Lagrangian associated with the constrained optimization problem**

$$\max_{x \in \Re^n} f(x), \quad \text{s.t.} \quad g(x) \leq 0$$

Necessary Optimality Conditions (Kuhn-Tucker Conditions):[(1)]

$$x \text{ is local extremum of } f(x) \text{ in } \{x : g(x) \leq 0\} \Rightarrow \begin{array}{c} D_x L(x;\lambda) = 0, \\ \lambda_i g_i(x) = 0, i \in \{1,\ldots,k\} \end{array}$$

The k relations $\lambda_i g_i(x) = 0, i \in \{1,\ldots,k\}$ **are also referred to as complementary slackness conditions. The variables** λ_i **are called Lagrange multipliers or dual variables.**

(1) The validity of these conditions depends on whether the constraint function g satisfies a "constraint qualification" in a neighborhood of the extremum, which means that in this neighborhood the constrains are nonredundant, so that Dg(x) is of full rank (i.e., of rank *k*) there.

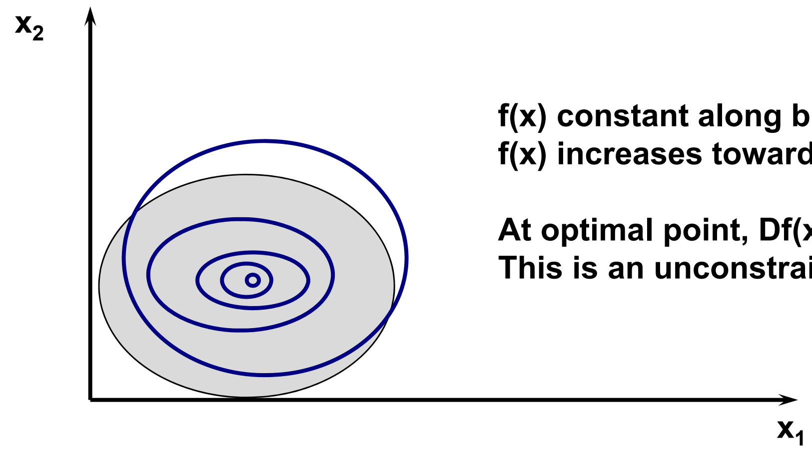
RELATION BETWEEN FIRST-ORDER CONDITIONS FOR CONSTRAINED AND UNCONSTRAINED OPTIMIZATION PROBLEMS

Consider the constrained optimization problem

$$\max_{x \in \Re^n} f(x), \quad \text{s.t.} \quad g(x) \leq 0$$

If, at an extremum x, the constraint is not binding, i.e., if g(x)<0, then complementary slackness implies that all Lagrange multipliers vanish.

Example



f(x) constant along blue curves.
f(x) increases toward peak at the middle.

At optimal point, Df(x) = 0
This is an unconstrained optimum.

CONSTRAINED OPTIMIZATION WITH MULTIPLE CONSTRAINTS
Intuition

Consider the following problem:

$\max_x f(x)$, s.t. $g_1(x) \leq 0$ and $g_2(x) \leq 0$

First Order Necessary condition:

$Df(x) = \lambda_1 Dg_1(x) + \lambda_2 Dg_2(x)$ [Df(x) lies between $Dg_1(x)$ and $Dg_2(x)$]

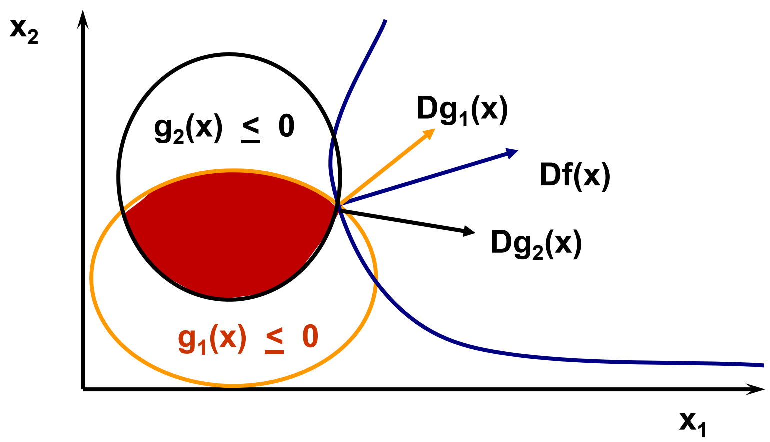
$g_1(x) \leq 0 \qquad \lambda_1 \geq 0$

$g_2(x) \leq 0 \qquad \lambda_2 \geq 0$

$\lambda_1 g_1(x) = 0$

$\lambda_2 g_2(x) = 0$

Red area is feasible set with two constraints



INTERPRETATION OF THE DUAL VARIABLES

λ corresponds to an increase in the optimized objective function f per unit relaxation of the constraint $g(x) \leq 0$. (Relaxation means $g(x) \leq b$, for very small vector $b >> 0$)

Show: $\Delta f = \lambda \,.\, b$

$\Delta f = \partial f / \partial x_1 \, \Delta x_1 + \ldots + \partial f / \partial x_n \, \Delta x_n$

$\Delta g = \partial g / \partial x_1 \, \Delta x_1 + \ldots + \partial g / \partial x_n \, \Delta x_n = b$

Calculate Δu, remembering first-order necessary optimality condition, $\partial f / \partial x_i = \lambda \, \partial g / \partial x_i$

$\Delta f = \lambda \, \partial g / \partial x_1 \Delta x_1 + \lambda \, \partial g / \partial x_2 \, \Delta x_2 + \ldots = \lambda \, \Delta g$

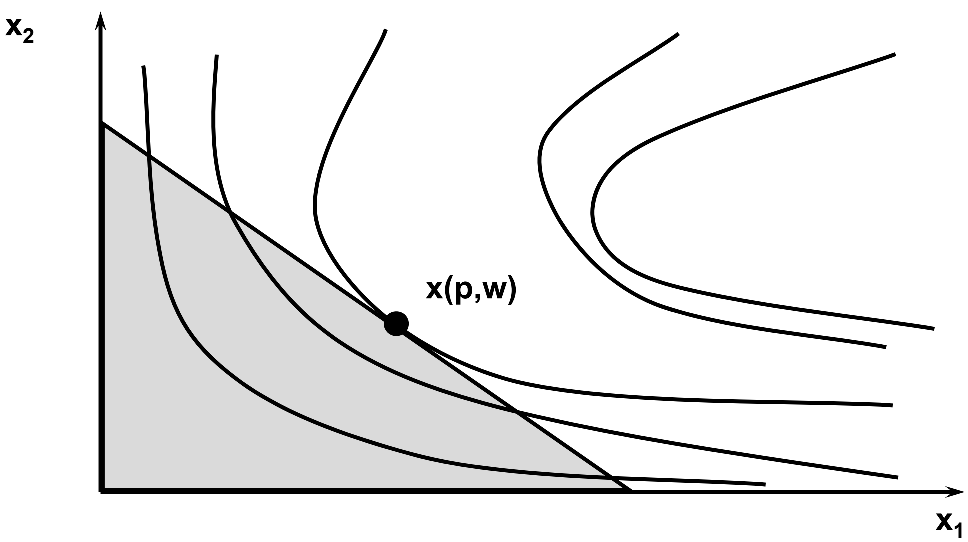
The dual variables (Lagrange multipliers) are equal to the value of being able to relax the constraints. They are often called the shadow prices of the problem.

CONSUMER SOLVES UTILITY MAXIMIZATION PROBLEM

Choice is the maximum utility alternative from feasible set, in this case, the budget set

Maximize $u(x)$
such that: $p \,.\, x \leq w$
$x \geq 0$

$x(p,w)$ denotes the optimal choice, or Walrasian demand function, given p and w.



CHARACTERIZING OPTIMAL CHOICE

Maximize $u(x)$
such that: $p \cdot x \leq w$
$x \geq 0$

Constraints can be written in standard form:
$p \cdot x - w \leq 0$ becomes $g_0(x) = p \cdot x - w \leq 0$
$-x \leq 0$ becomes $g_i(x) = -x_i \leq 0$ for each i in {1,…,L}

First-order necessary optimality conditions:

$$Du(x) = \sum_{i=0}^{L} \lambda_i Dg_i(x) = \lambda_0 p - \begin{bmatrix} \lambda_1 \\ \\ \lambda_L \end{bmatrix}$$

and

$$\lambda_0 (p \cdot x - w) = 0$$

$$\lambda_i x_i = 0, i \in \{1,\ldots,L\}$$

INTERPRETATION OF LAGRANGE MULITPLIERS

$Du(x) = \lambda_0 p - (\lambda_1,\ldots,\lambda_L)$
$(p \cdot x - w)\lambda_0 = 0$
with local nonsatiation, $p \cdot x = w$ and $\lambda_0 > 0$

$(\lambda_1,\ldots,\lambda_L) \cdot x = 0$
if $x_i > 0$, then $\lambda_i = 0$

Thus:

$\partial u/\partial x_i = \lambda_0 p_i$ if $x_i > 0$
$\partial u/\partial x_i \leq \lambda_0 p_i$ if $x_i = 0$

The marginal utility of each good that is purchased is equal to its price multiplied by the shadow price on wealth. If a good is not purchased, its marginal utility is smaller than its price multiplied by the shadow price on wealth.

INTERPRETATION (Cont'd)

$$\partial u/\partial x_i = \lambda_0 p_i \quad \text{if} \quad x_i > 0$$
$$\partial u/\partial x_i \leq \lambda_0 p_i \quad \text{if} \quad x_i = 0$$

For two goods that are both purchased (that is $x_i > 0$) :

$$\frac{\partial u/\partial x_i}{\partial u/\partial x_j} = \frac{\lambda p_i}{\lambda p_j} = \frac{p_i}{p_j}$$

Interpret:

$$\frac{\partial u/\partial x_i}{\partial u/\partial x_j} = - \partial x_j / \partial x_i |_{u \text{ constant}}$$

Marginal Rate of Substitution of good i for good j and is denoted MRS_{ij}. MRS_{ij} is the amount of good j the consumer would need to receive in order to exactly be compensated for a unit loss of good i.

INTERPRETATION (Cont'd)

For two goods that are both purchased (that is $x_i > 0$) :

$$\frac{\partial u/\partial x_i}{\partial u/\partial x_j} = \frac{\lambda p_i}{\lambda p_j} = \frac{p_i}{p_j}$$

Thus

$$MRS_{ij} = p_i / p_j$$

The price ratio is equal to the MRS. Amount of good j the consumer would need to receive to exactly be compensated for a unit loss of good i is equal to the price ratio.

Dollar value of good i lost is p_i. Dollar value of good j gained to exactly compensate is p_j MRS_{ij}

INTERPRETATION (Cont'd)

$$\partial u / \partial x_i = \lambda p_i \quad \text{if} \quad x_i > 0$$
$$\partial u / \partial x_i < \lambda p_i \quad \text{if} \quad x_i = 0$$

Assume good j is not purchased, but good i is. Then take ratio of two sides of equation:

$$\frac{\partial u/\partial x_i}{\partial u/\partial x_j} > \frac{\lambda p_i}{\lambda p_j} = \frac{p_i}{p_j}$$

Thus $MRS_{i\,j} > p_i / p_j$

If good j is not purchased, but good i is purchased
to be exactly compensated for a unit loss of good i, the person would need to get more than p_i/p_j units of good j.

AN INTERESTING OBSERVATION

Different consumers have different preferences. Thus, different consumers generally choose different consumption bundles.

<u>But</u> at the optimal consumption bundle, each consumer has the *same* $MRS_{i\,j}$ as any other consumer

- **Relative value of two goods (subjective sense) is identical among all people who buy positive quantities of both**
- **Everyone who buys i and j have same rate at which they are willing to substitute one product for another product**

EXAMPLE: CONSUMER WITH COBB-DOUGLAS UTILITY FUNCTION

Consider a consumer with Cobb-Douglas utility function

$$u(x_1, x_2) = (x_1)^{\alpha}(x_2)^{1-\alpha}$$

where $\alpha \in (0,1)$ is a given constant.

Given a price vector $p = (p_1, p_2)$, the consumer's utility maximization problem yields (using the Lagrangian methods described earlier) the Walrasian demand vector x(p,w) as a function of price and wealth:

$$x(p,w) = \arg \max_{(x_1,x_2)\in B(p,w)} \{\alpha \log x_1 + (1-\alpha)\log x_2\} = \underbrace{\left(\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}\right)}_{\text{"Walrasian Demand"}}$$

AGENDA

Administrivia & Course Overview

Preferences and Utility Representation

Some Properties

Utility Representation (Cont'd)

Demand Theory: Basics

A Little Refresher on Constrained Optimization

Key Concepts to Remember

KEY CONCEPTS TO REMEMBER

- Choice Set & Quasi-Order on Sets
- Preference Relation & Rational Preferences
- Utility Function
- Framing
- Ellsberg Paradox
- Properties of Choice
- Continuity & Convexity of Preferences
- Cardinal vs. Ordinal Properties
- Choice Set vs. Budget Set
- Utility Maximization Problem
- Walrasian Demand / Walras' Law
- Constrained Optimization / Necessary Optimality Conditions
- Lagrange Multipliers (Dual Variables, Shadow Prices)
- Cobb-Douglas Utility Function

APPENDIX

(Optional)

LEXICOGRAPHIC PREFERENCES CANNOT BE REPRESENTED BY A UTILITY FUNCTION (1/3)

Example 3 Consider a decision maker with *lexicographic preferences* $\preceq$, defined on the set $\mathcal{S} = [0,1] \times [0,1]$, such that

$$(s_1, s_2) \preceq (\hat{s}_1, \hat{s}_2) \quad \overset{\text{def}}{\Leftrightarrow} \quad s_1 < \hat{s}_1 \text{ or } (s_1 = \hat{s}_1 \text{ and } s_2 \leq \hat{s}_2).$$

The decision maker thus prefers an improvement in s_1 more than *any* improvement in s_2. Let us for a moment assume that there exists a utility function $u : \mathcal{S} \to \mathbb{R}$ that represents these preferences according to (4). We now show that this inevitably leads to a contradiction. Note first that $(s_1, 0) \prec (s_1, 1)$ and therefore $u((s_1, 0)) < u((s_1, 1))$ for all $s_1 \in [0,1]$. If we let

$$\Delta(s_1) = u((s_1, 1)) - u((s_1, 0)),$$

LEXICOGRAPHIC PREFERENCES CANNOT BE REPRESENTED BY A UTILITY FUNCTION (2/3)

then $\Delta(s_1) > 0$ for all $s_1 \in [0,1]$. As a result, the range $[0,1]$ of the first coordinate can be written as a union of the subsets $\mathcal{S}_{1k} = \{s_1 : \Delta(s_1) \geq 1/k\}$,

$$[0,1] = \bigcup_{k=1}^{\infty} \mathcal{S}_{1k}.$$

Since the interval $[0,1]$ is uncountable, some of the sets $\mathcal{S}_{1k}$ have to be uncountable as well. Let $\mathcal{S}_{1\bar{k}}$ be such an uncountable set for an appropriate $\bar{k} \in \{1, 2, \ldots\}$. Let $\bar{\Delta} = u((1,1)) - u((0,0))$ be the largest possible utility difference between any two elements in $\mathcal{S}$ and let $K > \bar{k}\bar{\Delta} + 1$ be an integer. Then for any K elements $\sigma_1, \ldots, \sigma_K \in \mathcal{S}_{1\bar{k}}$ with $\sigma_1 < \sigma_2 < \cdots < \sigma_K$ we have

$$u((\sigma_k, 0)) - u((\sigma_{k-1}, 0)) > u((\sigma_{k-1}, 1)) - u((\sigma_{k-1}, 0)) > 1/\bar{k}$$

for any $k \in \{2, \ldots, K\}$.

LEXICOGRAPHIC PREFERENCES CANNOT BE REPRESENTED BY A UTILITY FUNCTION (3/3)

Hence,

$$
\begin{aligned}
\bar{\Delta} &= u((1,1)) - u((0,0)) \\
&= [u((1,1)) - u((\sigma_K, 0))] + [u((\sigma_K, 0)) - u((\sigma_{K-1}, 0))] + \cdots \\
&\quad + [u((\sigma_2, 0)) - u((\sigma_1, 0))] + [u((\sigma_1, 0)) - u((0,0))] \\
&> 0 + 1/\bar{k} + \cdots + 1/\bar{k} + 0 = (K-1)/\bar{k} > \bar{\Delta},
\end{aligned}
$$

i.e., a contradiction. A utility representation of lexicographic preferences is therefore not possible. The intuition is that because of the nonseparability of the choice set with respect to $\preceq$, any finite difference in the first attribute must yield an unbounded utility difference, which results from adding up the uncountably many finite utility differences (generated by variations in s_2 for each fixed s_1) in between. $\square$