# ECOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

## College of Management of Technology

## MGT 621 – MICROECONOMICS

Thomas A. Weber *

Autumn 2023

Course Notes

This Version: September 11, 2023

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# LECTURE NOTES

# MGT 621 – MICROECONOMICS

Thomas A. Weber

*1. Theory of Choice*

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# AGENDA

**Administrivia & Course Overview**

**Preferences and Utility Representation**

**Some Properties**

**Utility Representation (Cont'd)**

**Demand Theory: Basics**

**A Little Refresher on Constrained Optimization**

**Key Concepts to Remember**

# INFRASTRUCTURE

**My Coordinates**

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# INFRASTRUCTURE (Cont'd)

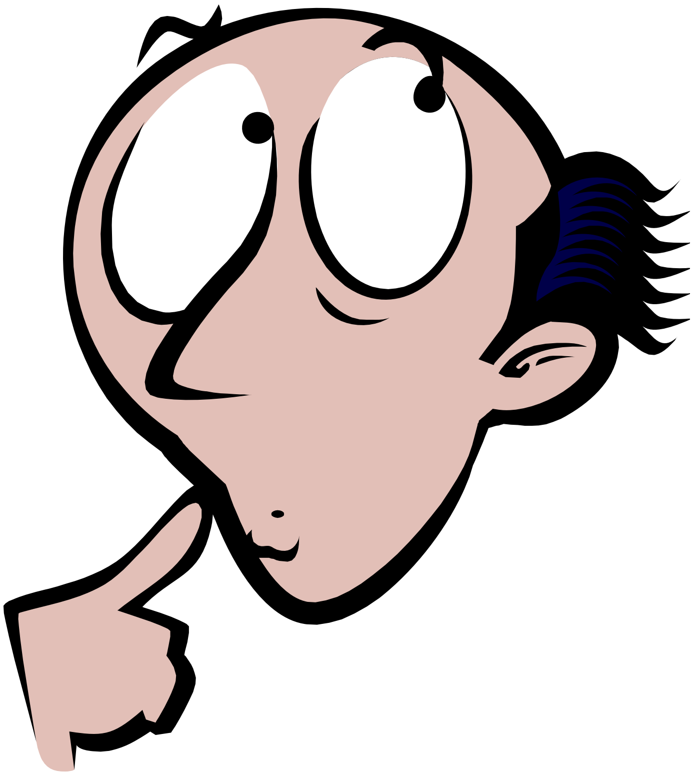
**Course Material & Information**

- Course website: http://econspace.net/MGT-621.html
  Access to content requires login
  Student ID: 621student
- Required Text:
  - [PR] Pindyck, R.S., Rubinfeld, D.L. (2012). *Microeconomics (8th Edition)*, Pearson/Prentice Hall, Upper Saddle River, NJ
  - All notes & additional readings will be posted
- Solid knowledge in calculus required
- Access to spreadsheet & math software (e.g., MS Excel, Matlab, Maple) may be useful for some homework and the course project
- Links to general information on course website

**Honor Code(!)**

# ADMINISTRIVIA

**Did we forget anything?**

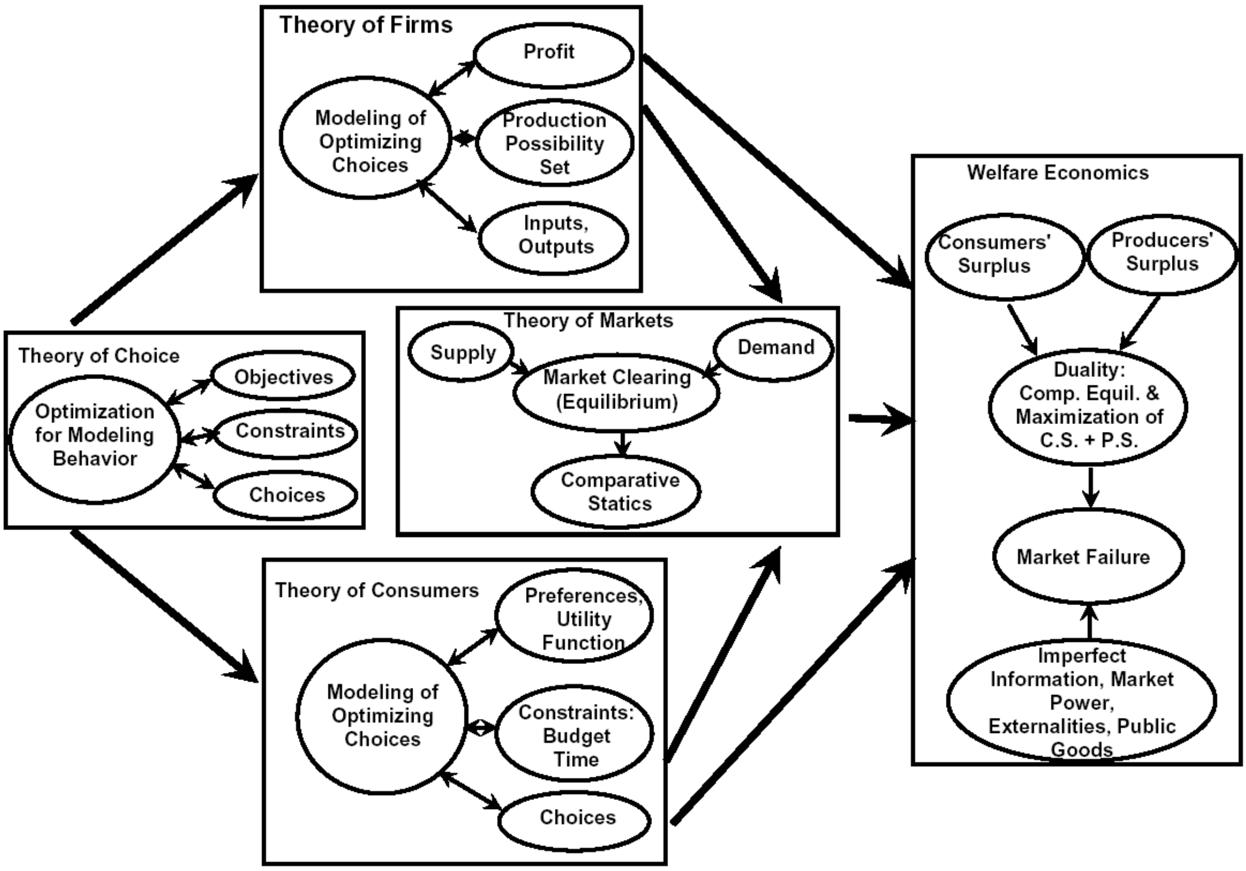




# ASSESSMENT

- **PROBLEM SETS (20%)**
  - **Reproductive & productive questions**
  - **Cooperation ok!!**
  - **Assignments need to be written up & turned in individually**
- **FINAL EXAM (40%)**
  - **Held on <u>Monday</u>, <u>October 2, 2023</u>; Room TBA; there is <u>no makeup</u>**
  - ***Any arrangements by <u>September 25</u>***
  - **3 hours (open book)**
  - **Covers everything discussed in the course**
- **COURSE PROJECT (40%)**
  - **<u>Report due on October 30</u> (before 5 pm; by email to the instructor)**

# MGT 621 MICROECONOMICS – OVERVIEW






# TOPICS IN THIS COURSE
## Tentative List

I. <u>Theory of Choice</u>
- Individual Decision Making
- Preferences and Utility Representation
- Consumer Choice (+ under uncertainty)
- Aggregate Demand

II. <u>Theory of the Firm</u>
- Production Sets
- Profit Maximization and Cost Minimization
- Aggregation

III. Market Equilibrium

IV. Market Failure

## TOPICS IN THIS COURSE (cont'd)
### Tentative List

I. Theory of Choice

II. Theory of the Firm

III. Market Equilibrium
- Competitive Markets
- Profit Maximization and Cost Minimization
- Aggregation

IV. Market Failure
- Monopoly
- Externalities
- Public Goods
- Regulation & Taxation



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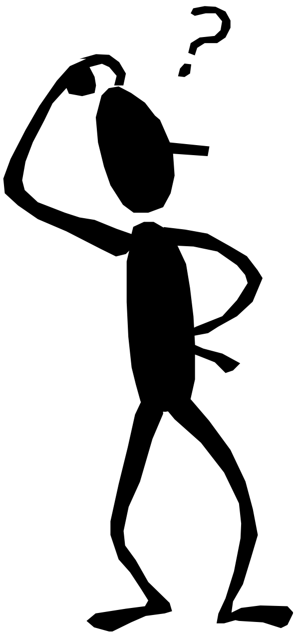
**Key Concepts to Remember**

# AN “ECONOMIC CHOICE PROBLEM”: BUYING A CAR


**What are your goals? … alternatives? … selection criteria?**

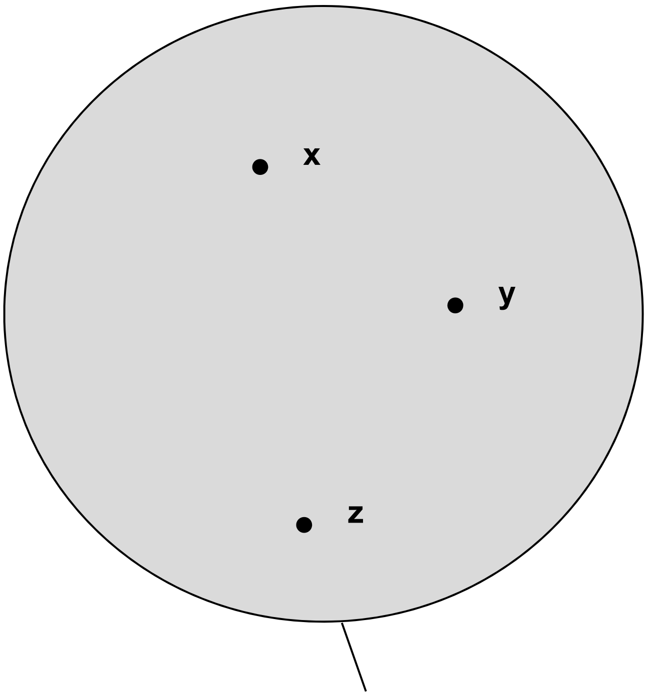


# PREFERENCES



***Homo Economicus* : “Joe”**

**(You? Me? Everybody else?)**




**Choice Set X = {x,y,z}**

**Contains all potentially feasible (mutually exclusive) alternatives.**

# PREFERENCES (Cont'd)

**To decide which alternative to choose Joe needs to be able to rank them, i.e., he needs to have a preference ordering of all elements of his choice set X.**

**Definition. A preference relation on X is a binary relation "$\preceq$" that for any two elements x,y in X compares them so that**

**(i)** $x \preceq y$ **: y is (weakly) preferred to x,**

**or**

**(ii)** $y \preceq x$ **: x is (weakly) preferred to y.**

**If both (i) and (ii) hold, then we say that there is indifference between x and y, denoted by $x \sim y$ or, equivalently, $y \sim x$. If (i) but not (ii), then $x \prec y$, and we say that y is strictly preferred to x.**

**Only if Joe has a preference relation on X, is he able to establish a preference ordering of all elements of his choice set X**

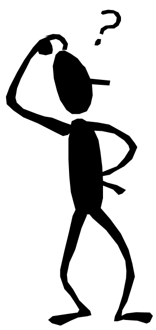


# PREFERENCES (Cont'd)

**Potential problem:**

**If Joe has a preference relation on X = {x,y,z}, he might have the following preference ordering between pairs of elements:**

1. $x \prec y$
2. $y \prec z$
3. $z \prec x$




**Example:**
- **x: apple**
- **y: banana**
- **z: orange**

**Problem?**

**Lack of transitivity!**

**(i.e., $x \prec y,\, y \prec z \;\Rightarrow\; x \prec z$ does not hold)**

# PREFERENCES (Cont'd)

**Lack of transitivity can be generated through aggregation of preferences of individuals with transitive preference relations on X: Arrow's Impossibility Theorem**

**Example: Consider *three alternatives* $x, y, z \in X$ and three agents ( $N = \{1,2,3\}$ ). Let the agents' preferences be as follows:**

- **Agent 1:** $x \succ_1 y, y \succ_1 z$
- **Agent 2:** $z \succ_2 x, x \succ_2 y$
- **Agent 3:** $y \succ_3 z, z \succ_3 x$

**If we use pairwise majority voting to aggregate the agents' preferences, then we obtain that socially $y \succ z, x \succ y, z \succ x$ ; in other words, social preferences would be intransitive**

**Arrow's Impossibility Theorem generalizes this result, and shows that dictatorship (or outside imposition) is required for a consistent aggregation of (at least 3 agents') preferences over (at least 3) independent alternatives.**



# MORE GENERALLY: QUASI-ORDER ON SETS

**Consider a set $X$ of alternatives (outcomes), which has at least two elements.**

**Definition: A quasi-order $R$ on $X$ is a binary relation that is complete, reflexive, and transitive, i.e.,**

- **For all $x, y \in X$ :** $xRy$ **or** $yRx$ **(*Completeness*)**
- **For all $x \in X$ :** $xRx$ **(*Reflexivity*)**
- **For all $x, y, z \in X$ :** $xRy, yRz \Rightarrow xRz$ **(*Transitivity*)**

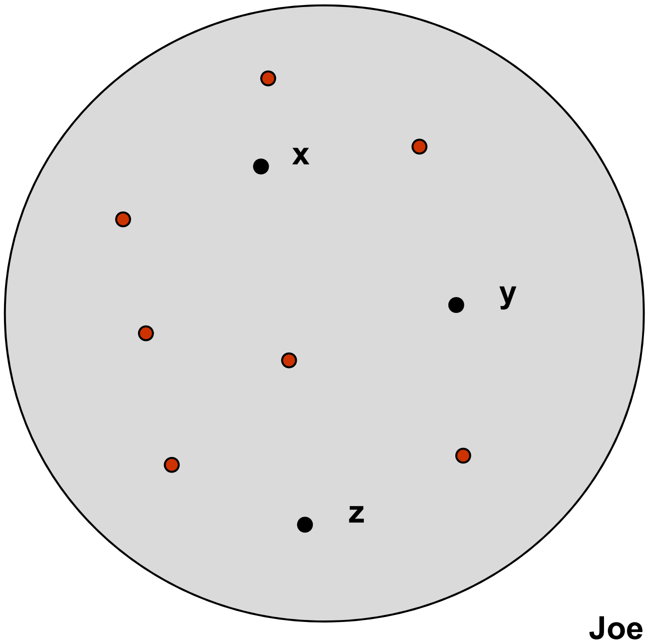
**If $xRy$, we say that $x$ is "weakly preferred" to $y$.**

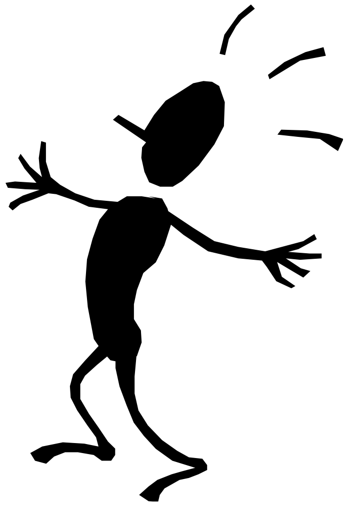
**Definition: A strict partial order $P$ is a binary relation that is irreflexive and transitive. For any $x, y \in X$ and quasi-order $R$ on $X$, we define $xPy$ as $(\text{not } yRx)$ . If $xPy$, we say that $x$ is "(strictly) preferred" to $y$ .[(1)]**

(1) In the same spirit we define $xIy$ as $(xRy \text{ and } yRx)$ and say, if $xIy$, that $x$ is "indifferent" to $y$.
Remark: For more on ordering alternatives and the representation of preferences, see Fishburn, P.C. (1970) *Utility Theory for Decision Making*," Wiley, New York, NY. Another useful reference is Kreps, D.M. (1988) *Notes on the Theory of Choice*, Westview Press, Boulder, CO.

# RATIONAL PREFERENCE ORDER

**Definition. A preference relation on X is rational if it is a quasi-order on X, i.e., if it is complete, reflexive, and transitive.**



**Joe can now make 'rational' choices …**

**… could they depend on the whole set X?**



# RATIONAL PREFERENCES – WHERE DO THEY COME FROM?

**Mainstream economic theory does not try to explain preferences, but typically takes preferences as data, that is, as fixed for the economic agent.**

**Preferences in fact result from many forces, e.g.,**

- **National culture**
- **Advertising**
- **Social institutions and norms**
- **Parental influence**
- **Education**
- **Religion**
- **Personal tastes**

**Preferences can be rational – that is, complete & transitive – and still be the result of the various forces. And they can be rational and change over time (e.g., under the influence of advertising)**

# A CLASS EXPERIMENT

1. You have been given $200 and have a choice between the following two options

   A: Win $150 with certainty

   B: Win $300 with probability .5

   Win $0 with probability .5

- Do you prefer A or B?

2. You have been given $500 and have a choice between the following two options

   C: Lose $150 with certainty

   D: Lose $300 with probability .5

   Lose $0 with probability .5

- Do you prefer C or D?



# RESULT: FRAMING GENERALLY DOES MATTER

| | | Gamble C | Loss Averse<br>Gamble D |
|---|---|---|---|
| Risk Averse | Gamble A | 35 | 28 |
| | Gamble B | 7 | 8 |

Rational choices

**Since, A = C and B = D, a rational agent's choice should be such that if A is preferred to B then C is preferred to D and vice versa.**

**However, the "modal choices" are (i.e., "most people prefer") A and D to avoid losses.**

Remark: The *reflection effect* refers to the observation that preferences tend to reverse when the lottery is "reflected" from the domain of gains (with respect to the *status quo*) to the domain of losses. For details, see Kahneman, D., Tversky, A. (1979) "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica*, Vol. 47, No. 2, pp. 263—291.

# ANOTHER EXAMPLE: ELLSBERG PARADOX

**An urn is known to contain 90 balls of which 30 are red and the other 60 black or yellow in unknown proportions. (Neither you nor the person with the urn knows the actual proportions.) One ball is to be drawn at random from the urn and your "reward" depends on the color of the ball drawn. You must choose between the following two bets, which have consequences as indicated.**

| | Red | Black | Yellow |
|---|---|---|---|
| a. Bet on red | $100 | $0 | $0 |
| b. Bet on black | $ 0 | $100 | $0 |

**Now under the same general conditions which bet would you choose in this second situation?**

| | Red | Black | Yellow |
|---|---|---|---|
| c. Bet on red and yellow | $100 | $0 | $100 |
| d. Bet on black and yellow | $0 | $100 | $100 |



# ELLSBERG PARADOX: CLASS RESULTS

| | Gamble c | Gamble d |
|---|---|---|
| Gamble a | 7 | 58 |
| Gamble b | 1 | 12 |

'consistent' choices

**If you prefer a to b then you should prefer c to d because yellow ball is irrelevant for each pair of decisions.**

**The "modal choices" are (i.e., "most people prefer") a and d to avoid ambiguity (→ ambiguity aversion) … we will deal with choice under uncertainty later.**

## UTILITY REPRESENTATION OF PREFERENCES

**Idea: Joe's rational preference relation on a nonempty choice set X could be represented by dots on the real line if there is a "utility function" u that maps every element x of X to a real number u(x), such that preferred elements get always mapped to larger real numbers.**

**Then instead of making a pairwise comparison between elements of X, Joe could 'simply' maximize his utility function u on X.**

**Definition. A function $u : X \to \mathbb{R}$ is a utility function that represents the preference relation $\preceq$ on X if for any x,y in X:**

$$x \preceq y \quad \Leftrightarrow \quad u(x) \leq u(y)$$



## UTILITY REPRESENTATION (Cont'd)

**For a utility representation of a preference relation to exist, the preference relation must necessarily be rational!**

**Proposition. If the function $u : X \to \mathbb{R}$ represents the preference relation $\preceq$ on X, then $\preceq$ is rational.**

**Proof (in 2 Steps)**

1. **Consider any x,y in X. Then, either $u(x) \leq u(y)$ or $u(y) \leq u(x)$ Since u represents $\preceq$, it is therefore either $x \preceq y$ or $y \preceq x$, so that $\preceq$ is complete. It is also reflexive (trivial).**
2. **Consider any x,y,z in X, such that $x \preceq y$ and $y \preceq z$. Thus, $u(x) \leq u(y) \leq u(z)$, which implies that $x \preceq z$. Hence, the preference relation $\preceq$ is also transitive.**

**QED**

# UTILITY REPRESENTATION MAY NOT EXIST!

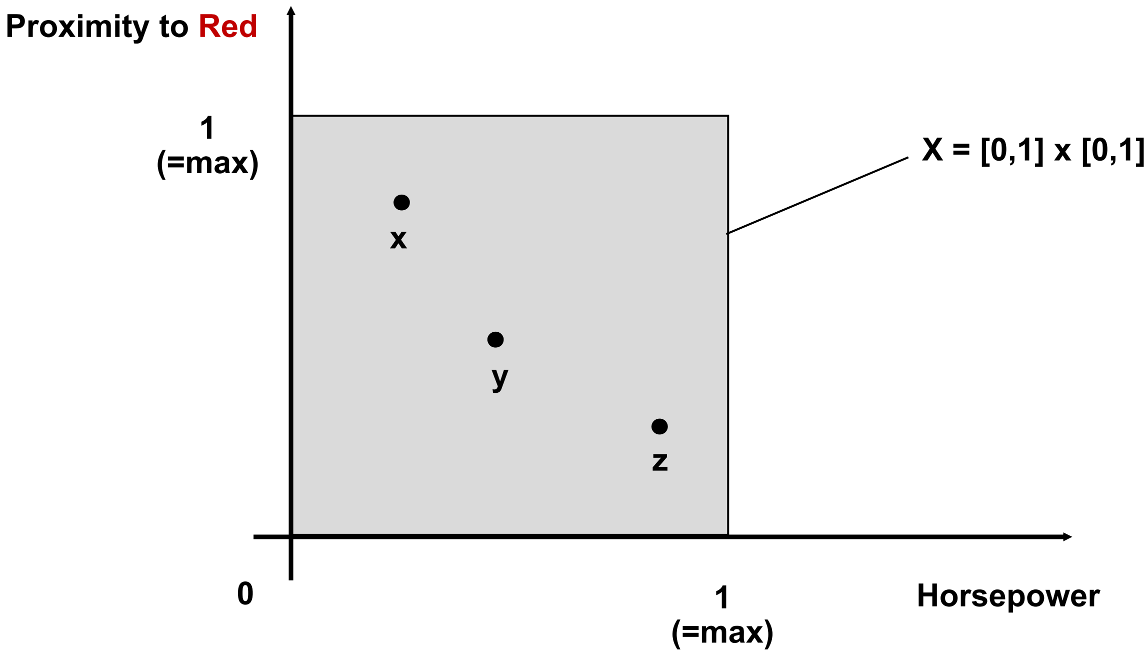
**Example: Preferences for a used car.**

**Joe would like to buy a Ford Mustang. He cares about two attributes: horsepower and color. Of two given models, he would always prefer the more powerful one. If they have the same power, then he would take the one that has a color closest to red.**

$\longrightarrow$ **Lexicographic Preferences**



# PREFERENCES FOR USED CAR (Cont'd)




**Question. What is Joe's preference ordering?**

**Right,** $x \prec y \prec z$ **!**

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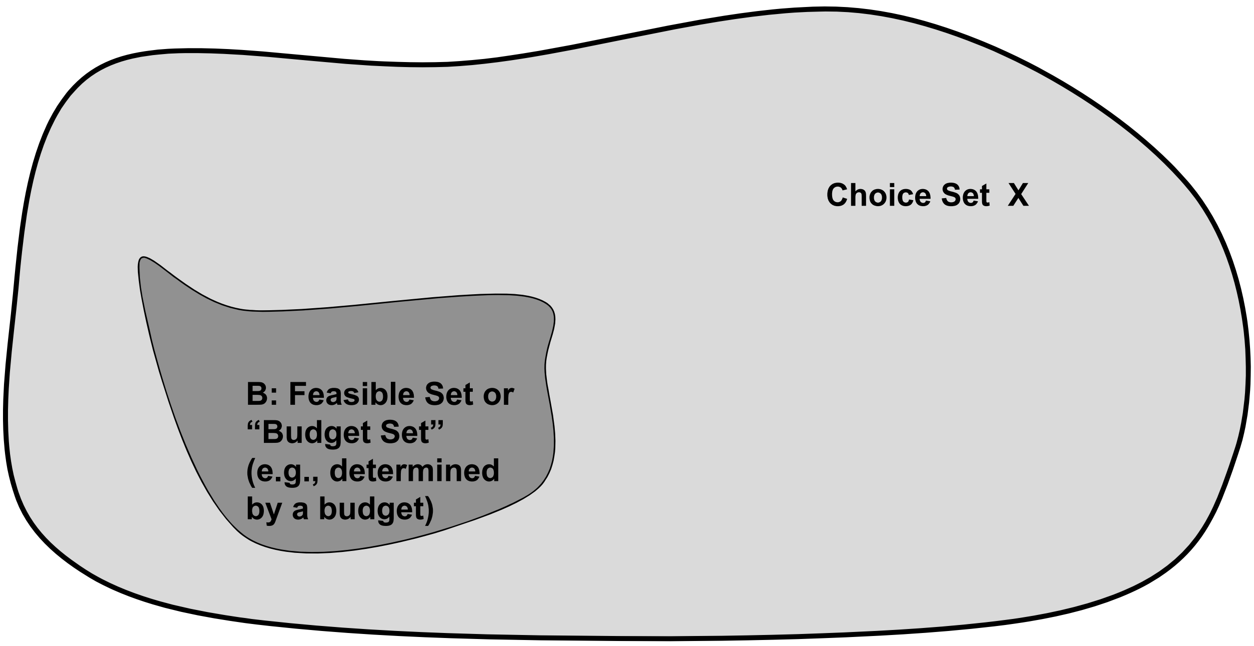
**Key Concepts to Remember**

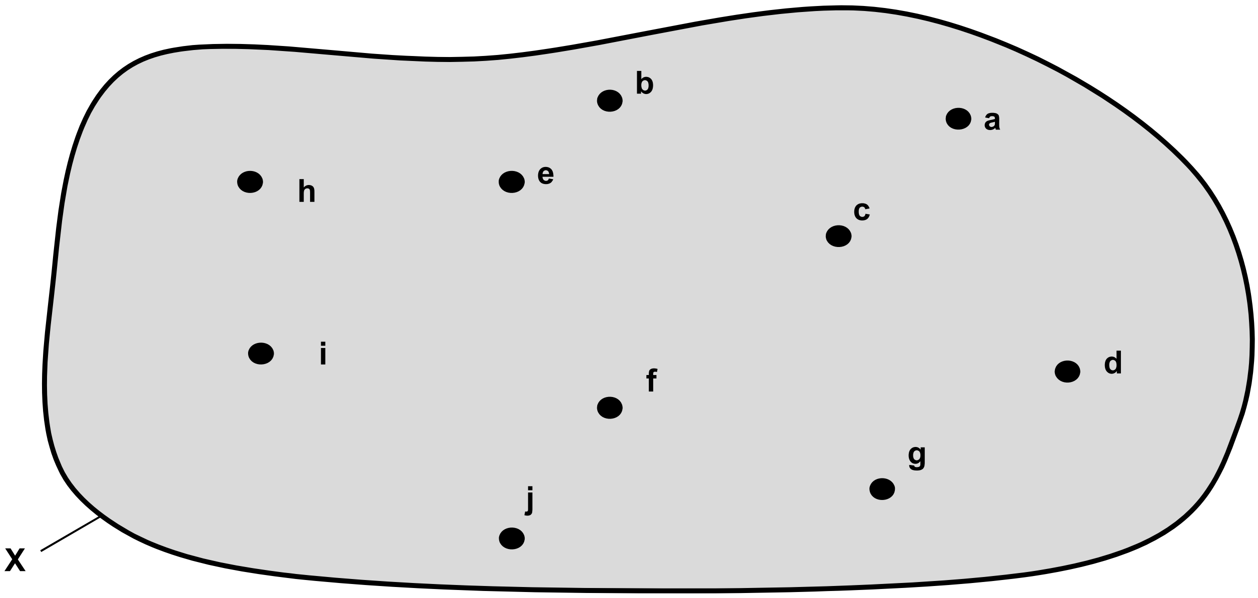


# ONLY A SUBSET OF THE CHOICE SET MAY BE FEASIBLE




$$B \subset X$$

# COMPLETE PREFERENCE RELATION …

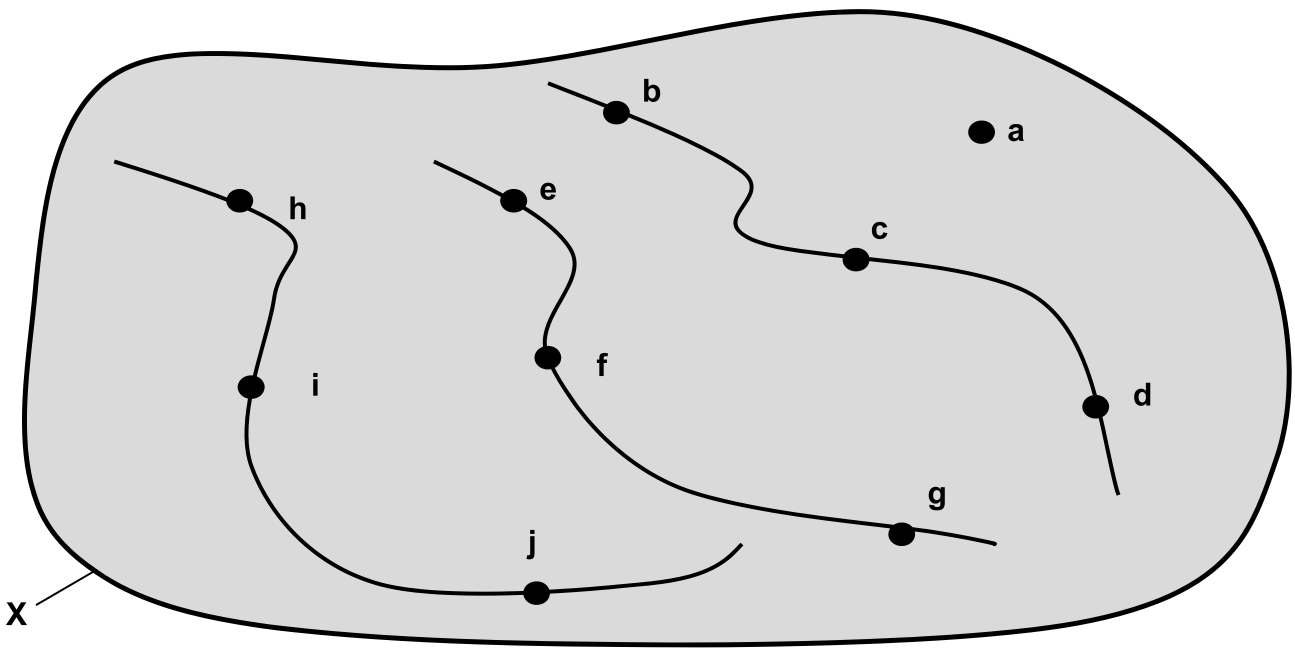



$a \succeq c \succeq f \succeq j$

$b \sim c \sim d$ $e \sim f \sim g$ $h \sim i \sim j$



# … ALLOWS TO DEFINE "EQUIVALENCE CLASSES" OR "INDIFFERENCE CURVES" WHEN THERE IS A UTILITY FUNCTION

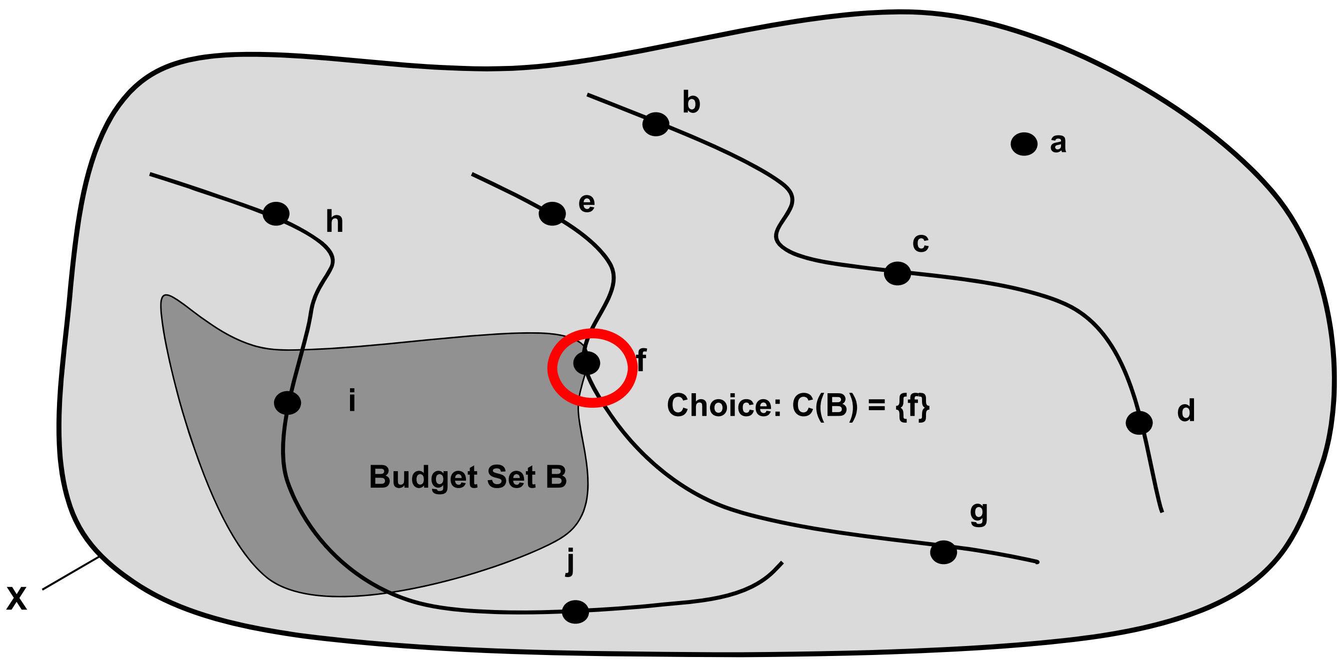



$a \succeq c \succeq f \succeq j$

$b \sim c \sim d$ $e \sim f \sim g$ $h \sim i \sim j$

Remark: It is not really necessary to be able to connect all the points of a choice set that an agent is indifferent about by "continuous" indifference curves. These "curves" might not look like curves at all, and do not really have to. What is important, is that different indifference curves do not intersect!

# RATIONAL CHOICE: MOST PREFERRED ALTERNATIVE (HERE AT POINT f)




| a ≿ c ≿ f ≿ j |
|---|

| b ~ c ~ d | e ~ f ~ g | h ~ i ~ j |
|---|---|---|



Remark: A choice (set) is generally be set valued. For example, C(X\{a}) = {b,c,d}.



# SOME PROPERTIES OF CHOICE

**Independence of irrelevant alternatives**

- If we reduce the budget set, eliminating points that are not chosen, then the optimal point – the choice point – will not change

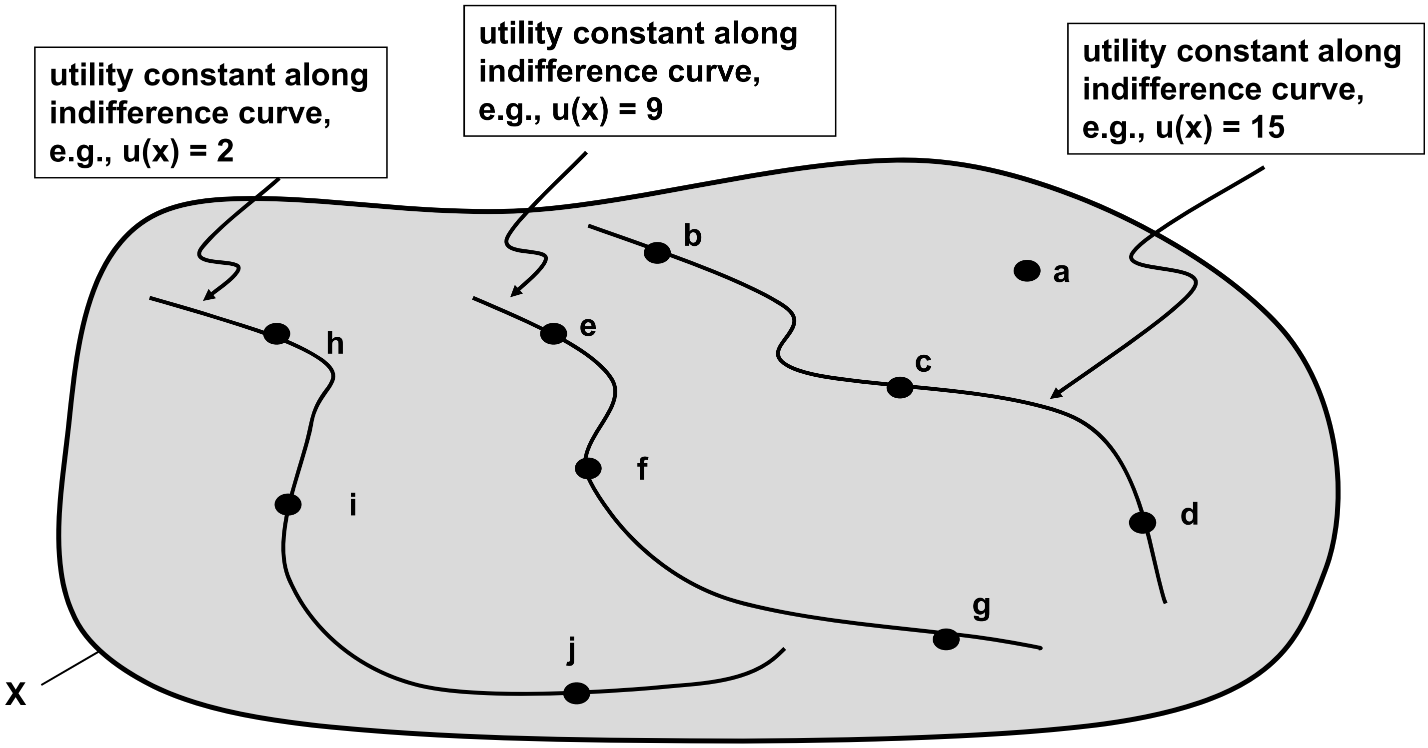
**Intensity of preferences is irrelevant to choice**

- Saying that ‘C is MUCH preferred to F’ or that ‘C is slightly preferred to F’ has no relevance to what point will be chosen

**Choice is invariant with respect to changes that leave feasible set unchanged**

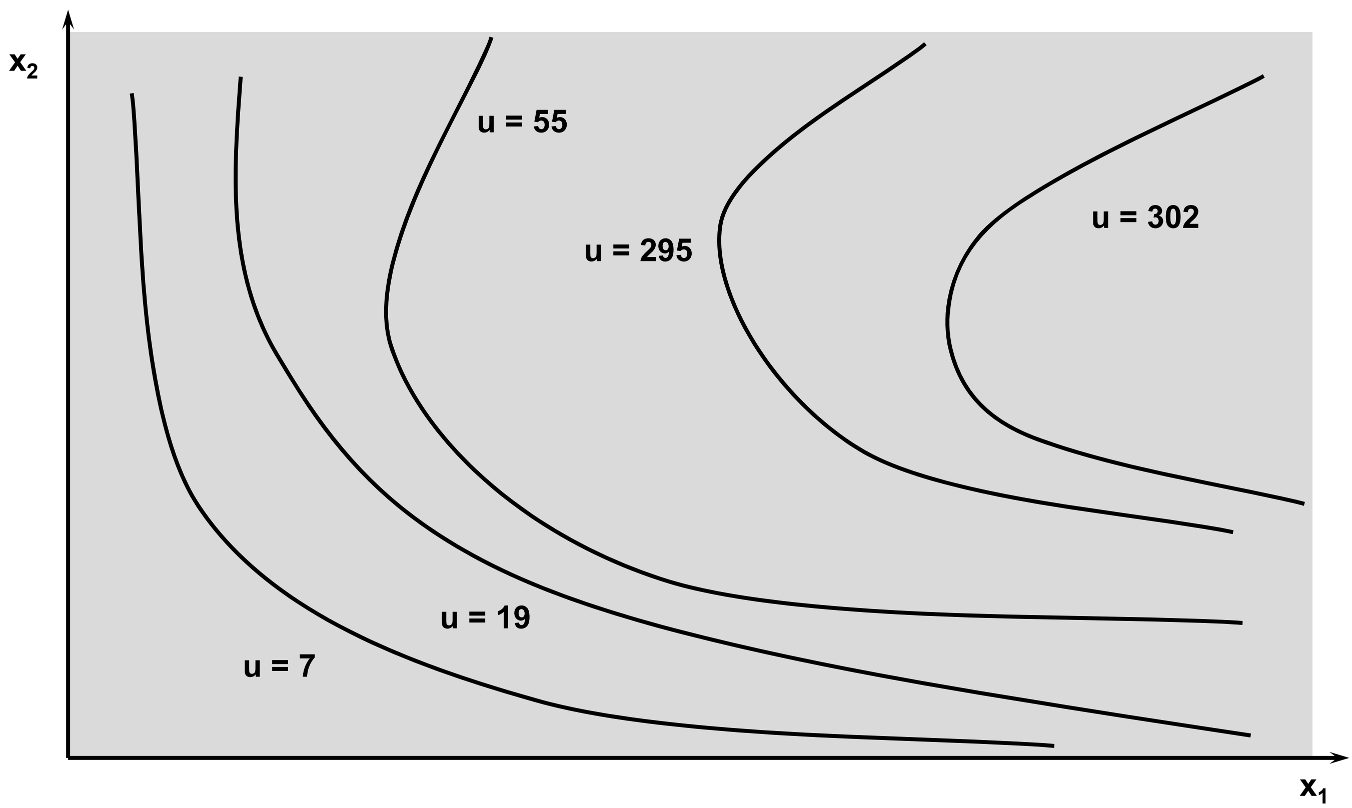
- Expansion or contraction of choice set (X) has no impact on choice if expansion or contraction does not impact feasible set
- Rescaling of problem parameters that leave the feasible set unchanged will not impact choice

# REPRESENTATION OF PREFERENCE RELATION BY UTILITY FUNCTION

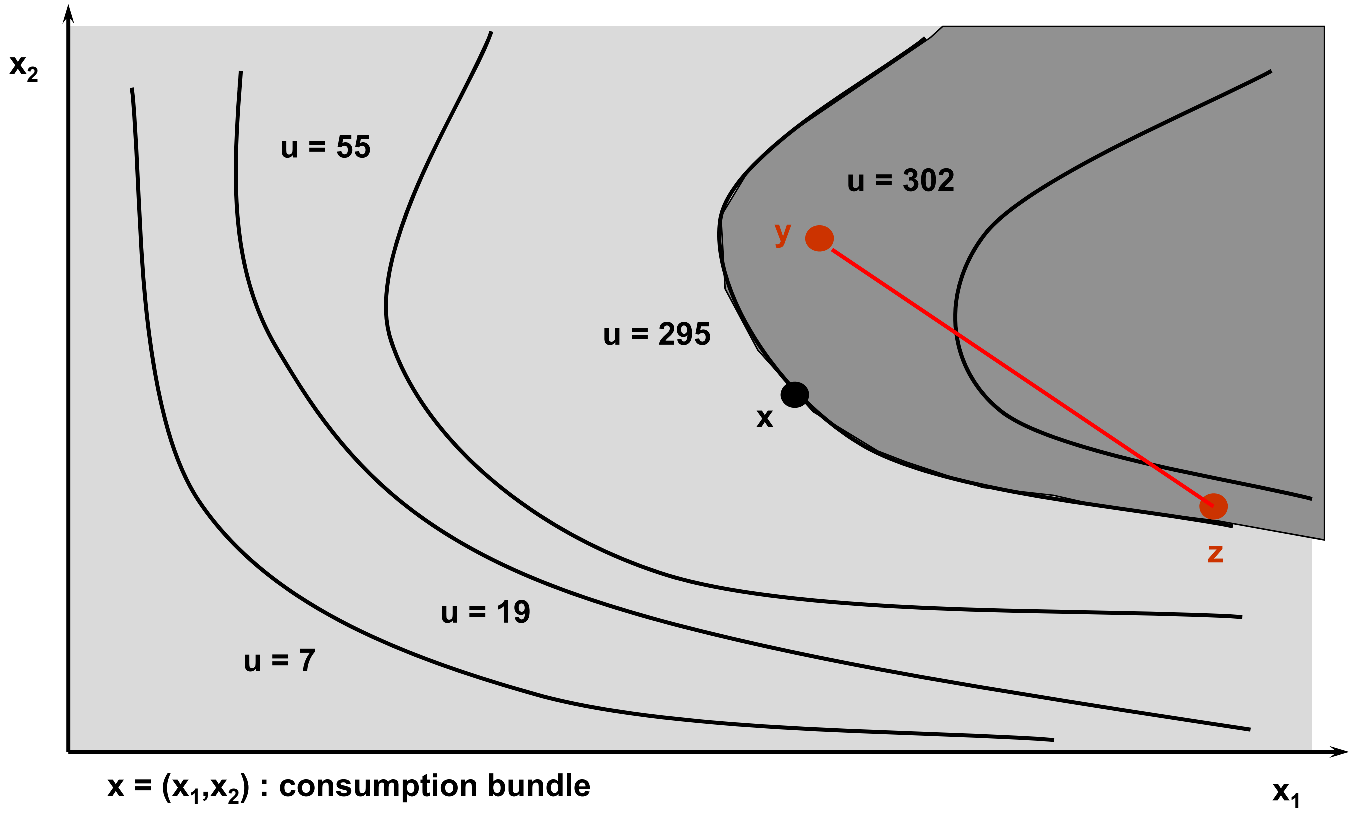





# EXAMPLE: CONSUMPTION SET WITH INDIFFERENCE CURVES



# CONVEXITY OF PREFERENCES




$x = (x_1,x_2)$ : consumption bundle



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# THEORY OF THE CONSUMER: PREFERENCES

All consumer preferences assumed to be rational

- Complete
- Reflexive
- Transitive

Preferences also assumed to be continuous

- Preference order does not jump around discontinuously

  $\{x: x \succeq y\}$ and $\{x: y \succeq x\}$ are both closed sets

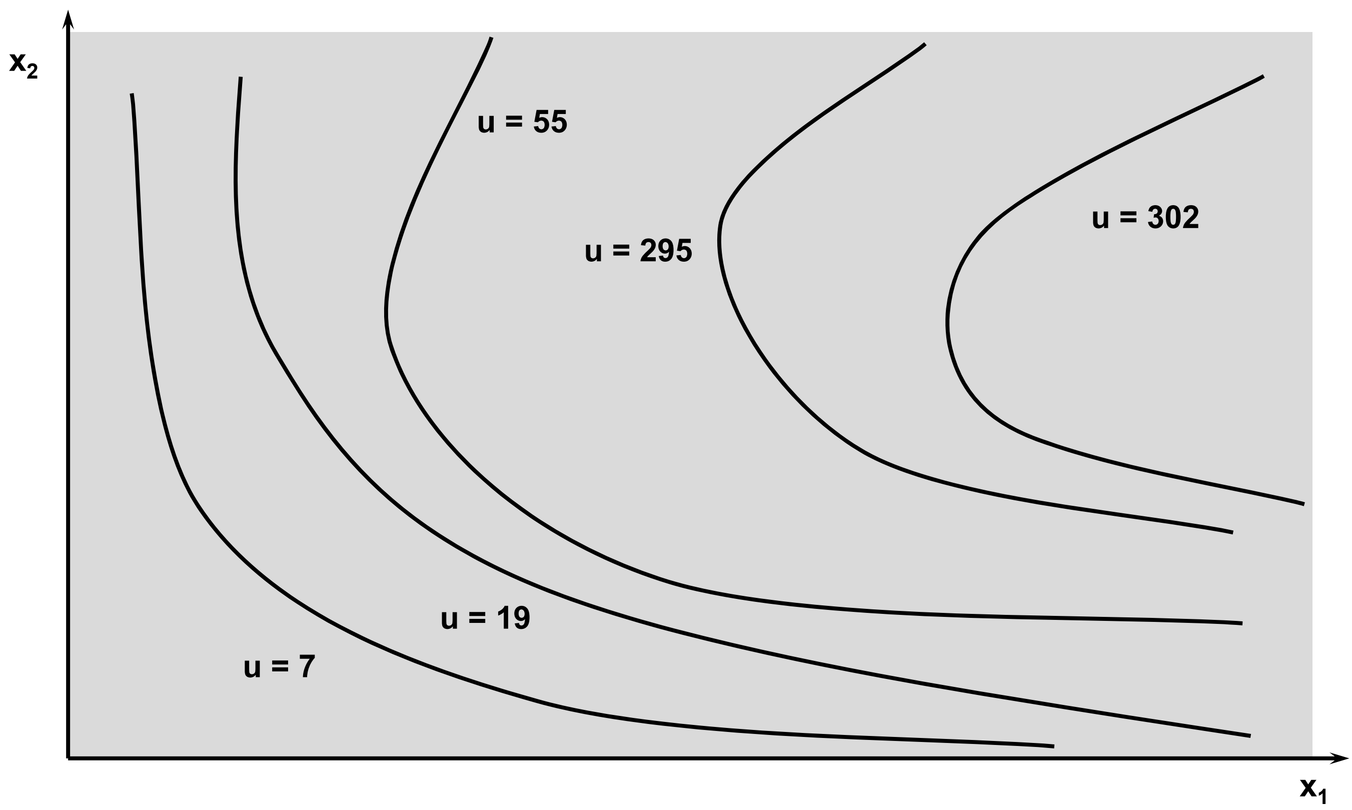
- Exclude situation: consumer prefers x(n) to y for sequence of x(n) converging to limit x(∞), but strictly prefers y to x(∞) (1)

**Theorem. If preferences are rational and continuous, then there exists a continuous utility function u(x) that describes preferences.**

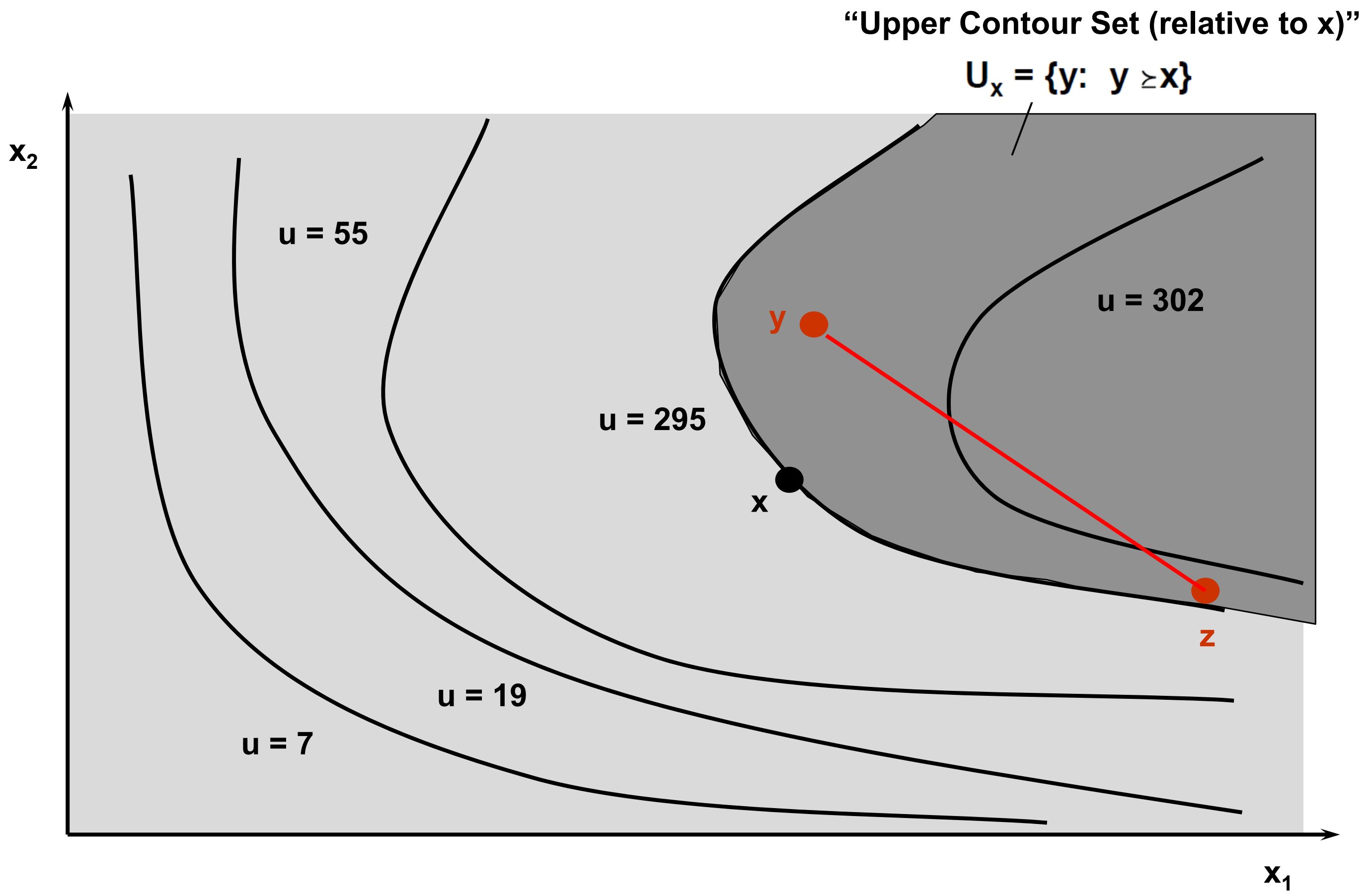
(1) Example: Take lexicographic preferences on the square X = [0,1] x [0,1] (as in the first lecture) and let x(n) = (1/n,1/n), y=(0,1).



# TYPICAL CONSUMPTION SET WITH INDIFFERENCE CURVES



# CONVEXITY OF PREFERENCES






# CONVEXITY

**Definition.** A rational preference relation $\preceq$ on X is **convex** if the upper contour set $U_x = \{y : x \preceq y\}$ is convex for any x in X, i.e.,

$$y, z \in U_x \quad \Rightarrow \quad \theta y + (1-\theta) z \in U_x \quad \forall \theta \in (0,1)$$

**Proposition.** A utility representation of a convex preference relation is quasi-concave (i.e., single-peaked).

# ORDINAL VS. CARDINAL PROPERTIES

**A utility representation u(x) for a given rational preference relation $\succeq$ on X is generally *not unique*.**

**The preference relation $\succeq$ fixes only ordering of elements of the choice X, and is therefore called ordinal.**

**Given the utility representation u(x) of $\succeq$ on X, the function v(x) = $\phi$(u(x)) is also a utility representation of $\succeq$ on X, as long as the (real-valued) transformation $\phi$ is increasing.**

**Each specific utility representation of $\succeq$ on X is called cardinal.**

**Thus, while the ordinal properties of utility functions are invariant with respect to increasing transformations, their cardinal properties are not!**



# UTILITY FUNCTIONS RELEVANT FOR CONSUMERS AND FIRMS

### Theory of Consumers

- **Need assumptions about preferences to ensure utility function exists.**
- **Normally only ordinal properties (which express ordering of options) of utility functions are important.**
- **For theories of consumer choice under uncertainty, cardinal properties are important. (Cardinal properties express how much better one option is than another.)**

### Theory of Firms

- **Preferences assumed for firms – profit – can always be written as a function, a profit function.**
- **Profit function plays the same role as utility function.**
- **For theory of firm behavior under uncertainty, we can use utility function with cardinal properties.**

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# THEORY OF THE CONSUMER
## Choice Set = Consumption Set

**Consumer modeled as choosing among bundles of commodities ("market baskets")**
**Example: $x_1$ shirts, $x_2$ lbs of beef, $x_3$ gallons of gasoline**

**$x = (x_1, x_2, x_3)$ is vector of these quantities (L-dimensional vector if there are L commodities)**

**Choice set X (= "Consumption Set") contains all feasible (not necessarily affordable!) bundles $x \in X$.**

**Standard Assumptions**

- **We typically include x = 0**
  - **Sometimes x=0 may include necessary commodities for survival**
  - **Sometimes choice set may be discrete, e.g., when only integer amounts of consumption are possible**

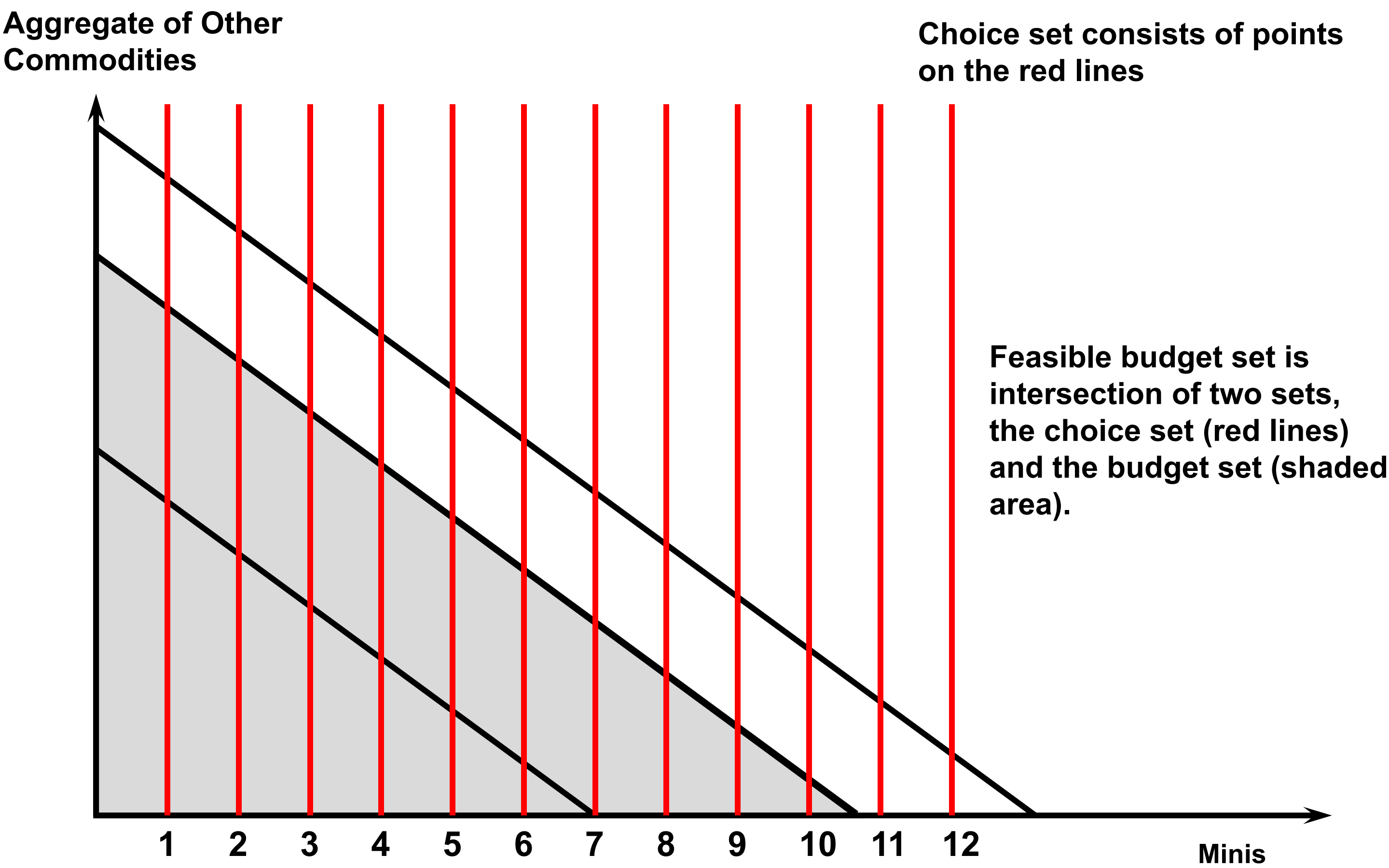
- **We often assume that preferences are (locally) nonsatiated. This means that a bigger bundle (if available to the consumer) is always strictly preferred.**

## PROBLEM IN REALITY?

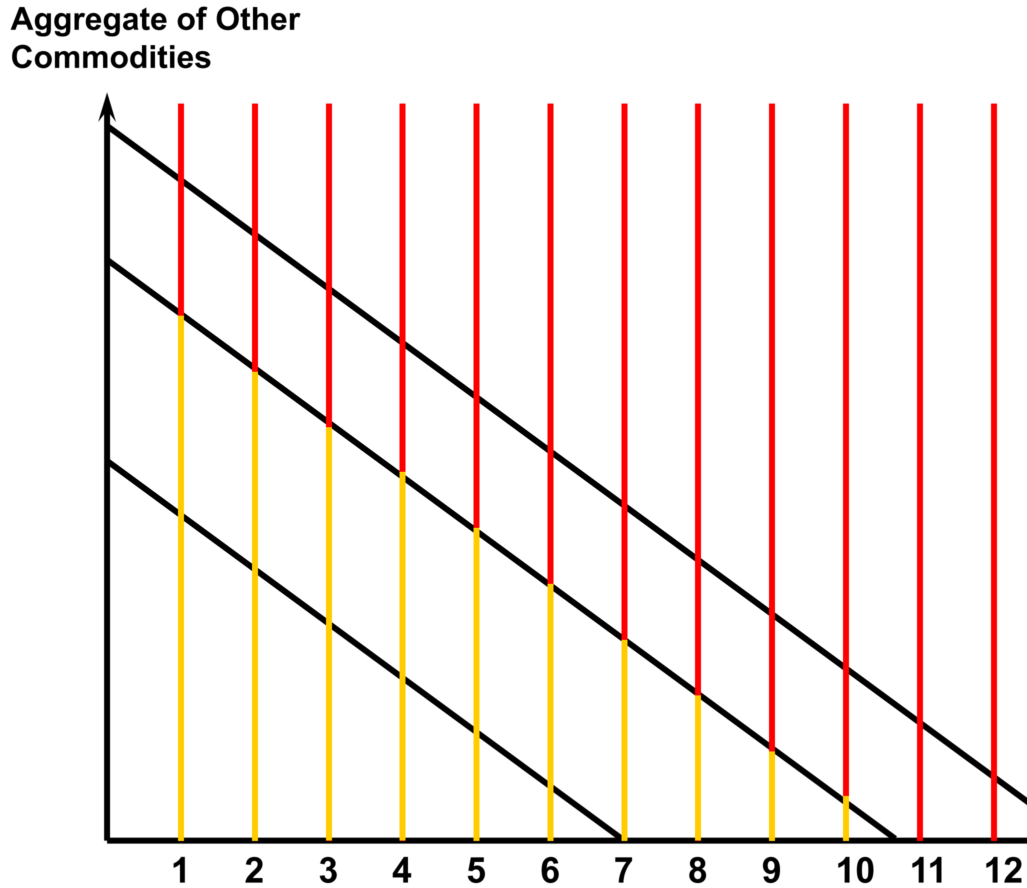

Indivisibilities … !



## BUDGET SET FOR DISCRETE COMMODITIES



# BUDGET SET FOR DISCRETE COMMODITIES (Cont'd)




**Feasible set (budget set) is set of orange lines**

**Feasible budget set is not convex if choice set is not convex. Math becomes more difficult. We often pretend that one can purchase fractions of Minis.**



# PROBLEM IN REALITY?

# THEORY OF THE CONSUMER
## Budget Set

**A consumer's choices are constrained to consumption bundles s/he can afford.**

- **Commodities traded at prices $p_1, p_2, \ldots, p_L$**
- **Prices represented by an L-dimensional price vector $p = (p_1, p_2, \ldots, p_L)$**
- **Assume that consumer cannot influence prices**

**Consumption bundle is affordable if total cost does not exceed the consumer's nonnegative wealth (income), represented by w.**

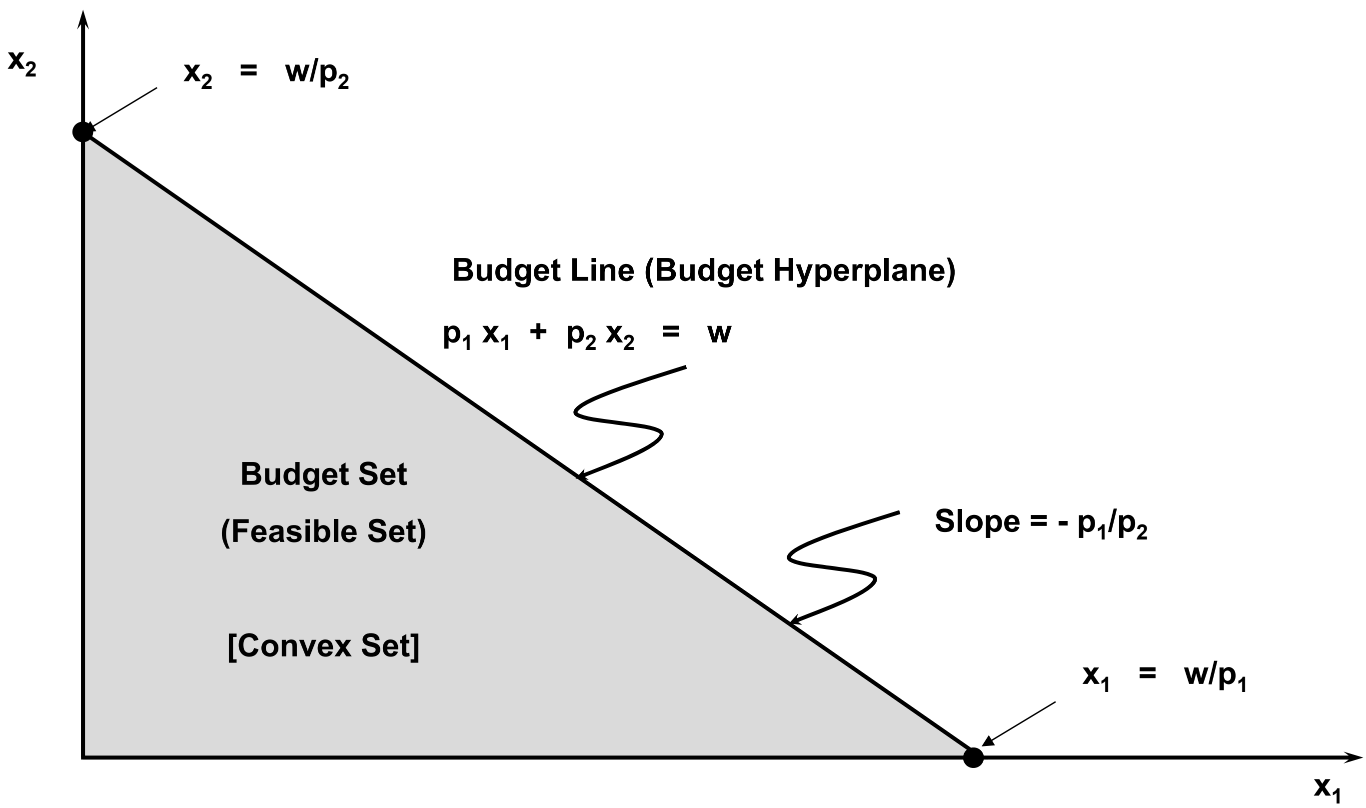
$$p \cdot x \leq w$$

$$p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots + p_L x_L \leq w$$

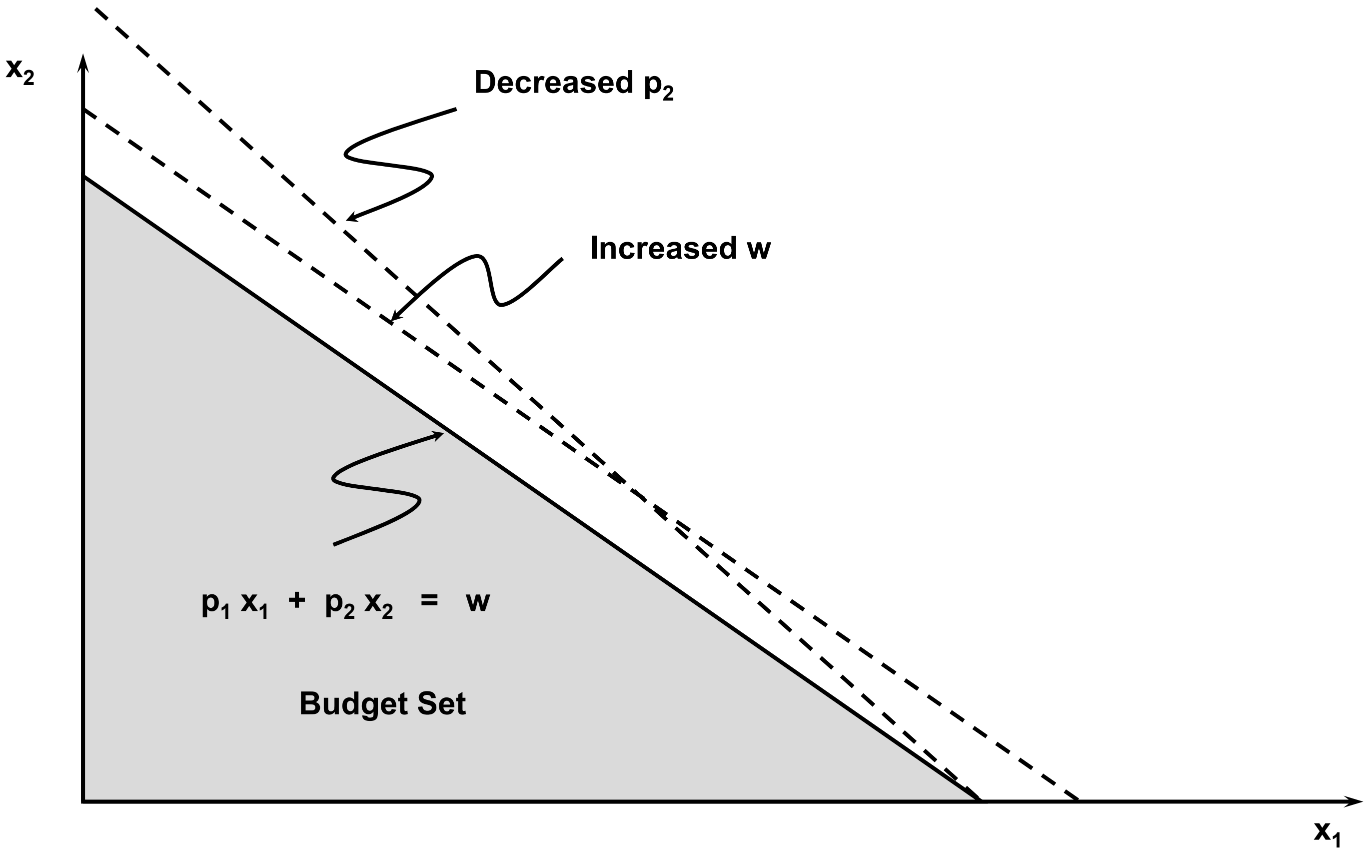
**Set of bundles x in X that satisfy this constraint are known as the budget set B(p,w).**



# BUDGET SET



# BUDGET SET DEPENDS ON PRICES AND WEALTH






# RATIONAL CHOICE = UTILITY MAXIMIZATION PROBLEM

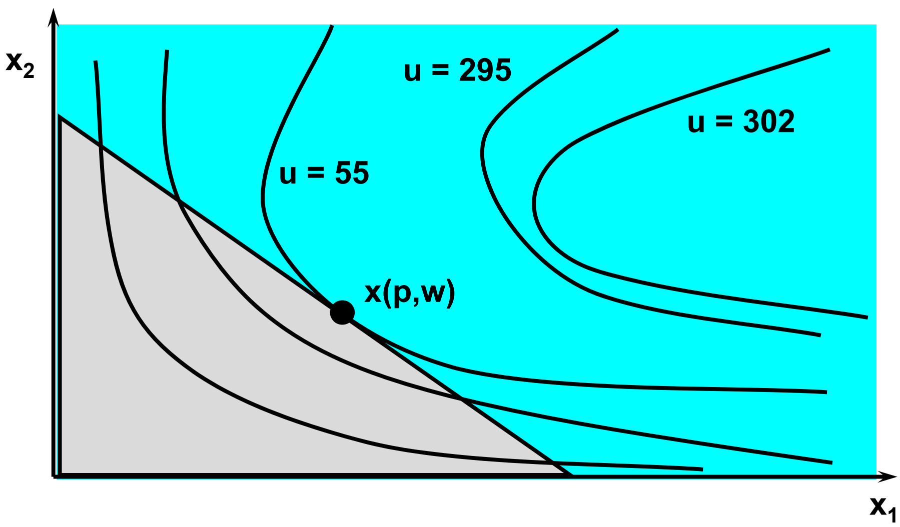
**Consumer chooses by maximizing utility over all alternatives in budget set, i.e., s/he solves**

| | |
|---|---|
| **Maximize** | **u(x)** |
| **such that** | $p \cdot x \leq w$ |

**x(p,w) denotes the optimal choice, and is referred to as the demand function**

**Example**

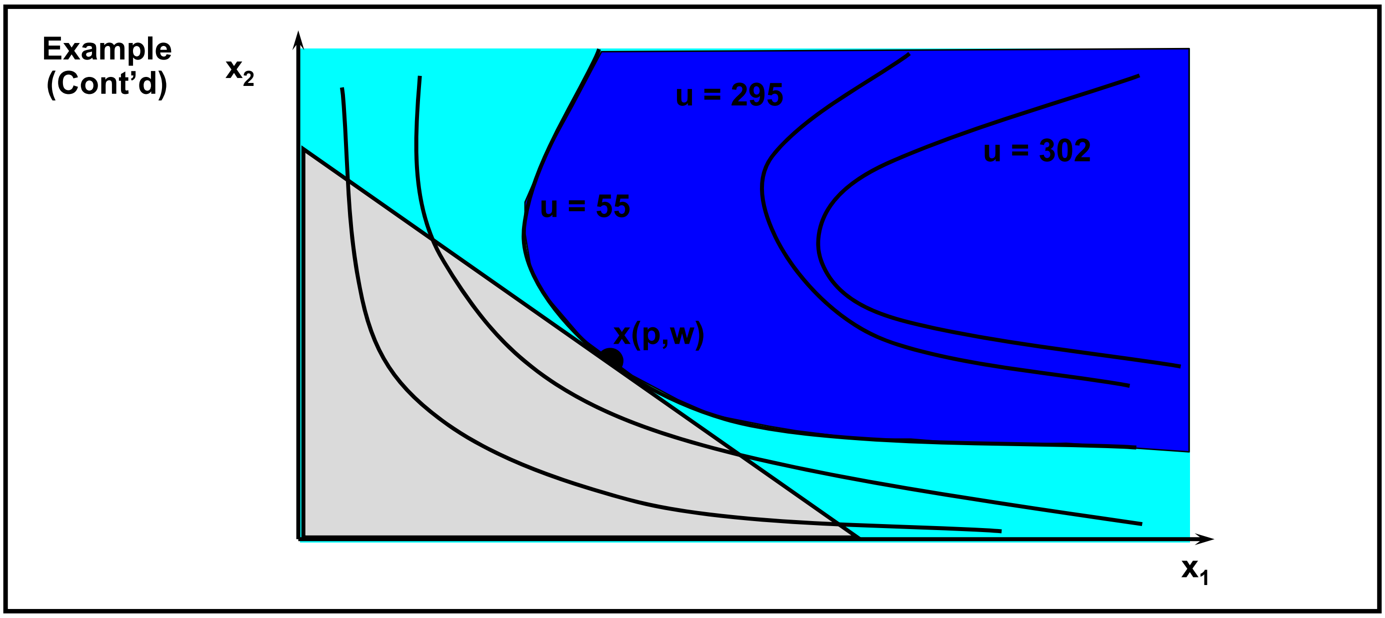
**u(x(p,w)) = 55**



# UTILITY MAXIMIZATION PROBLEM (Cont'd)

**x(p,w) denotes the optimal choice, or ("Walrasian") demand function.**

- **No feasible point (x) has u(x) > 55**
- **No feasible point strictly preferred to x(p,w).**
- **Set of preferred points (= upper contour set) and feasible set have no common interior points.**



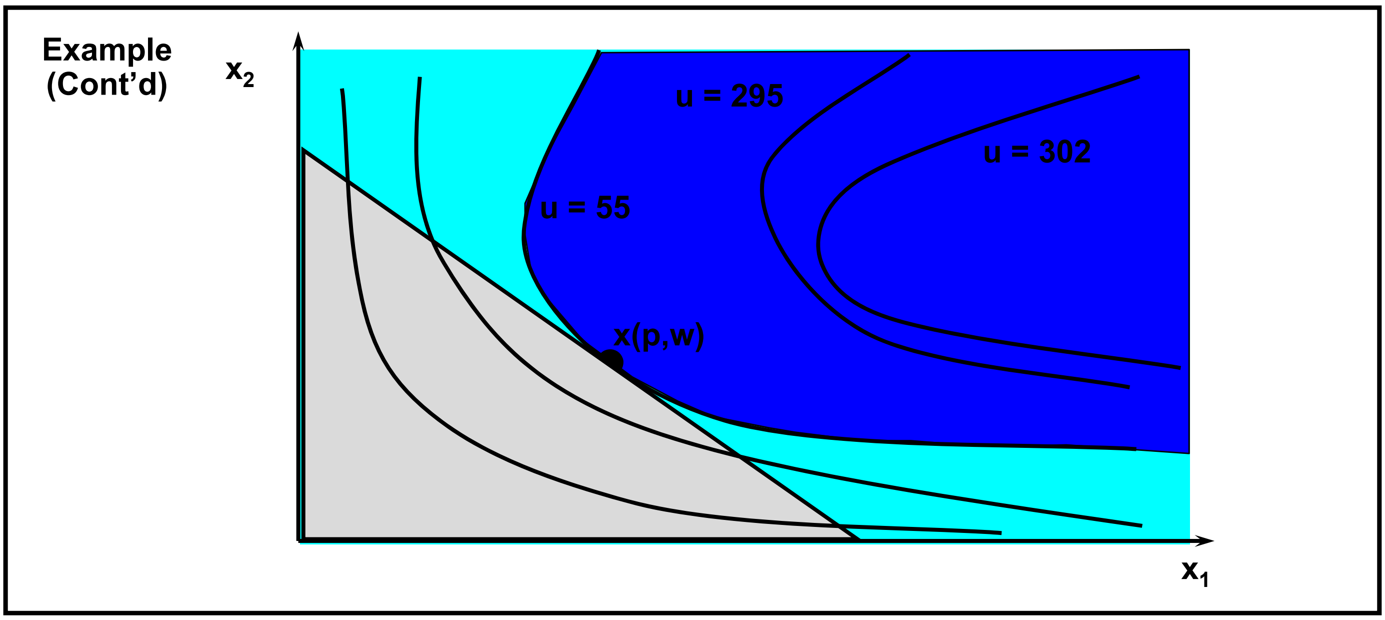



# EQUIVALENT EXPENDITURE MINIMIZATION PROBLEM (Cont'd)

**Equivalently, x(p,w) solves a second problem:**

$$\begin{aligned} &\text{Minimize} \quad p \cdot x \\ &\text{such that} \;\; u(x) \geq 55 \end{aligned}$$

**That is, minimize expenditure, under constraint that u(x) ≥ 55**



# PROPERTIES OF THE CONSUMER DEMAND FUNCTION

**Homogeneity of degree zero** in p,w:

$x(\alpha p, \alpha w) = x(p,w)$ for any $\alpha > 0$

**Walras' Law**:

$p \cdot x(p,w) = w$

(holds if preference are locally nonsatiated)

**Convexity**:

If preferences are convex, then x(p,w) is a convex set

**Uniqueness**:

If preferences are <u>strictly</u> convex, then x(p,w) is a single point



# AGENDA

Administrivia & Course Overview

Preferences and Utility Representation

Some Properties

Utility Representation (Cont'd)

Demand Theory: Basics

**A Little Refresher on Constrained Optimization**

Key Concepts to Remember

# CONSTRAINED OPTIMIZATION

**Let $f : X \to \Re$ be a real-valued function, where X is a nonempty, compact subset of $\Re^n$,[1] where n is a positive integer.**

**We would like to maximize f(x) on X, i.e., solve the problem**

**Objective Function (Maximand)**

$$\max_{x \in X} f(x)$$

**Constraint Set**

**(Constrained Optimization Problem)**

**Remark:**

**The utility maximization problem is of this form, with f(x) = u(x) and $X = \{x \in R_+^L : p \cdot x \le w\}$ for some price vector $p = (p_1, ..., p_L) >> 0$, wealth w>0, and number of commodities L>0.**

(1) A subset of a finite-dimensional Euclidean space is *compact* if it is closed and bounded. A set is *closed* if the limit of any (sub-)sequence constructed from elements of the set also lies in the set. A solution to a constrained optimization problem on a compact set exists, provided the objective function f is continuous.



# OPTIMALITY CONDITIONS WHEN CONSTRAINT NOT BINDING
## One-Dimensional Case

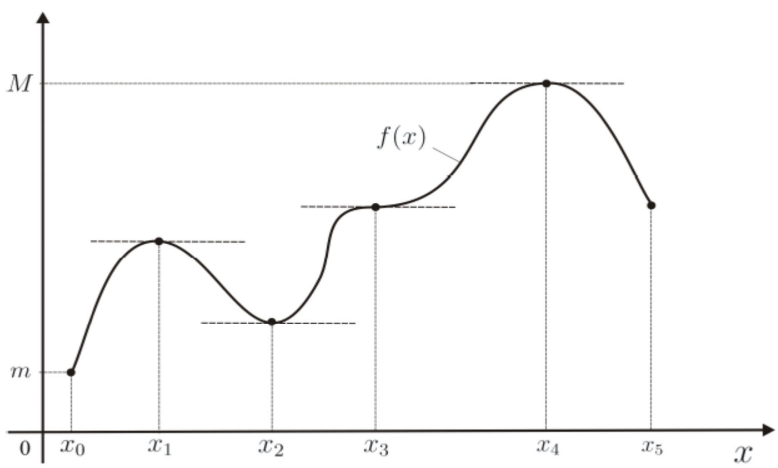
**Fermat's Rule (= First-Order Necessary Optimality Condition ["FOC"])**

$$x \text{ local extremum } \Rightarrow f'(x) = 0$$

**Example:**

$X = [x_0, x_5] \subset \Re$

**f differentiable**




**Note that the FOC is satisfied for local maxima and local minima in the interior of X.**

# OPTIMALITY CONDITIONS (Cont'd)
## One-Dimensional Case

**In order to guarantee that one has arrived at an interior maximizer (= optimal x that maximizes the objective function and does not lie on the boundary of X), one can use additional optimality conditions: second-order optimality conditions**

$$x \text{ local maximizer} \Rightarrow f''(x) \le 0$$
$$x \text{ local minimizer} \Rightarrow f''(x) \ge 0$$

**Second-Order Necessary Optimality Condition**

$$f''(x) < 0 = f'(x) \Rightarrow x \text{ local maximizer}$$
$$f''(x) > 0 = f'(x) \Rightarrow x \text{ local minimizer}$$

**Second-Order Sufficient Optimality Condition ["SOC"]**

**Examples (Gap Between Necessary and Sufficient Second-Order Optimality Conditions):**

**(a) The function** $f(x) = 1 - x^4$ **has a maximum at x = 0 which does not satisfy SOC.**

**(b) The function** $f(x) = 1 - x^3$ **has no extremum even though it satisfies** $f''(0) \le 0 = f'(0)$ **.**



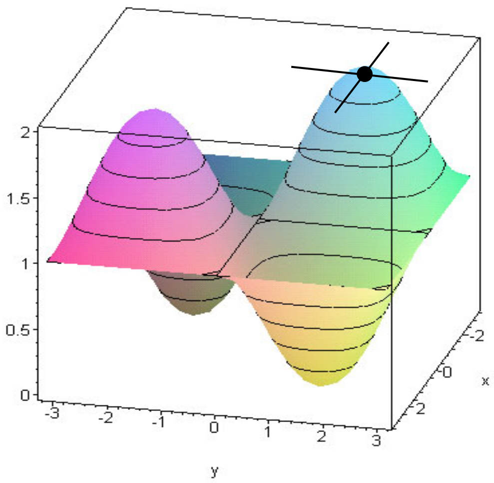
# OPTIMALITY CONDITIONS WHEN CONSTRAINT NOT BINDING
## Multidimensional Case

**Fermat's Rule generalizes to the case of multiple dimensions (n>1)**

$$x \text{ local extremum } \Rightarrow Df(x) = (\partial f(x)/\partial x_1, \ldots, \partial f(x)/\partial x_n) = 0$$

**Example:**

$$X = [-\pi, \pi]^2 \subset \Re^2$$
$$f(x) = 1 - \sin(x_1)\cos(x_2)$$




**Again, the FOC is satisfied for local maxima and local minima in the interior of X.**

# CONSTRAINED OPTIMIZATION

**Let $f : X \to \Re$ be a real-valued objective function, $X = \{x \in \Re^n : g(x) \le 0\}$ be a nonempty compact constraint set, where $g : X \to \Re^k$ is a vector-valued function.**

**The standard constrained optimization problem is then often written in the form**

$$\max_{x \in \Re^n} f(x), \quad \text{s.t.} \quad g(x) \le 0$$

**Idea:**

**Relax this problem by introducing *k* additional variables ("Lagrange Multipliers"), one for each constraint component.**

**Then find critical points (= points that satisfy FOC) of "Lagrangian" L (= relaxed objective function), where**

$$L(x;\lambda) = f(x) - \lambda \cdot g(x)$$



# CONSTRAINED OPTIMIZATION: INTUITION

**Interpret Dg(x) as a vector perpendicular to frontier (determined g(x) = 0), pointing in direction of increasing g(x).**

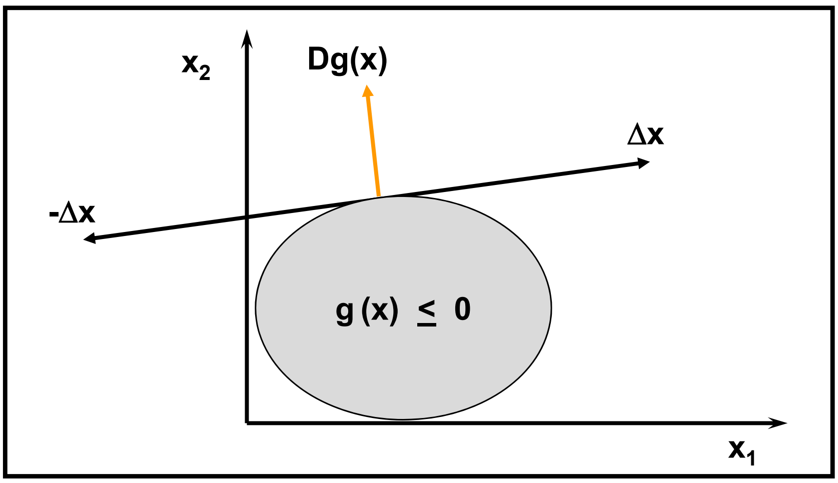
**Choose $\Delta$x that is tangent to frontier. For tiny movements along $\Delta$x or along -$\Delta$x, the function g(x) does not change in value. Thus,**

**$\Delta x_1\, \partial g/\partial x_1 + \cdots + \Delta x_n\, \partial g/\partial x_n = 0$ or $\Delta x \cdot Dg(x) = 0$**

**Any two vectors whose inner product is zero must be perpendicular to each other. Thus, $\Delta$x and Dg(x) are perpendicular to each other.**

**Now take $\Delta$x = Dg(x) (assumed nonzero). Then, $\Delta g = \Delta x \cdot Dg(x) > 0$, because all components of $\Delta$g are $(\partial g/\partial x_i)^2 > 0$.**

**Hence, g(x) is increasing in direction of Dg(x).**



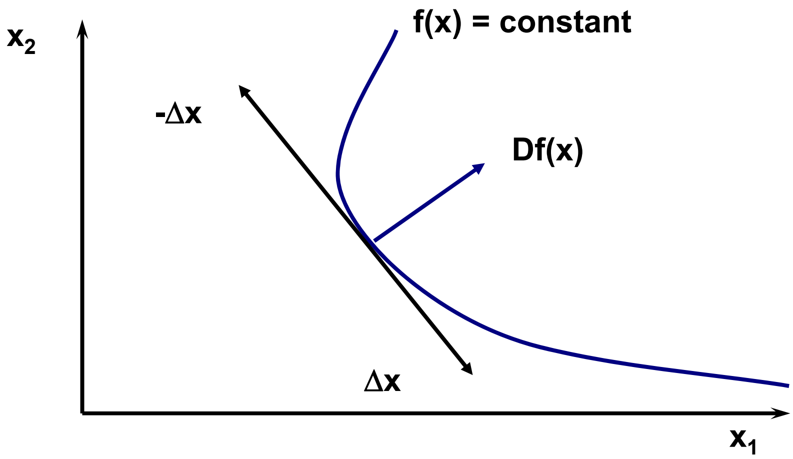
# CONSTRAINED OPTIMIZATION: INTUITION (Cont'd)

**Interpret Df(x) as a vector perpendicular to the level set {y : f(x) = f(y) }**

**This vector is pointed in the direction of increasing value of f(x).**

**Choose $\Delta x$ that is tangent to the level set, f(x) = constant. For tiny movements along $\Delta x$ or along $-\Delta x$, f(x) does not change in value. Thus $\Delta x_1 \, \partial f / \partial x_1 + \cdots + \Delta x_n \, \partial f / \partial x_n = 0$ or $\Delta x \cdot Df(x) = 0$.**

**Thus, $\Delta x$ and Df(x) are perpendicular to each other.**



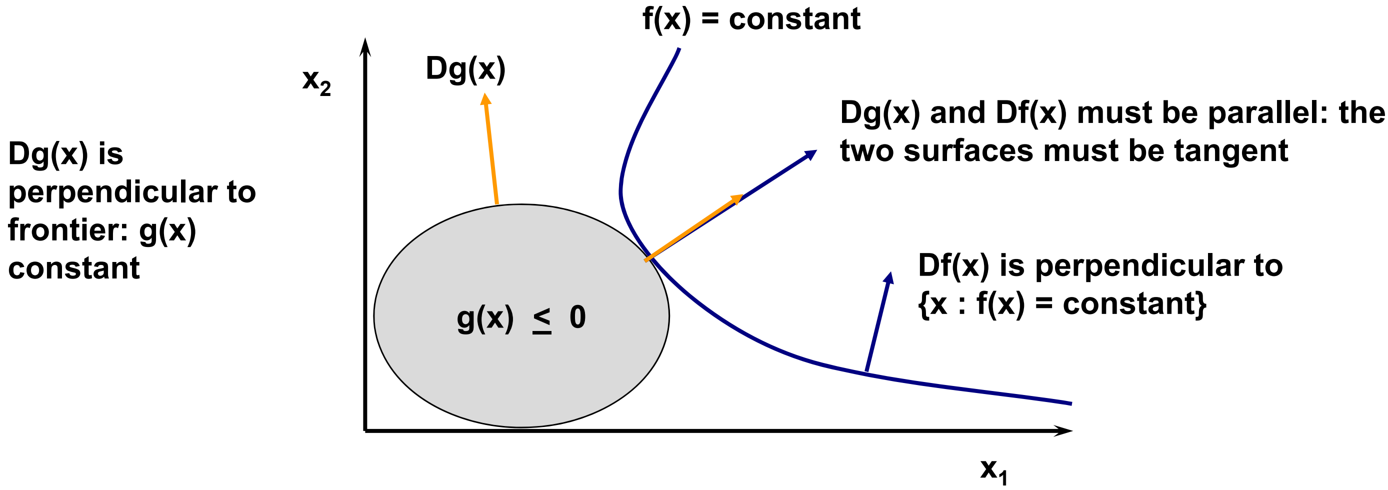



# INTUITION OF FIRST-ORDER CONDITION

**First-Order Necessary Optimality Condition for Constrained Optimization:**

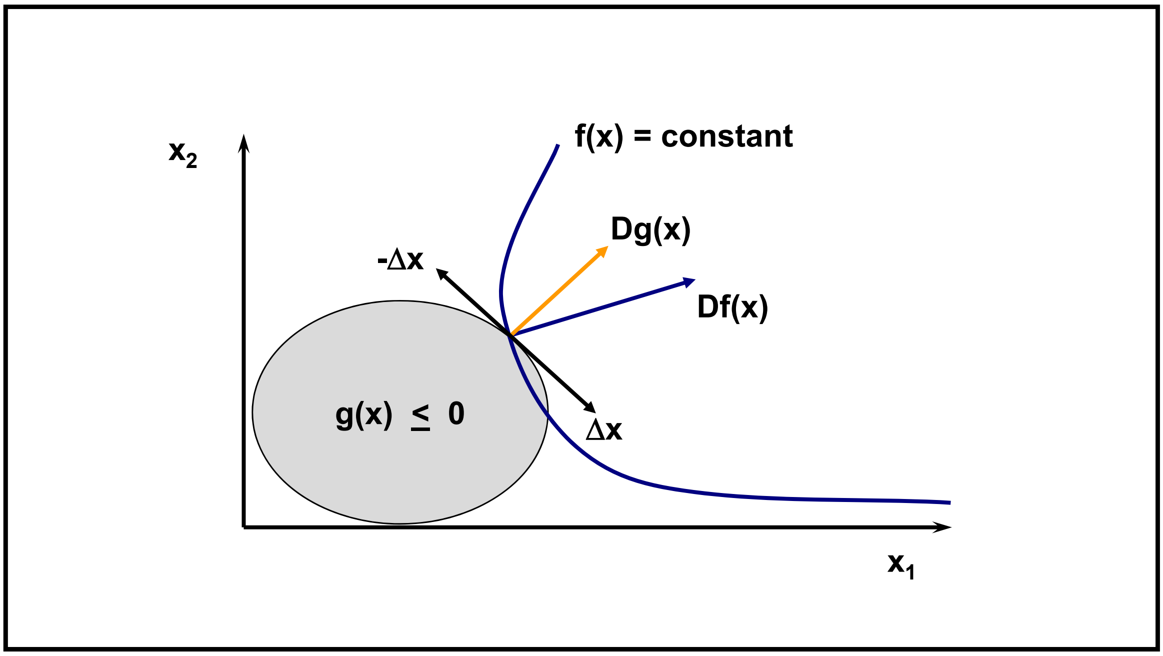
**If the constraint g(x)=0 is binding, then a level set of f must be tangent to the constraint set at an extremal point. This implies that the gradient of f and the gradient of the constraint function g need to be parallel.**

> **$Df(x) = \lambda \, Dg(x)$ for some $\lambda \geq 0$ ($\lambda$ is a scalar)**
>
> **If constraint is binding (i.e., if g(x) = 0), then $\lambda > 0$**



# INTUITION OF FIRST-ORDER CONDITION

**If Dg(x) and Df(x) are not parallel, there are feasible points with greater f(x). They can be found by moving tiny distance in direction ∆x or -∆x.**






# FIRST-ORDER NECESSARY OPTIMALITY CONDITION FOR CONSTRAINED OPTIMIZATION PROBLEM

## Formal Statement

**Let** $L(x;\lambda) = f(x) - \lambda \cdot g(x)$ **be the Lagrangian associated with the constrained optimization problem**

$$\max_{x \in \Re^n} f(x), \quad \text{s.t.} \quad g(x) \le 0$$

**Necessary Optimality Conditions (Kuhn-Tucker Conditions):(1)**

$$x \text{ is local extremum of } \; f(x) \; \text{ in } \; \{x{:}g(x) \le 0\} \Rightarrow \begin{array}{c} D_x L(x;\lambda) = 0, \\ \lambda_i g_i(x) = 0, i \in \{1,\ldots,k\} \end{array}$$

**The k relations** $\lambda_i g_i(x) = 0, i \in \{1,\ldots,k\}$ **are also referred to as complementary slackness conditions. The variables** $\lambda_i$ **are called Lagrange multipliers or dual variables.**

(1) The validity of these conditions depends on whether the constraint function g satisfies a "constraint qualification" in a neighborhood of the extremum, which means that in this neighborhood the constrains are nonredundant, so that Dg(x) is of full rank (i.e., of rank *k*) there.

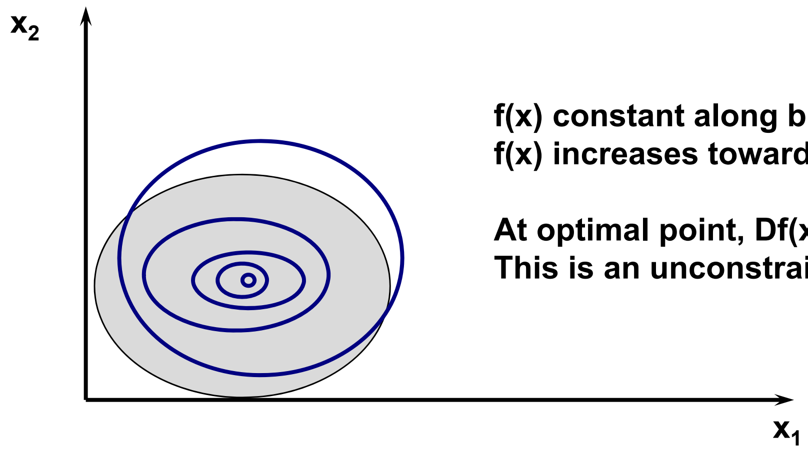
# RELATION BETWEEN FIRST-ORDER CONDITIONS FOR CONSTRAINED AND UNCONSTRAINED OPTIMIZATION PROBLEMS

**Consider the constrained optimization problem**

$$\max_{x \in \Re^n} f(x), \quad \text{s.t.} \quad g(x) \le 0$$

**If, at an extremum x, the constraint is not binding, i.e., if g(x)<0, then complementary slackness implies that all Lagrange multipliers vanish.**

**Example**




**f(x) constant along blue curves.**
**f(x) increases toward peak at the middle.**

**At optimal point, Df(x) = 0**
**This is an unconstrained optimum.**



# CONSTRAINED OPTIMIZATION WITH MULTIPLE CONSTRAINTS
## Intuition

**Consider the following problem:**

**$\max_x f(x)$, s.t. $g_1(x) \le 0$ and $g_2(x) \le 0$**

**First Order Necessary condition:**

**$Df(x) = \lambda_1 Dg_1(x) + \lambda_2 Dg_2(x)$ [ Df(x) lies between $Dg_1(x)$ and $Dg_2(x)$ ]**

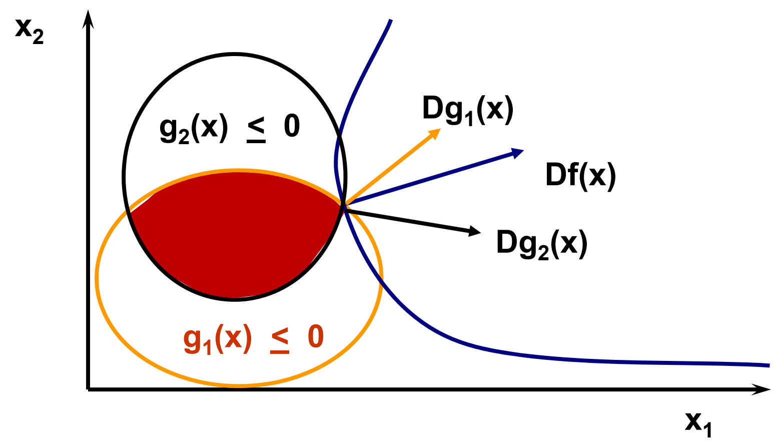
**$g_1(x) \le 0$ $\lambda_1 \ge 0$**

**$g_2(x) \le 0$ $\lambda_2 \ge 0$**

**$\lambda_1 g_1(x) = 0$**

**$\lambda_2 g_2(x) = 0$**

**Red area is feasible set with two constraints**



# INTERPRETATION OF THE DUAL VARIABLES

$\lambda$ corresponds to an increase in the optimized objective function f per unit relaxation of the constraint $g(x) \leq 0$. (Relaxation means $g(x) \leq b$, for very small vector $b >> 0$)

**Show:** $\Delta f = \lambda . b$

$\Delta f = \partial f / \partial x_1 \Delta x_1 + \ldots + \partial f / \partial x_n \Delta x_n$

$\Delta g = \partial g / \partial x_1 \Delta x_1 + \ldots + \partial g / \partial x_n \Delta x_n = b$

Calculate $\Delta u$, remembering first-order necessary optimality condition, $\partial f / \partial x_i = \lambda \, \partial g / \partial x_i$

$\Delta f = \lambda \, \partial g / \partial x_1 \Delta x_1 + \lambda \, \partial g / \partial x_2 \Delta x_{2} + \ldots = \lambda \Delta g$

> The dual variables (Lagrange multipliers) are equal to the value of being able to relax the constraints. They are often called the **shadow prices** of the problem.

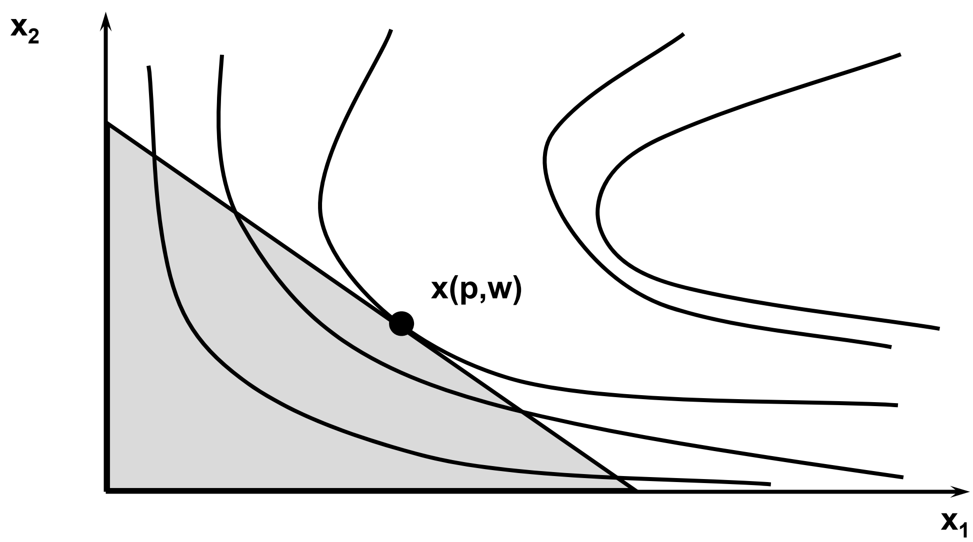


# CONSUMER SOLVES UTILITY MAXIMIZATION PROBLEM

Choice is the maximum utility alternative from feasible set, in this case, the budget set

Maximize $u(x)$
such that: $p . x \leq w$
$x \geq 0$

$x(p,w)$ denotes the optimal choice, or Walrasian demand function, given p and w.



# CHARACTERIZING OPTIMAL CHOICE

**Maximize u(x)**
**such that:** $p \cdot x \leq w$
$x \geq 0$

**Constraints can be written in standard form:**

| | | | |
|---|---|---|---|
| $p \cdot x - w \leq 0$ | becomes | $g_0(x) = p \cdot x - w \leq 0$ | |
| $-x \leq 0$ | becomes | $g_i(x) = -x_i \leq 0$ | for each i in {1,…,L} |

**First-order necessary optimality conditions:**

$$Du(x) = \sum_{i=0}^{L} \lambda_i Dg_i(x) = \lambda_0 p - \begin{bmatrix} \lambda_1 \\ \\ \lambda_L \end{bmatrix}$$

**and**

$$\lambda_0 (p \cdot x - w) = 0$$

$$\lambda_i x_i = 0, i \in \{1, \ldots, L\}$$



# INTERPRETATION OF LAGRANGE MULITPLIERS

$Du(x) = \lambda_0 p - (\lambda_1, \ldots, \lambda_L)$
$(p \cdot x - w) \lambda_0 = 0$
**with local nonsatiation,** $p \cdot x = w$ **and** $\lambda_0 > 0$

$(\lambda_1, \ldots, \lambda_L) \cdot x = 0$
**if** $x_i > 0$, **then** $\lambda_i = 0$

**Thus:**

$\partial u / \partial x_i = \lambda_0 p_i$ if $x_i > 0$
$\partial u / \partial x_i \leq \lambda_0 p_i$ if $x_i = 0$

**The marginal utility of each good that is purchased is equal to its price multiplied by the shadow price on wealth. If a good is not purchased, its marginal utility is smaller than its price multiplied by the shadow price on wealth.**

# INTERPRETATION (Cont'd)

$$\partial u/\partial x_i = \lambda_0 p_i \quad \text{if} \quad x_i > 0$$
$$\partial u/\partial x_i \leq \lambda_0 p_i \quad \text{if} \quad x_i = 0$$

For two goods that are both purchased (that is $x_i > 0$) :

$$\frac{\partial u/\partial x_i}{\partial u/\partial x_j} = \frac{\lambda\, p_i}{\lambda\, p_j} = \frac{p_i}{p_j}$$

Interpret:

$$\frac{\partial u/\partial x_i}{\partial u/\partial x_j} = - \partial x_j / \partial x_i \,|_{u \text{ constant}}$$

**Marginal Rate of Substitution** of good i for good j and is denoted $MRS_{i\,j}$. $MRS_{i\,j}$ is the amount of good j the consumer would need to receive in order to exactly be compensated for a unit loss of good i.





# INTERPRETATION (Cont'd)

For two goods that are both purchased (that is $x_i > 0$) :

$$\frac{\partial u/\partial x_i}{\partial u/\partial x_j} = \frac{\lambda\, p_i}{\lambda\, p_j} = \frac{p_i}{p_j}$$

Thus

$$MRS_{i\,j} = p_i / p_j$$

The price ratio is equal to the MRS. Amount of good j the consumer would need to receive to exactly be compensated for a unit loss of good i is equal to the price ratio.

Dollar value of good i lost is $p_i$. Dollar value of good j gained to exactly compensate is $p_j\, MRS_{i\,j}$

## INTERPRETATION (Cont'd)

$$\boxed{\begin{array}{llll} \partial u/\partial x_i = \lambda p_i & \text{if} & x_i > 0 \\ \partial u/\partial x_i < \lambda p_i & \text{if} & x_i = 0 \end{array}}$$

**Assume good j is not purchased, but good i is. Then take ratio of two sides of equation:**

$$\frac{\partial u/\partial x_i}{\partial u/\partial x_j} > \frac{\lambda p_i}{\lambda p_j} = \frac{p_i}{p_j}$$

**Thus** $\quad MRS_{i\,j} > p_i / p_j$

**If good j is not purchased, but good i is purchased**
**to be exactly compensated for a unit loss of good i, the person would need to get more than $p_i/p_j$ units of good j.**



## AN INTERESTING OBSERVATION

**Different consumers have different preferences. Thus, different consumers generally choose different consumption bundles.**

**<u>But</u> at the optimal consumption bundle, each consumer has the *same* $MRS_{i\,j}$ as any other consumer**

- **Relative value of two goods (subjective sense) is identical among all people who buy positive quantities of both**
- **Everyone who buys i and j have same rate at which they are willing to substitute one product for another product**

# EXAMPLE: CONSUMER WITH COBB-DOUGLAS UTILITY FUNCTION

**Consider a consumer with Cobb-Douglas utility function**

$$u(x_1, x_2) = (x_1)^{\alpha}(x_2)^{1-\alpha}$$

**where $\alpha \in (0,1)$ is a given constant.**

**Given a price vector $p = (p_1, p_2)$, the consumer's utility maximization problem yields (using the Lagrangian methods described earlier) the Walrasian demand vector x(p,w) as a function of price and wealth:**

$$x(p,w) = \arg \max_{(x_1,x_2)\in B(p,w)} \{\alpha \log x_1 + (1-\alpha)\log x_2\} = \underbrace{\left(\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}\right)}$$

***"Walrasian Demand"***



# AGENDA

**Administrivia & Course Overview**

**Preferences and Utility Representation**

**Some Properties**

**Utility Representation (Cont'd)**

**Demand Theory: Basics**

**A Little Refresher on Constrained Optimization**

**Key Concepts to Remember**

# KEY CONCEPTS TO REMEMBER

- Choice Set & Quasi-Order on Sets
- Preference Relation & Rational Preferences
- Utility Function
- Framing
- Ellsberg Paradox
- Properties of Choice
- Continuity & Convexity of Preferences
- Cardinal vs. Ordinal Properties
- Choice Set vs. Budget Set
- Utility Maximization Problem
- Walrasian Demand / Walras' Law
- Constrained Optimization / Necessary Optimality Conditions
- Lagrange Multipliers (Dual Variables, Shadow Prices)
- Cobb-Douglas Utility Function



# APPENDIX

# (Optional)

# LEXICOGRAPHIC PREFERENCES CANNOT BE REPRESENTED BY A UTILITY FUNCTION (1/3)

**Example 3** Consider a decision maker with *lexicographic preferences* $\preceq$, defined on the set $\mathcal{S} = [0,1] \times [0,1]$, such that

$$(s_1, s_2) \preceq (\hat{s}_1, \hat{s}_2) \quad \overset{\text{def}}{\Leftrightarrow} \quad s_1 < \hat{s}_1 \text{ or } (s_1 = \hat{s}_1 \text{ and } s_2 \leq \hat{s}_2).$$

The decision maker thus prefers an improvement in $s_1$ more than *any* improvement in $s_2$. Let us for a moment assume that there exists a utility function $u : \mathcal{S} \to \mathbb{R}$ that represents these preferences according to (4). We now show that this inevitably leads to a contradiction. Note first that $(s_1, 0) \prec (s_1, 1)$ and therefore $u((s_1, 0)) < u((s_1, 1))$ for all $s_1 \in [0,1]$. If we let

$$\Delta(s_1) = u((s_1, 1)) - u((s_1, 0)),$$



# LEXICOGRAPHIC PREFERENCES CANNOT BE REPRESENTED BY A UTILITY FUNCTION (2/3)

then $\Delta(s_1) > 0$ for all $s_1 \in [0,1]$. As a result, the range $[0,1]$ of the first coordinate can be written as a union of the subsets $\mathcal{S}_{1k} = \{s_1 : \Delta(s_1) \geq 1/k\}$,

$$[0,1] = \bigcup_{k=1}^{\infty} \mathcal{S}_{1k}.$$

Since the interval $[0,1]$ is uncountable, some of the sets $\mathcal{S}_{1k}$ have to be uncountable as well. Let $\mathcal{S}_{1\bar{k}}$ be such an uncountable set for an appropriate $\bar{k} \in \{1, 2, \ldots\}$. Let $\bar{\Delta} = u((1,1)) - u((0,0))$ be the largest possible utility difference between any two elements in $\mathcal{S}$ and let $K > \bar{k}\bar{\Delta} + 1$ be an integer. Then for any $K$ elements $\sigma_1, \ldots, \sigma_K \in \mathcal{S}_{1\bar{k}}$ with $\sigma_1 < \sigma_2 < \cdots < \sigma_K$ we have

$$u((\sigma_k, 0)) - u((\sigma_{k-1}, 0)) > u((\sigma_{k-1}, 1)) - u((\sigma_{k-1}, 0)) > 1/\bar{k}$$

for any $k \in \{2, \ldots, K\}$.

# LEXICOGRAPHIC PREFERENCES CANNOT BE REPRESENTED BY A UTILITY FUNCTION (3/3)

Hence,

$$\begin{aligned}
\bar{\Delta} &= u((1,1)) - u((0,0)) \\
&= [u((1,1)) - u((\sigma_K,0))] + [u((\sigma_K,0)) - u((\sigma_{K-1},0))] + \cdots \\
&\quad + [u((\sigma_2,0)) - u((\sigma_1,0))] + [u((\sigma_1,0)) - u((0,0))] \\
&> 0 + 1/\bar{k} + \cdots + 1/\bar{k} + 0 = (K-1)/\bar{k} > \bar{\Delta},
\end{aligned}$$

i.e., a contradiction. A utility representation of lexicographic preferences is therefore not possible. The intuition is that because of the nonseparability of the choice set with respect to $\preceq$, any finite difference in the first attribute must yield an unbounded utility difference, which results from adding up the uncountably many finite utility differences (generated by variations in $s_2$ for each fixed $s_1$) in between. $\square$

# MGT 621 – MICROECONOMICS

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*2. Demand Theory*

Autumn 2023

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# AGENDA

**Some Special Utility Functions**

**Wealth Effects**

**Price Effects**

**Demand Aggregation**

**Standard Welfare Measures**

**Welfare Changes**

**Key Concepts to Remember**

# SOME SPECIAL UTILITY FUNCTIONS

**Let** $\alpha_i, \rho > 0$ **and** $k : \Re \to \Re$ **be an increasing function.**

1. **Cobb-Douglas:**

$$u(x) = k\left(\prod_{i=1}^{L} x_i^{\alpha_i}\right)$$

**Preferences are strongly monotonic and strictly convex on** $\Re^L_{++}$

2. **Constant-Elasticity-of-Substitution (CES)**

$$u(x) = k\left(\left[\sum_{i=1}^{L} \alpha_i x_i{}^{\rho}\right]^{1/\rho}\right)$$

**CES preferences are strongly monotonic and strictly convex on** $\Re^L_{++}$**. Cobb-Douglas is a special case for** $\rho = 1$ **.** [(1)]

3. **Fixed-Coefficient (Leontief)**

$$u(x) = k\left(\min_{i \in \{1,\ldots,L\}} \{\alpha_i x_i\}\right)$$

**Preferences are continuous, monotonic (not strongly), convex (but not strictly)**

(1) The "elasticity of substitution" (introduced by John Hicks) between good 1 and 2 is $E_{12} = - d \ln(x_1/x_2) / d \ln(u_{x1}/u_{x2})$ ( $= E_{21}$).



# AGENDA

Some Special Utility Functions

**Wealth Effects**

Price Effects

Demand Aggregation

Standard Welfare Measures

Welfare Changes

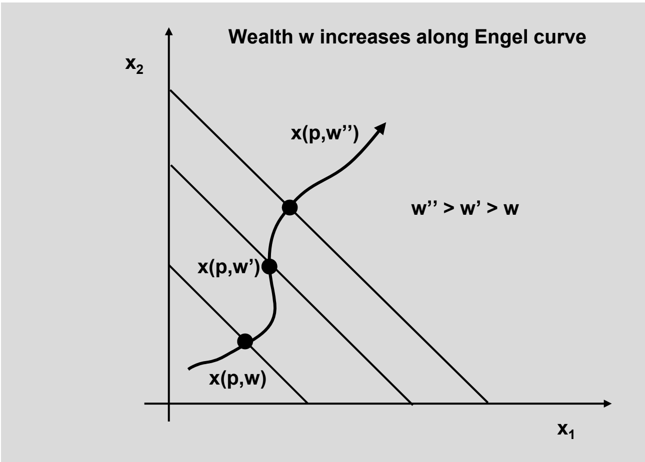
Key Concepts to Remember

# COMPARATIVE STATICS: WEALTH EFFECT

**Definition**. Let x(p,w) be a consumer's Walrasian demand function.

For any fixed price p, the function f(w) = x(p,w) is called a wealth expansion function; its graph is a **wealth expansion path** (also known as **Engel curve**).

The derivative of x(p,w) with respect to wealth, $f'(w) = D_w x(p,w) = \left[ \frac{\partial x_i(p,w)}{\partial w} \right]_{i=1}^{L}$ , is called the **wealth** (or **income**) **effect** (on demand).




**Good i** is called **normal** if $\frac{\partial x_i(p,w)}{\partial w} \geq 0$ , otherwise it is called **inferior** (at (p,w)).

**Demand** is called **normal** if all goods are normal at any (p,w).



# INCOME ELASTICITY

Income elasticity of demand for good i:

- differentiate Walrasian demand $x_i$ with respect to w, then multiply by $(w/x_i)$:

$$e_i = (w/x_i)\ \partial x_i/\partial w$$

# AVERAGE INCOME ELASTICITY

$\Sigma \{ p_i x_i \} = w$

**Differentiate with respect to w**

$\Sigma \{ p_i \partial x_i / \partial w \} = 1$

**Multiply each term by $(x_i w / x_i w)$ and regroup**

$$\Sigma \underbrace{[ (p_i x_i)/w ]}_{\text{Fraction of income spent on good i}} \underbrace{[ (\partial x_i/\partial w)(w/x_i) ]}_{\text{Income elasticity of demand for good i}} = 1$$

**Weighted average of income elasticities equals 1. Weights are fractions of income spent on various goods**



# AGENDA

**Some Special Utility Functions**

**Wealth Effects**

**Price Effects**

**Demand Aggregation**

**Standard Welfare Measures**

**Welfare Changes**

**Key Concepts to Remember**

# PRICE ELASTICITY

**(Own-)Price elasticity of demand for good i**

- **differentiate Walrasian demand for good i with respect to $p_i$, then multiply by $(p_i/x_i)$:**

$$\varepsilon_{ii} = (p_i/x_i)\ (\partial x_i / \partial p_i)$$

**Cross-Price elasticity of demand for good i with respect to changes in the price of good j**

- **differentiate Walrasian demand for good i with respect to $p_j$, then multiply by $(p_j/x_i)$:**

$$\varepsilon_{ij} = (p_j/x_i)\ (\partial x_i / \partial p_j)$$



# PRICE / INCOME ELASTICITIES FOR DIFFERENT COMMODITIES

| Commodity | Price Elasticity | Income Elasticity |
|---|---|---|
| Electricity | - 1.2 | + 0.2 |
| Beef (Meat) | - 0.9 | + 0.4 |
| Women's hats | - 3.0 | |
| Sugar | - 0.3 | |
| Corn | - 0.5 | |
| Potatoes | - 0.3 | |
| Movies | - 3.7 | |
| Flour | | - 0.4 |
| Restaurant Meals | | + 1.5 |
| Margarine | | - 0.2 |
| Butter | | + 0.4 |
| Furniture | | + 1.5 |
| Milk | | + 0.1 |

## ELASTICITIES FOR DIFFERENT COMMODITIES (Cont'd)
### Cross-Price Elasticities

| Commodity | With Respect to Price of | Cross-Price Elasticity |
|---|---|---|
| Electricity | Natural Gas | + 0.2 |
| Natural Gas | Fuel Oil | + 0.4 |
| Beef | Pork | + 0.3 |
| Pork | Beef | + 0.1 |
| Margarine | Butter | + 0.8 |
| Butter | Margarine | +0.7 |
| Gasoline | Automobiles | Negative |
| Solar Panels | Electricity | Positive |
| Software | Computers | Negative |
| Hotel Rooms | Airline travel | Negative |

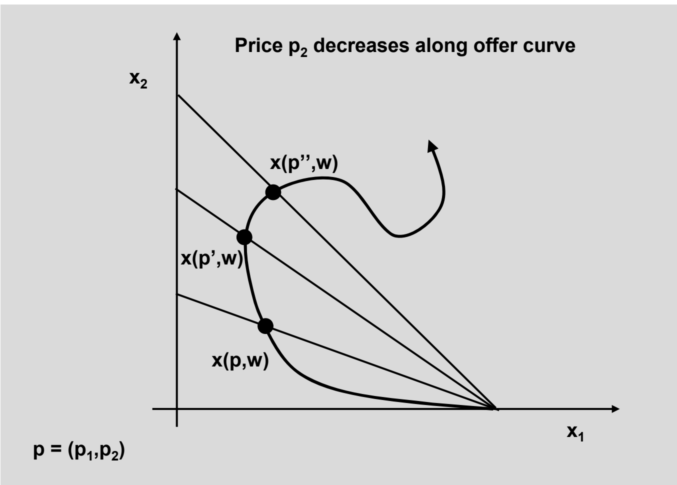


## COMPARATIVE STATICS: PRICE EFFECT

**Definition**. Let x(p,w) be a consumer's Walrasian demand function.

For any fixed wealth w, the graph of the function g(p) = x(p,w) is called the consumer's **offer curve**.

The derivative of x(p,w) with respect to price, $Dg(p) = D_p x(p,w) = \left[\frac{\partial x_i(p,w)}{\partial p_j}\right]_{i,j=1}^{L}$, is called the **price effect** on demand.




**Good i** is called a **Giffen good (at (p,w))** if $\frac{\partial x_i(p,w)}{\partial p_i} > 0$, otherwise it is called a non-Giffen good (= standard case).

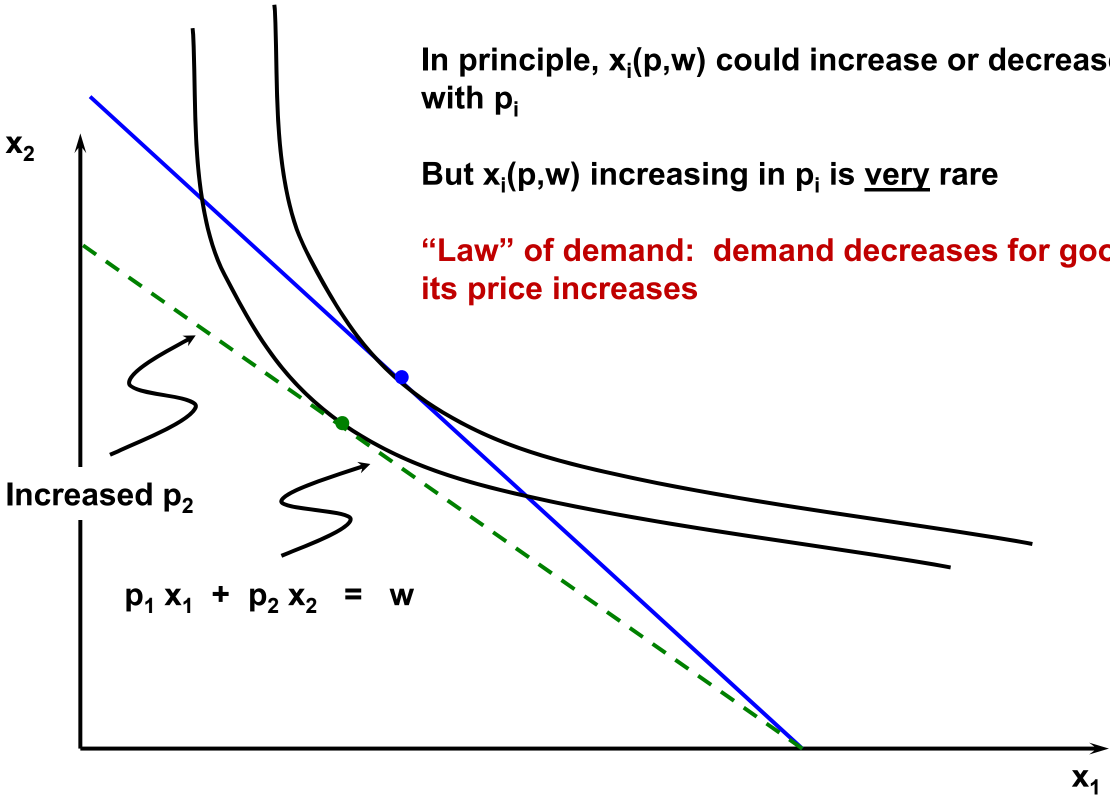
**Demand** exhibits **own-price effects** (of price of good i on demand of good i), and **cross-price effects** (of price of good i on demand of good j).

# COMPARATIVE STATICS WITH RESPECT TO PRICES

In principle, $x_i(p,w)$ could increase or decrease with $p_i$

But $x_i(p,w)$ increasing in $p_i$ is <u>very</u> rare

**"Law" of demand: demand decreases for good if its price increases**



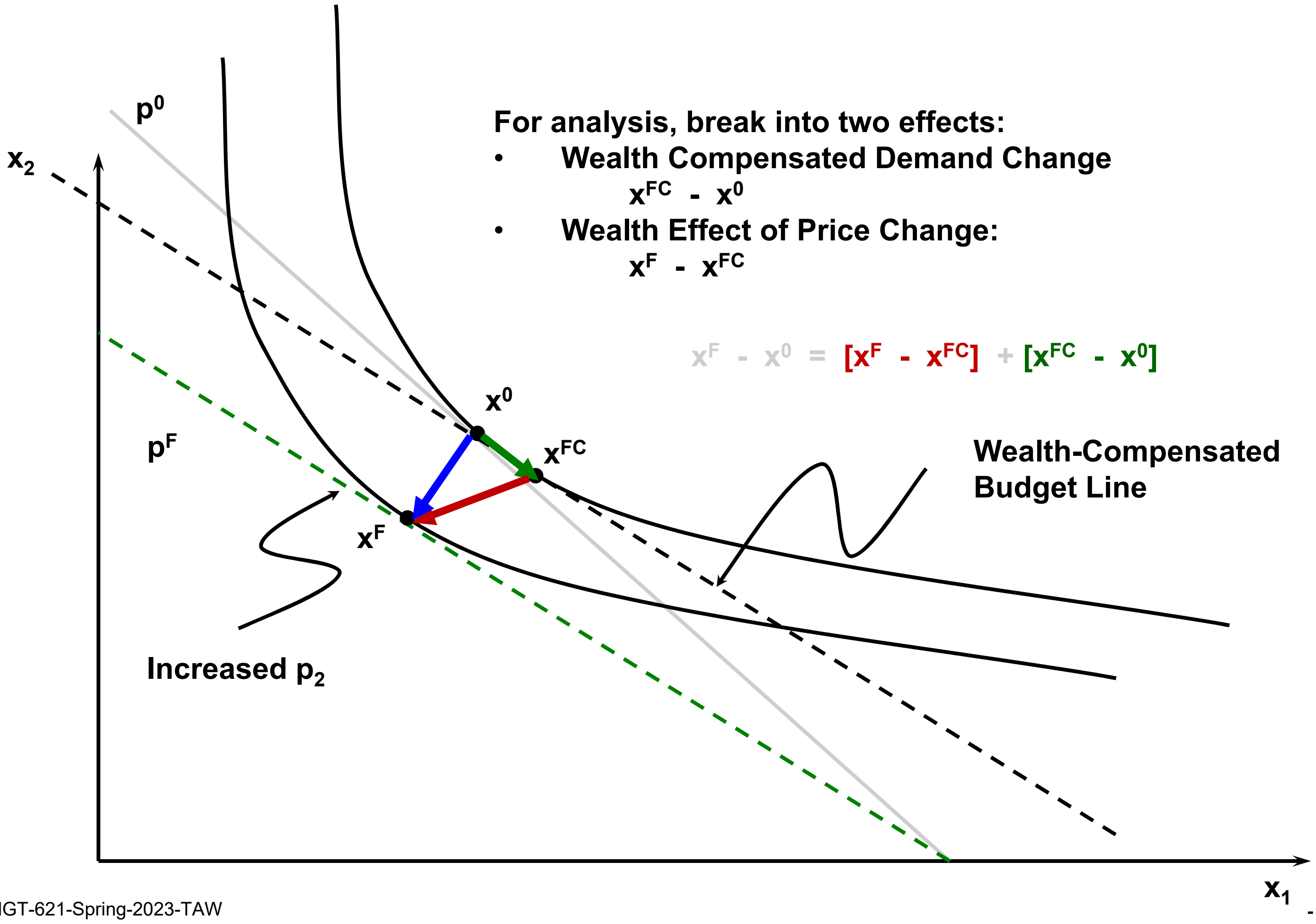



# RESPONSE TO A PRICE INCREASE

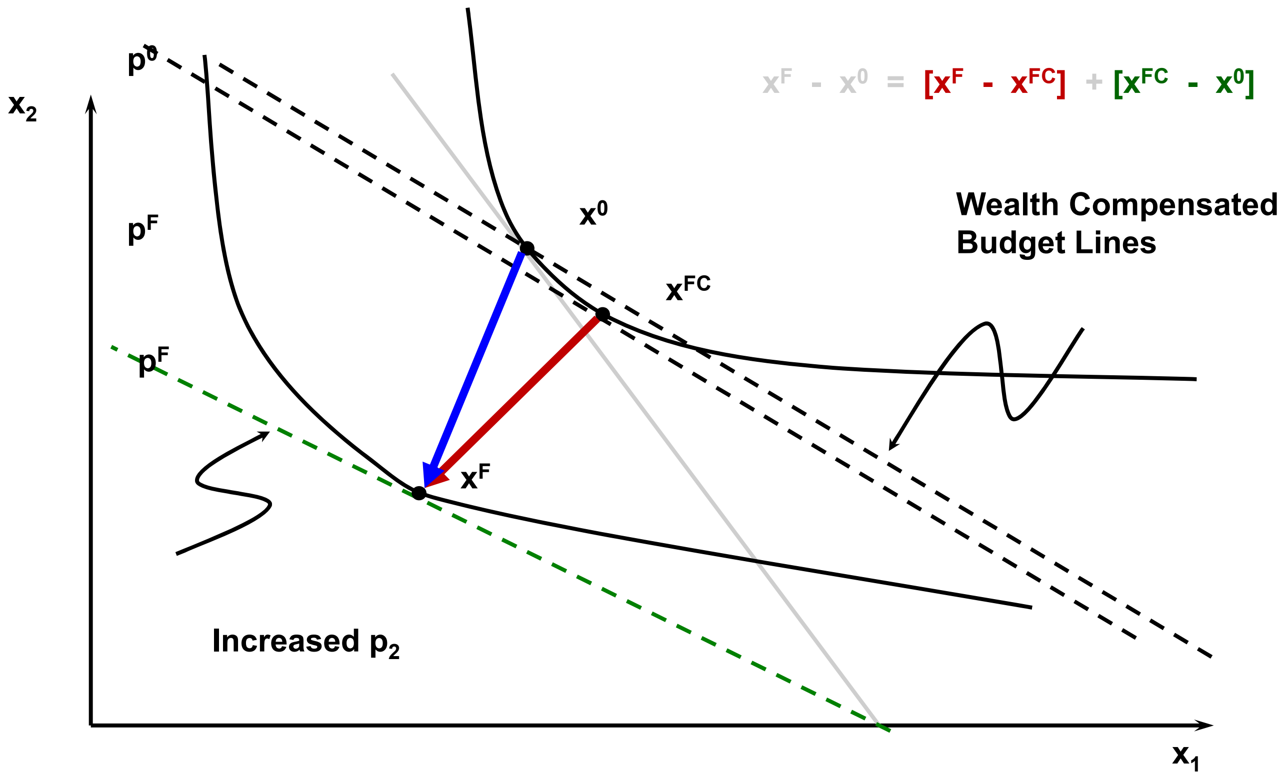
For analysis, break into two effects:

- Wealth Compensated Demand Change
  $x^{FC} - x^0$
- Wealth Effect of Price Change:
  $x^F - x^{FC}$

$$x^F - x^0 = [x^F - x^{FC}] + [x^{FC} - x^0]$$

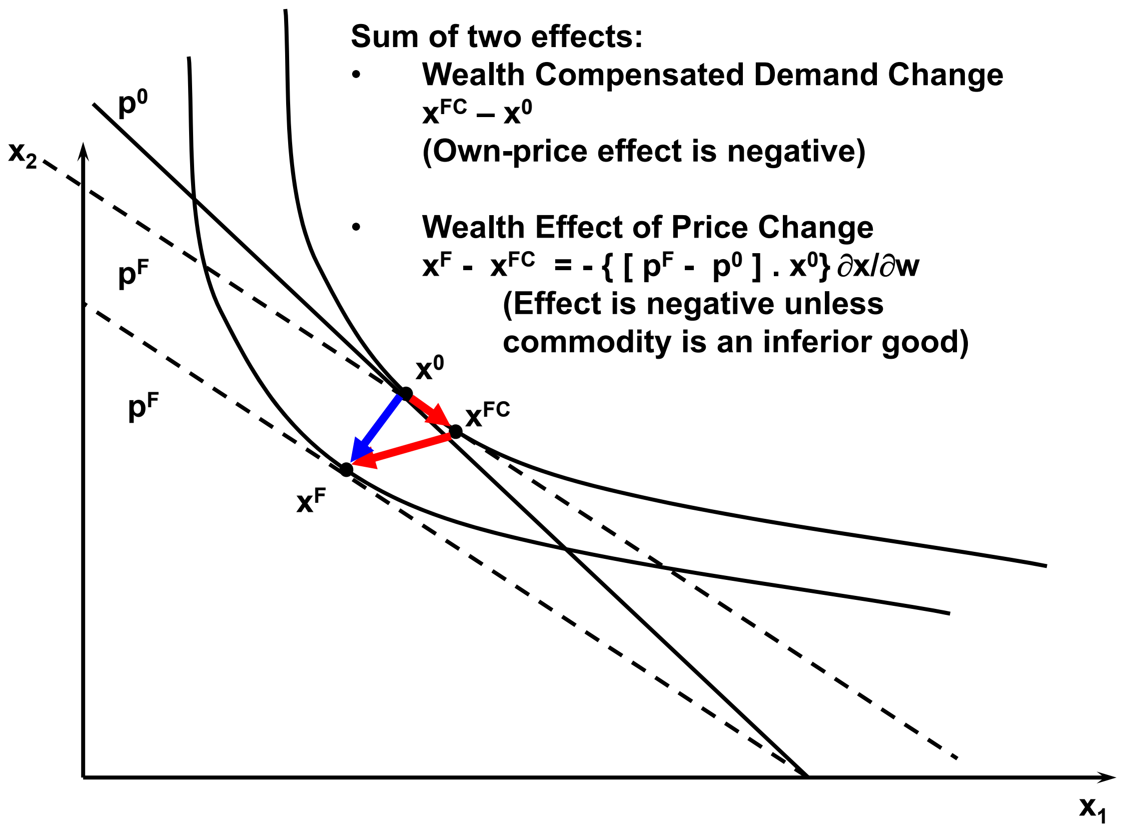


# RESPONSE TO A PRICE INCREASE (Cont'd)






# RESPONSE TO A PRICE INCREASE (Cont'd)




Sum of two effects:

- Wealth Compensated Demand Change
  $x^{FC} - x^0$
  (Own-price effect is negative)
- Wealth Effect of Price Change
  $x^F - x^{FC} = -\{ [ p^F - p^0 ] . x^0\} \partial x/\partial w$
  (Effect is negative unless commodity is an inferior good)

# INDIRECT UTILITY AND EXPENDITURE FUNCTION

**Definition. A consumer's indirect utility function v(p,w) corresponds to her maximized utility over her Walrasian budget set B(p,w),**

$$v(p,w) = \max_{x \in B(p,w)} u(x)$$

**Hence, if x(p,w) is the consumer's Walrasian demand function, then $v(p,w) = u(x(p,w))$ . Note that v(p,w) is strictly increasing in w, as long as the preferences are locally nonsatiated; hence, v(p,w) for a fixed p, has an inverse with respect to w.**

**Now, given any utility level U, we can define the expenditure function e(p,U) in terms of the consumer's indirect utility implicitly, by setting,**

$$v(p, e(p,U)) = U$$

**i.e., the expenditure function defines the minimum expenditure necessary to achieve a given utility level U. In other words,**

$$e(p,U) = \min_{x \in \{\hat{x} \in \Re_+^L : u(\hat{x}) \geq U\}} p \cdot x$$



# HICKSIAN DEMAND FUNCTION

**Definition. The Hicksian demand function h(p,U) for a given price p and utility level U is given by the Walrasian demand function evaluated at the price p and the minimum wealth necessary to achieve U, i.e.,**

$$h(p,U) = x(p, e(p,U)) \in \arg \min_{x \in \{\hat{x} \in \Re_+^L : u(\hat{x}) \geq U\}} p \cdot x \qquad \textbf{(*)}$$

**From (*) we can conclude that**

$$\frac{\partial h_j(p,U)}{\partial p_i} = \frac{\partial x_j(p,e(p,U))}{\partial p_i} + \frac{\partial x_j(p,e(p,U))}{\partial w} \boxed{\underbrace{\frac{\partial e(p,U)}{\partial p_i}}_{h_i(p,U)}}$$

**Roy's Identity**

**(follows directly from application of envelope theorem)**

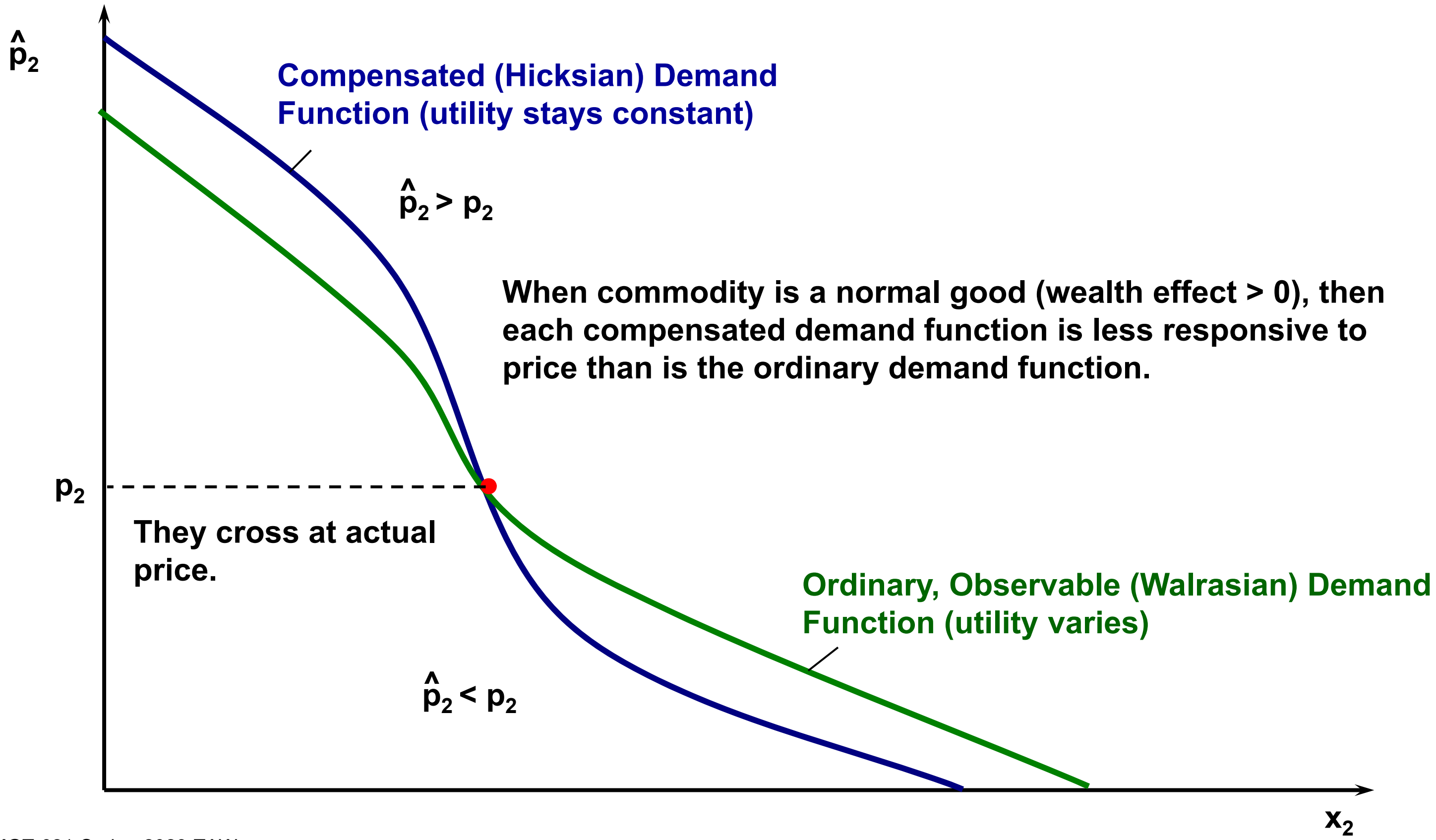
**whence**

$$\boxed{\underbrace{\frac{\partial x_j(p,e(p,U))}{\partial p_i}}_{\text{Price Effect}} = \underbrace{\frac{\partial h_j(p,U)}{\partial p_i}}_{\text{Wealth-Compensated Demand Change}} - \underbrace{\frac{\partial x_j(p,e(p,U))}{\partial w} h_i(p,U)}_{\text{Wealth Effect}}}$$

**Slutsky Equation**

# WALRASIAN VS. COMPENSATED (HICKSIAN) DEMAND FUNCTION

Consider a change from $p_2$ to $\hat{p}_2$ (for a normal good)






# PROOF OF ROY'S IDENTITY

**Differentiate expenditure function with respect to one price component,**

$$\frac{\partial e(p,U)}{\partial p_i} = \frac{\partial}{\partial p_i}\left[\min_{x\in\{\hat{x}\in\Re_+^L : u(\hat{x})\geq U\}} p\cdot x\right] \underset{\uparrow}{=} \frac{\partial}{\partial p_i}\left[p\cdot x - \lambda(U-u(x))\right]\Big|_{x=x(p,e(p,U))} = x_i(p,e(p,U)) = h_i(p,U)$$

**Envelope Theorem**

**This is Roy's identity.** **QED**

**We conclude that** $\dfrac{\partial h_j(p,U)}{\partial p_i} = \dfrac{\partial^2 e(p,U)}{\partial p_i \partial p_j}$

**The matrix** $S(p,U) = \left[\dfrac{\partial h_j(p,U)}{\partial p_i}\right]_{i,j=1}^{L} = \left[\dfrac{\partial^2 e(p,U)}{\partial p_i \partial p_j}\right]_{i,j=1}^{L}$ **is called Slutsky matrix.**

# PROPERTIES OF THE SLUTSKY MATRIX

**Proposition. The Slutsky matrix S(p,U) is symmetric, negative semidefinite, and satisfies S(p,U) p = 0.**

**Proof.**

**(i) S is symmetric as long as expenditure function is twice continuously differentiable (theorem by Cauchy [sometimes attributed to H.A. Schwarz]).**

**(ii) The negative semi-definiteness of S (i.e., the fact that $D_p h(p,U) \le 0$) follows from the "law of compensated demand", which states that**

$$(p'-p)(h(p',U)-h(p,U)) \le 0 \qquad \textbf{(\#)}$$

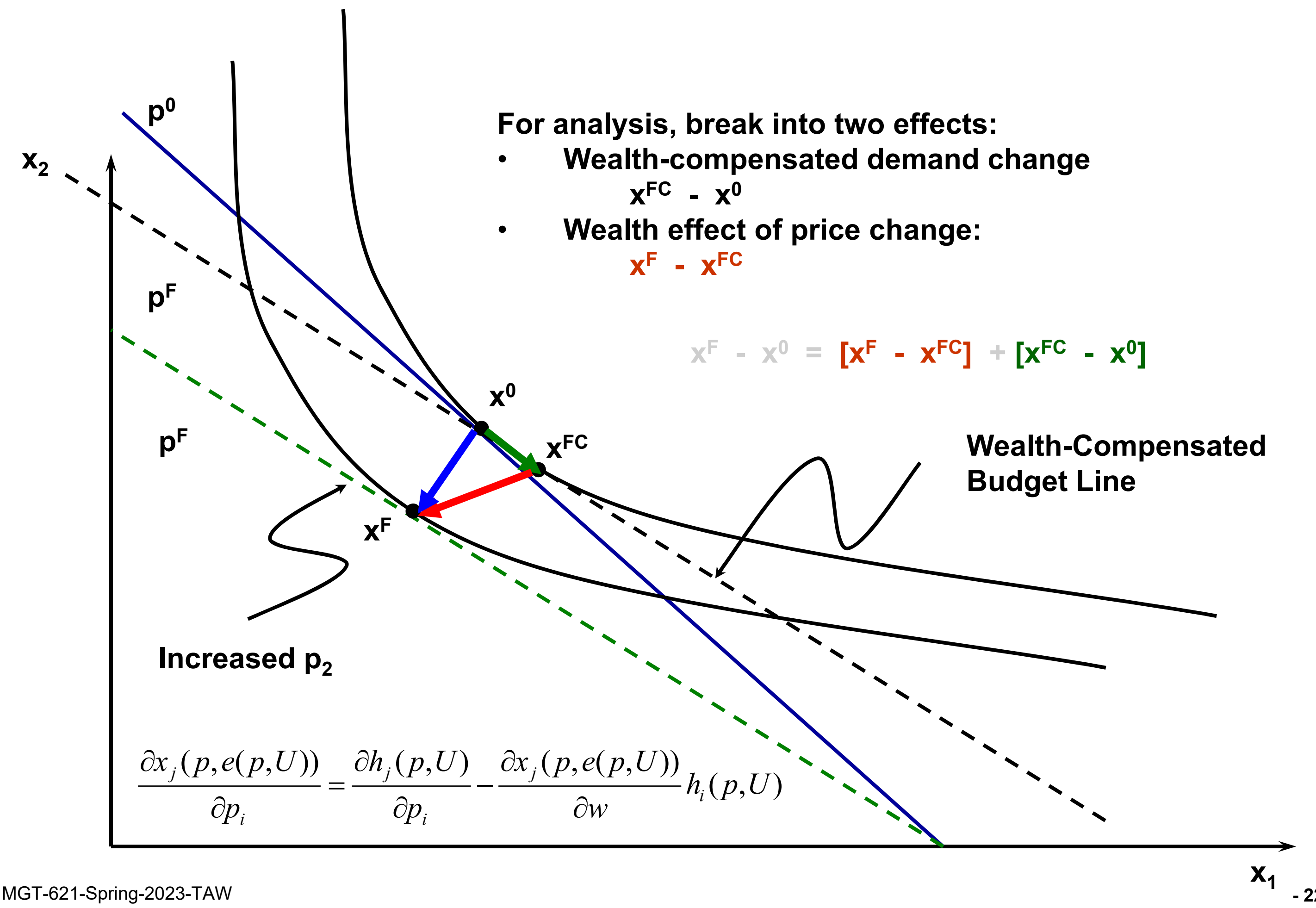
**Relation (#) holds, because $p' \cdot h(p',U) \le p' \cdot h(p,U)$ and $p \cdot h(p,U) \le p \cdot h(p',U)$.**

**(iii) Note that Hicksian demand h(p,U) is homogeneous of degree zero in p (prove this as an exercise!), so that**

$$\left.\frac{\partial h(\alpha\, p,U)}{\partial \alpha}\right|_{\alpha=1} = D_p h(p,U)p = S(p,U)p = 0 \qquad \textbf{QED}$$



# RESPONSE TO A PRICE INCREASE




$$\frac{\partial x_j(p,e(p,U))}{\partial p_i} = \frac{\partial h_j(p,U)}{\partial p_i} - \frac{\partial x_j(p,e(p,U))}{\partial w} h_i(p,U)$$

# INTERPRETATION OF THE SLUTSKY EQUATION

**Decompose change of demand:** $x^F - x^0 = [x^{FC} - x^0] + [x^F - x^{FC}]$

**For very small price changes $\Delta p$, obtain:**

$$\boxed{\begin{array}{lcl} [x^{FC} - x^0] & = & S(p,w) \cdot \Delta p \\ [x^F - x^{FC}] & = & -\partial x/\partial w \; \{x^0 \cdot \Delta p\} \end{array}}$$

$$\Delta x = S(p,w) \,.\, \Delta p - \partial x/\partial w \; \{x \cdot \Delta p\}$$

**This is the Slutsky Equation. Most often it is written in terms of partial derivatives of $x_i$ with respect to $p_j$ (for small change $x = x^0$):**

$$\boxed{\partial x_i/\partial p_j = S_{ij} - \partial x_i/\partial w \; x_j}$$

**$S_{ij}$ is (i,j)-th element of Slutsky substitution matrix, derivative of wealth compensated demand $x^{FC}_i$ with respect to $p_j$. Referred to as "substitution effect" of a price change.**

**$-\partial x_i/\partial w \; x_j$ is referred to as the "income effect" of a price change.**



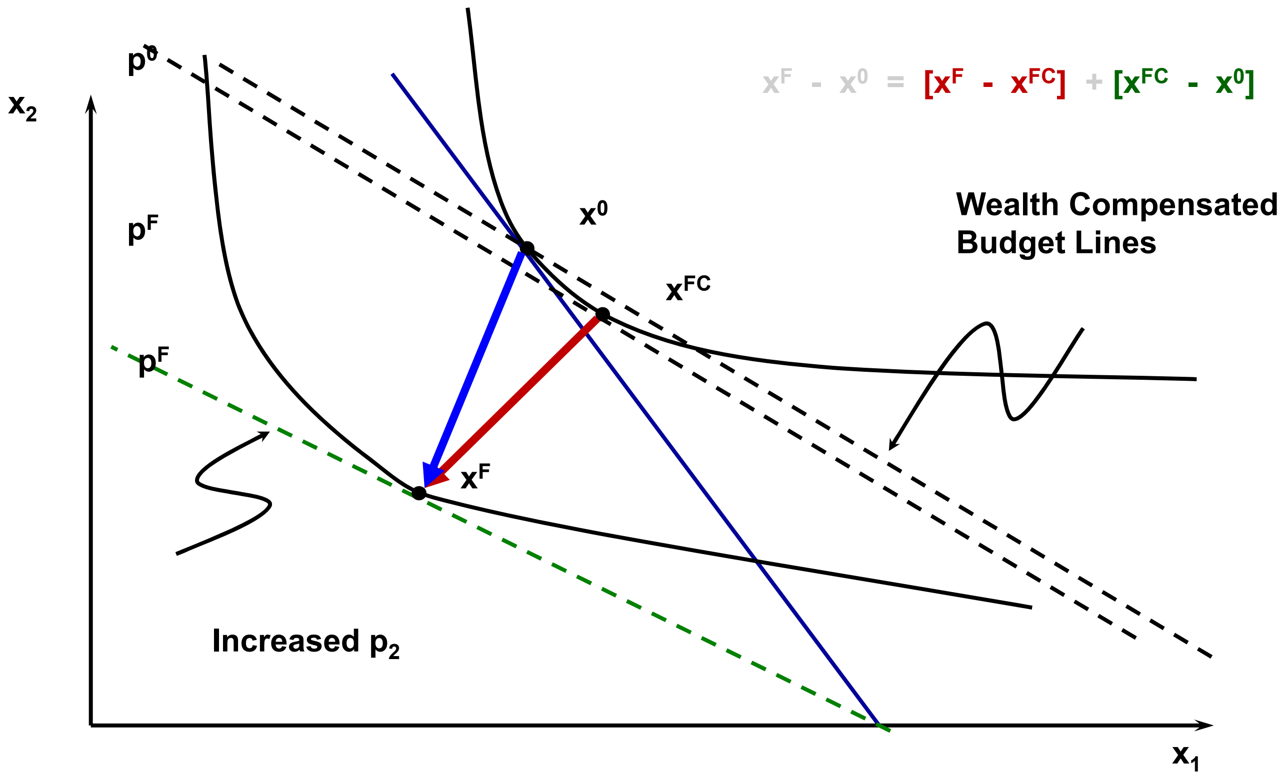
# INTERPRETATION OF THE SLUTSKY EQUATION (Cont'd)

$$D_p x(p,w) = \underbrace{S(p,w)}_{\text{“Substitution Effect”}} - \underbrace{\frac{\partial x(p,w)}{\partial w} x(p,w)}_{\text{“Income Effect”}}$$

**Note that (compared to earlier slides)**

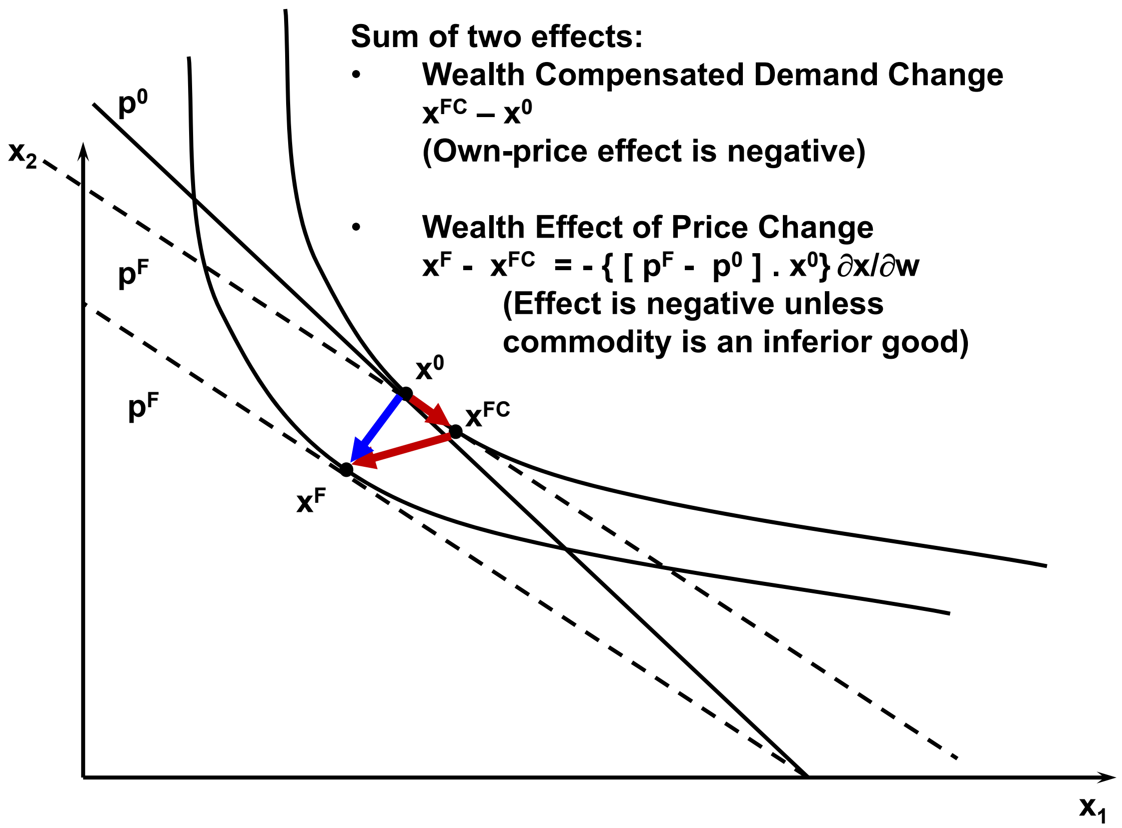
$$w = e(p,U)$$
$$x(p,w) = h(p,U)$$

# RESPONSE TO A PRICE INCREASE (Cont'd)






# RESPONSE TO A PRICE INCREASE (Cont'd)




Sum of two effects:

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  (Own-price effect is negative)
- Wealth Effect of Price Change
  $x^F - x^{FC} = - \{ [ p^F - p^0 ] . x^0 \} \partial x / \partial w$
  (Effect is negative unless commodity is an inferior good)

# AGENDA

Some Special Utility Functions

Wealth Effects

Price Effects

**Demand Aggregation**

Standard Welfare Measures

Welfare Changes

Key Concepts to Remember



# MARKET DEMAND FUNCTIONS

**Demand Aggregated Over Many Individuals: Market Demand Function**

**If following properties hold for each individual demand function:**

- Continuity
- Homogeneity of Degree Zero
- Walras' Law

**Then they hold for the market demand function**

# MARKET DEMAND FUNCTIONS (Cont'd)

**Is it possible to find aggregate demand function D(p,w) (for n individuals), such that**

$$\boxed{D(p,w) = \sum_{k=1}^{n} x^k(p,w^k) \quad \text{for} \quad w = \sum_{k=1}^{n} w^k \quad ?}$$

**In general, if everyone faces the same price vector p, then aggregate demand can be written as a function of p, but NOT necessarily also as a function of aggregate income w, unless**

$$\sum_{k=1}^{n} \frac{\partial x^k}{\partial w^k} dw^k = 0$$

**for any small wealth change dw that leaves aggregate wealth the same, i.e., for which**

$$dw = (dw^1, \ldots, dw^n)$$

$$\sum_{k=1}^{n} dw^k = 0$$

**In other words, all the** $\partial x^k / \partial w^k$ **have to be the same across all consumers.**

**Wealth effects must compensate each other in the aggregate, no matter how the wealth is re-distributed among the individuals!**



# MARKET DEMAND FUNCTIONS (Cont'd)

**Proposition. A (necessary and) sufficient condition for demand aggregation to be possible is when preferences are such that each consumer k's indirect utility $v^k$ is quasilinear ("of the Gorman form"), i.e.,**

$$\boxed{v^k(p,w^k) = a^k(p) + b(p)w^k}$$

**Proof: (sufficiency only)**

**By the definition of indirect utility it is** $v^k(p, e^k(p,u)) = u$

**Thus,** $v_p^k(p,w^k) + v_{w^k}^k(p,w^k)e_p^k(p,u^k(x^k(p,w^k))) = v_p^k(p,w^k) + v_{w^k}^k(p,w)h^k(p,u^k(x^k(p,w^k)))$

$$= \underbrace{v_p^k(p,w^k)}_{a_p^k(p)+b'(p)w^k} + \underbrace{v_{w^k}^k(p,w^k)}_{b(p)} x^k(p,w^k)$$

$$= 0 \qquad (= \partial u / \partial p)$$

**And therefore** $\dfrac{\partial x^k(p,w^k)}{\partial w^k} = -\dfrac{b'(p)}{b(p)}$ **is the same for any consumer k, no matter what his or her wealth level $w^k$.**

**QED**

# MARKET DEMAND FUNCTIONS (Cont'd)

**A market demand function is useful for making statements about the consumer responses to changes in price and/or aggregate income.**

**Example.**

**Sometimes the market demand function is also useful to explain other aggregate effects, such as the "bandwaggon effect," under which the demand for a good depends on the expectation about how many consumers will adopt the product.**



# NETWORK EXTERNALITIES

**Externalities exist when the action of one agent directly affects the environment of another agent; *network externalities* are externalities between participants of a common network**

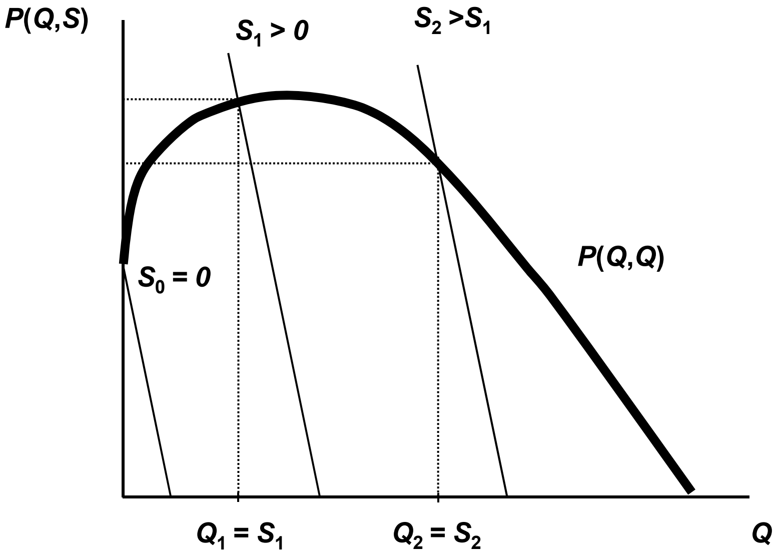
**"How much would you pay for the first fax machine?"**

**Complementarity**

- ***Direct*** **(e.g., in 2-way networks, "exchange transactions")**
- ***Indirect (e.g., Microsoft Word)***
- **Necessary Conditions:**
  - **Compatibility**
  - **Interoperability**

**Aggregate Demand depends on the Expected Demand**

# GENERATING FULFILLED EXPECTATIONS DEMAND CURVE
## Demand in the Presence of Network Externalities




**Bandwaggon Effect**



# DEMAND CURVE SHIFTS DUE TO NETWORK EXTERNALITIES
## Fulfilled-Expectations Demand

**Quantitative Approach**

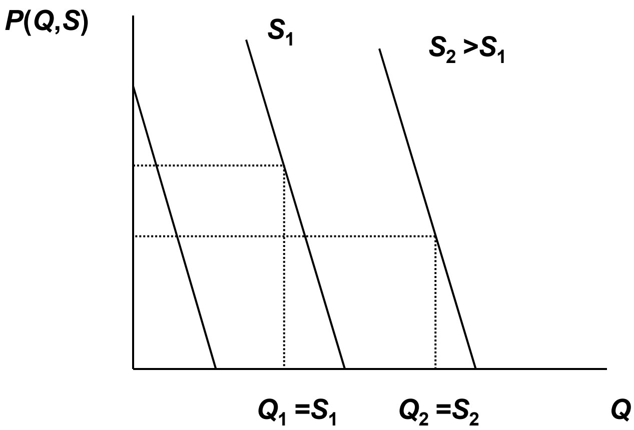
**$P(Q,S) = P(Q,0) + f(S)$**

**P** Willingness to pay

**f(S)** Externality function, $f(0) = 0$, $f'' < 0 < f'$

**Q** Output

**S** Expected demand / Installed base




Source: N. Economides

# NETWORK EXPANSION PATH CAN HAVE SEVERAL FULFILLED EXPECTATIONS EQUILIBRIA

**'Chicken and Egg' Paradox**

**If the installed base is too small, customers may not be willing to purchase the product.**

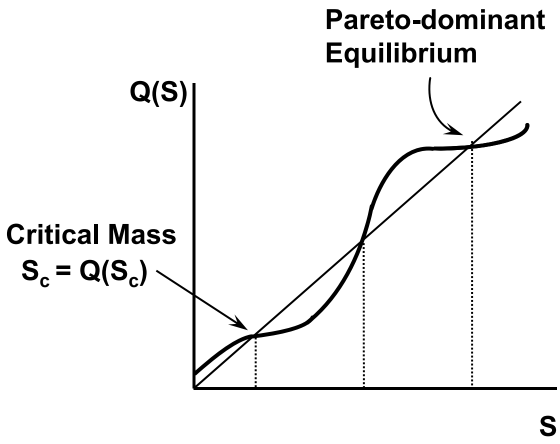
**For certain products, small networks are not observed (*Discontinuous Network Expansion Path*)**

**There are conditions under which a critical mass point exists for a network good**

**Neither existence nor the size of the minimum feasible network depends on the market structure**

**Network sizes resulting from perfect competition, oligopoly or monopoly are different (Economides & Himmelberg, 1994)**

**Network Expansion Path**






# WHAT IS THE CRITICAL MASS?
## Let's compute it!

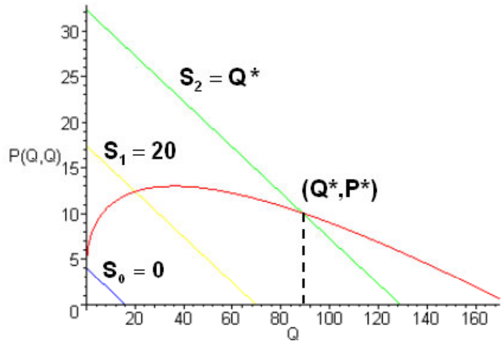
<u>**Example**</u>(1)

| | |
|---|---|
| **Willingness to Pay:** | $P(Q,S) = 4 - Q/4 + 3\sqrt{S}$ |
| **Profit in Fulfilled-Expectations Equilibrium (S=Q):** | $\Pi(Q) = Q \cdot P(Q,Q) - 2Q = \left(4 - Q/4 + 3\sqrt{Q}\right)Q - 2Q$ |
| **Maximization yields:** | $Q^* = S^* = 88.8$, $P^* = P(Q^*,S^*) = 10.1$ |
| **Network Expansion Path:** | $P^* = P(Q,S) = 4 - Q/4 + 3\sqrt{S} \Rightarrow S(Q) = (P^* - 4 + Q/4)^2$ |
| **Critical Mass:** | $Q_c = \min\{Q' : S(Q') = Q' \geq 0\} \Rightarrow Q_c = 6.6$ |

**Note that $Q_c/Q^* = 6.6/88.8 = 7.4\%$ is significant. Thus, in order to have a chance to achieve the optimum, the firm has to instill the belief that in equilibrium more than the critical mass of users (i.e., more than 6.6 Million) will eventually adopt.**

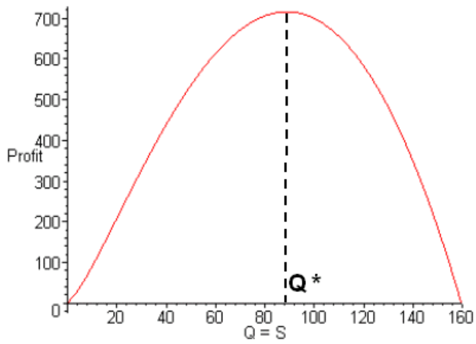
(1) Quantities measured in millions of units. Number of potential customers is 174.5 million (there willingness to pay in the fulfilled-expectations equilibrium will be zero, i.e., p(174.5,174.5) = 0)

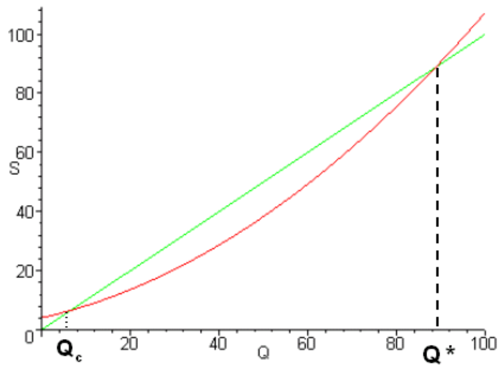
# WHAT IS THE CRITICAL MASS?
## Let's compute it! (Cont'd)




$$\boxed{P(Q,S) = 4 - Q/4 + 3\sqrt{S}}$$

$$\boxed{\Pi(Q) = Q \cdot P(Q,Q) - 2Q}$$






# BUSINESS IMPLICATIONS FOR SELLERS OF NETWORK GOODS

- **Penetration pricing (initially possibly < 0) to reduce adoption costs for the consumer. Pulling one consumer over is likely to induce further consumers ("herding") to adopt, also due to the effect of network externalities**
- **Growth is a strategic imperative**
  - **Production-side economies can help: e.g., lower marginal costs lowers optimal (monopoly) price, which in turn lowers critical mass**
  - **Demand-side economies are most important for achieving market dominance**
- **Strategic pre-announcements to reduce uncertainty. Market uncertainty can prevent the consumers from exploiting beneficial network externalities since consumers fear being stranded with a new technology(1)**
- **Tradeoff current and future benefits through lock-in. Difficult tradeoff and frequent cause of business failure:**
  - **Myopia (too high prices) vs. overestimating future benefits (too low prices)**

(1) Cf. Farrel, J., and Saloner, G., Installed Base and Compatibility: Innovation, Product Preannouncements, and Predation, AER 76(5):940—955, 1986.

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# MEASURING WELFARE CHANGES

Consider a simple example of valuing a nonmarket good (e.g., a national park).

- Assume that there are N standard market goods and one nonmarket good.
- A consumer has preferences represented by a smooth increasing utility function u(x,q), where x denotes the consumption in the market goods and q the consumption of the nonmarket good
- The consumer's income (wealth) is y > 0

Given any q, the consumer's indirect utility function is

$$v(p,q,y) = \max_{x \in \{\hat{x} \in \Re_+^N : p \cdot \hat{x} \le y\}} u(x,q)$$

where p is the price vector for the market goods.

**Question**. How much is an exogenous change of q from $q^0$ to $q^1 > q^0$ worth to the consumer?

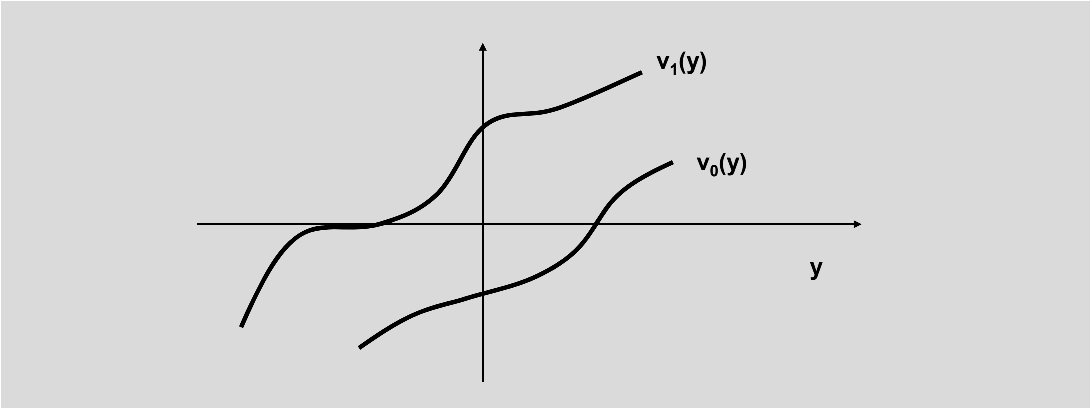
# MEASURING WELFARE CHANGES (Cont'd)

One can interpret $q^0$ and $q^1$ as two "states" of the economy, and the consumer has some value for the change of the state (assume that $q^1 > q^0$, without loss of generality).

Let

$$v_i(y) = v(p, q^i, y)$$

denote the consumer's indirect utility as a function of his income y for i in {0,1}.(1)

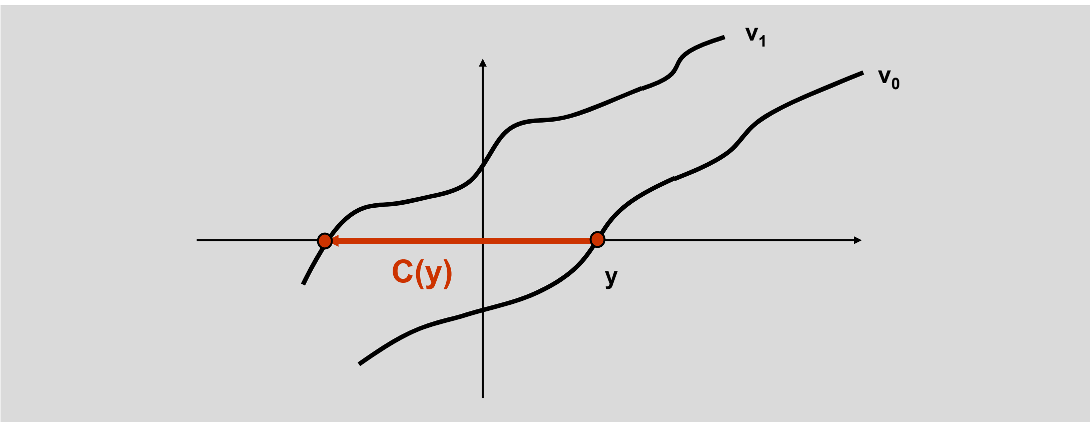



From standard demand theory we know that $v_i(y)$ is strictly increasing in y.

(1) We suppress the dependence on the constant price vector for simplicity. More generally, $v_i(y)$ can denote the consumer's (indirect) utility function in state i of the economy as a function of his or her wealth. Thus, the analysis here can also be applied when the state of the economy is defined by different commodity price vectors (corresponding to the treatment in standard economics textbooks such as MWG).



# MEASURING WELFARE CHANGES (Cont'd)



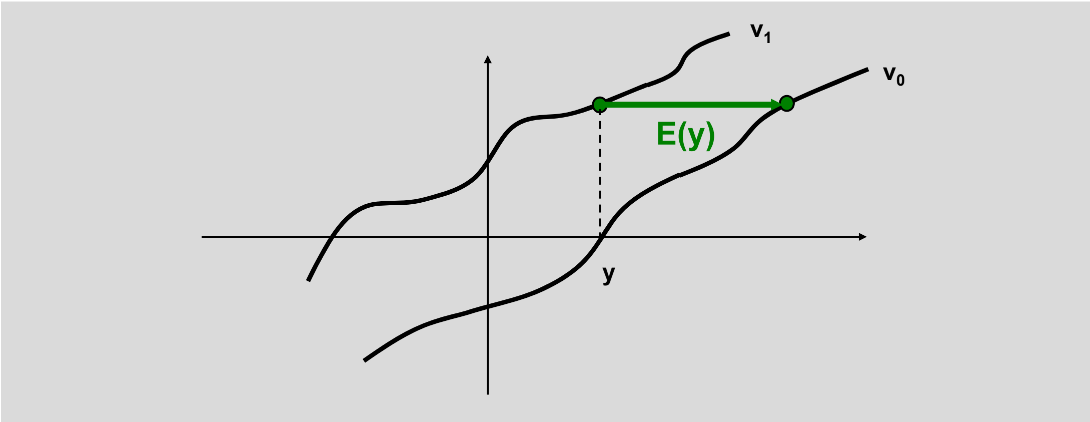

**Question I.** How much would a consumer of income y be **willing to pay** for the transition from $q^0$ to $q^1$?

**Answer I.** A income change of –C(y) would "compensate" the consumer for having $q^1$ instead of $q^0$. [The word compensate means here to bring the consumer back to the original utility level before the change.]

$$\boxed{v_1(y - C(y)) = v_0(y)}$$

**C(y) is the Compensating Variation**

# MEASURING WELFARE CHANGES (Cont'd)




**Question II.** How much would a consumer of income y be willing to accept for the transition from $q^1$ to $q^0$?

**Answer II.** A income change of +E(y) would make the consumer feel "equivalent" between having $q^0$ (at income y+E(y)) and having $q^1$ (at income y).

$$v_1(y) = v_0(y + E(y))$$

**E(y) is the Equivalent Variation**



# COMPENSATING AND EQUIVALENT VARIATIONS

**Definition.** Let $v_i(y)$ be a consumer's increasing (indirect) utility function for an economy in state $i \in \{0,1\}$ as a function of income y.

(i) The **compensating variation C(y)** is defined as the consumer's maximum ***willingness to pay*** to transition from state 0 to state 1, i.e.,

$$C(y) = \sup\{c \in \Re : v_0(y) \le v_1(y - c)\}$$

(ii) The **equivalent variation E(y)** is defined as the consumer's minimum ***willingness to accept*** to transition from state 1 to state 0, i.e.,

$$E(y) = \inf\{e \in \Re : v_1(y) \le v_0(y + e)\}$$

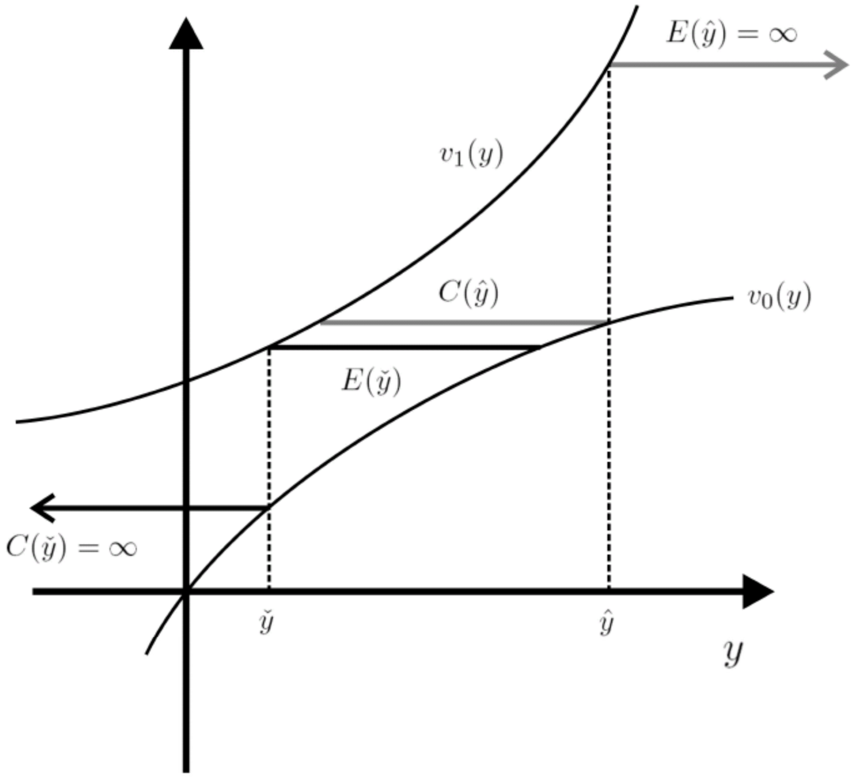
**Remark.** If C(y) and E(y) are bounded, we have that

$$v_0(y) = v_1(y - C(y)) \quad \text{and} \quad v_0(y + E(y)) = v_1(y),$$

corresponding to the standard definition of these two welfare measures.

# COMPENSATING AND EQUIVALENT VARIATIONS

## They can be very different!






# WHAT IS THE RELATION BETWEEN C(y) AND E(y)?

**The answer is simple. Since both $v_0$ and $v_1$ are invertible functions, we obtain from the definition of C and E that**

$$C(y) = y - v_1^{-1}(v_0(y)) \quad \text{and} \quad E(y) = v_0^{-1}(v_1(y)) - y$$

**This immediately implies that C and E are independent of the particular utility representation of the consumer's preferences (why?).**

**As a result, we could choose the utility representation such that $v_0(y) = y$ , so that $v_1(y) = E(y) + y$ (from the definition of E(y)). Thus, from the definition of C(y) we know that we simply need to form the inverse of $v_1$ to find C(y), so that**

$$\boxed{C(y) = y - w_{01}(y) = E(w_{01}(y)),}$$

**where the compensated income $w_{01}(y)$ is such that $y = w_{01}(y) + E(w_{01}(y))$ .**

**Similarly, one can show that**

$$\boxed{E(y) = w_{10}(y) - y = C(w_{10}(y)),}$$

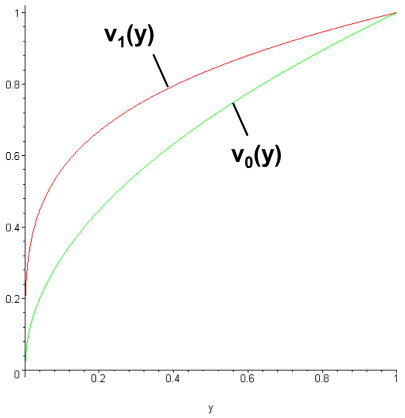
**where the compensated income $w_{10}(y)$ is such that $y = w_{10}(y) - C(w_{10}(y))$**

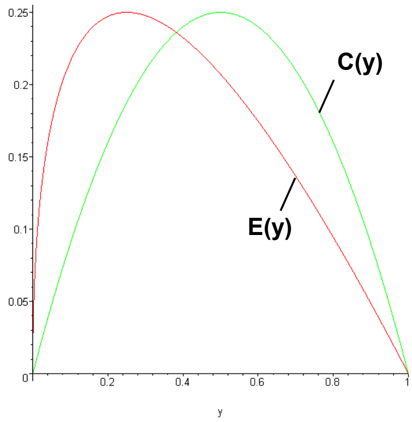
# EXAMPLE: COMPUTATION OF C(y) AND E(y)

**Consider a consumer with indirect utility functions $v_0(y) = y^\alpha$ and $v_1(y) = y^\beta$, where $\alpha = ½$ and $\beta = ¼$, and the income y lies in [0,1].[1]**

$$C(y) = y - v_1^{-1}(v_0(y)) = y - (\sqrt{y})^4 = (1-y)y$$

$$E(y) = v_0^{-1}(v_1(y)) - y = \left(y^{1/4}\right)^2 - y = \sqrt{y} - y$$

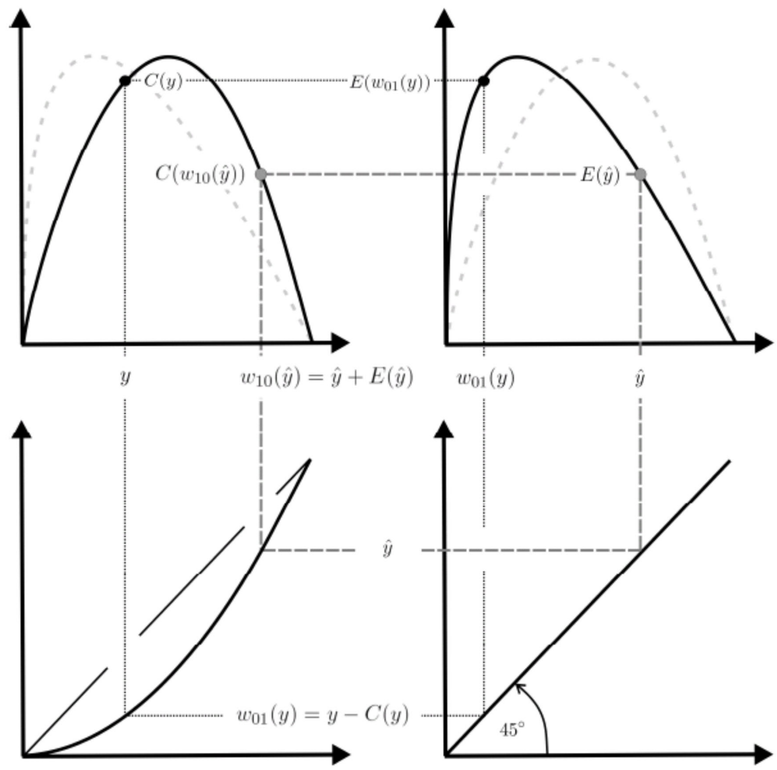



(1) These indirect utility functions can be obtained after solving the utility maximization problem for appropriate Cobb-Douglas utilities.



# EXAMPLE (Cont'd)




$$C(y) = E(w_{01}(y))$$
$$E(\hat{y}) = C(w_{10}(\hat{y}))$$
$$w_{10}(\hat{y}) = w_{01}^{-1}(\hat{y}) = \hat{y} + E(\hat{y})$$
$$w_{01}(y) = y - C(y)$$

# EXAMPLE: TRANSFER OF A NONMARKET GOOD

**Assume that there are two consumers, the first has welfare measures $C(y), E(y)$, while the second has the welfare measures $\hat{C}(y), \hat{E}(y)$. For simplicity, we assume that both start with the same income level y. The first consumer holds one unit of a nonmarket good, while the second consumer possesses none.**

**Questions.** **(i) At what transfers t will there be a transaction of the nonmarket good?**

**(ii) Is it possible that after the first transfer takes place, another such transfer occurs moving the good back to the first consumer?**

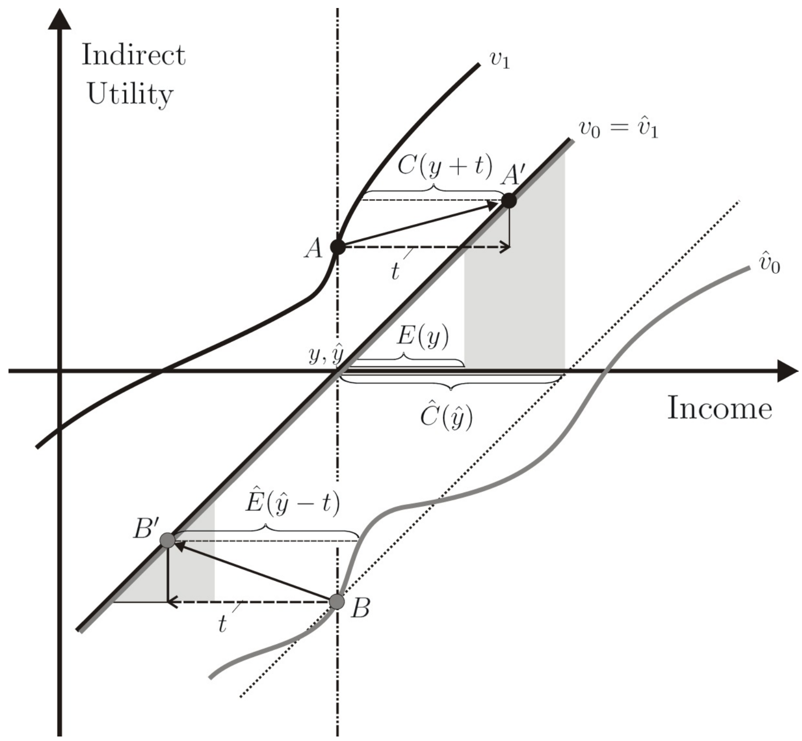
**Answers.**

**(i) A necessary and sufficient condition for a transfer is that $\hat{C}(y) \geq E(y)$**

**(ii) A necessary and sufficient condition for a second transfer (after the good had been exchanged under (i) at price t) is that $C(y+t) \geq \hat{E}(y-t)$. This can never happen if the first transaction realized gains from trade!**



# TRANSFER OF A NONMARKET GOOD (Cont'd)




**The first transfer leads to a Pareto improvement! A second transfer cannot generate another (strict) Pareto improvement.**

# AGENDA

Some Special Utility Functions

Wealth Effects

Price Effects

Demand Aggregation

Standard Welfare Measures

**Welfare Changes**

Key Concepts to Remember



# HOW TO COMPUTE E(y) AND C(y) FOR PRICE CHANGES?

**Price change from $p$ to $\hat{p}$**

$$C(y) = e(p, v(p, y)) - e(\hat{p}, v(p, y))$$
$$= y - e(\hat{p}, v(p, y))$$

**Compensating Variation**

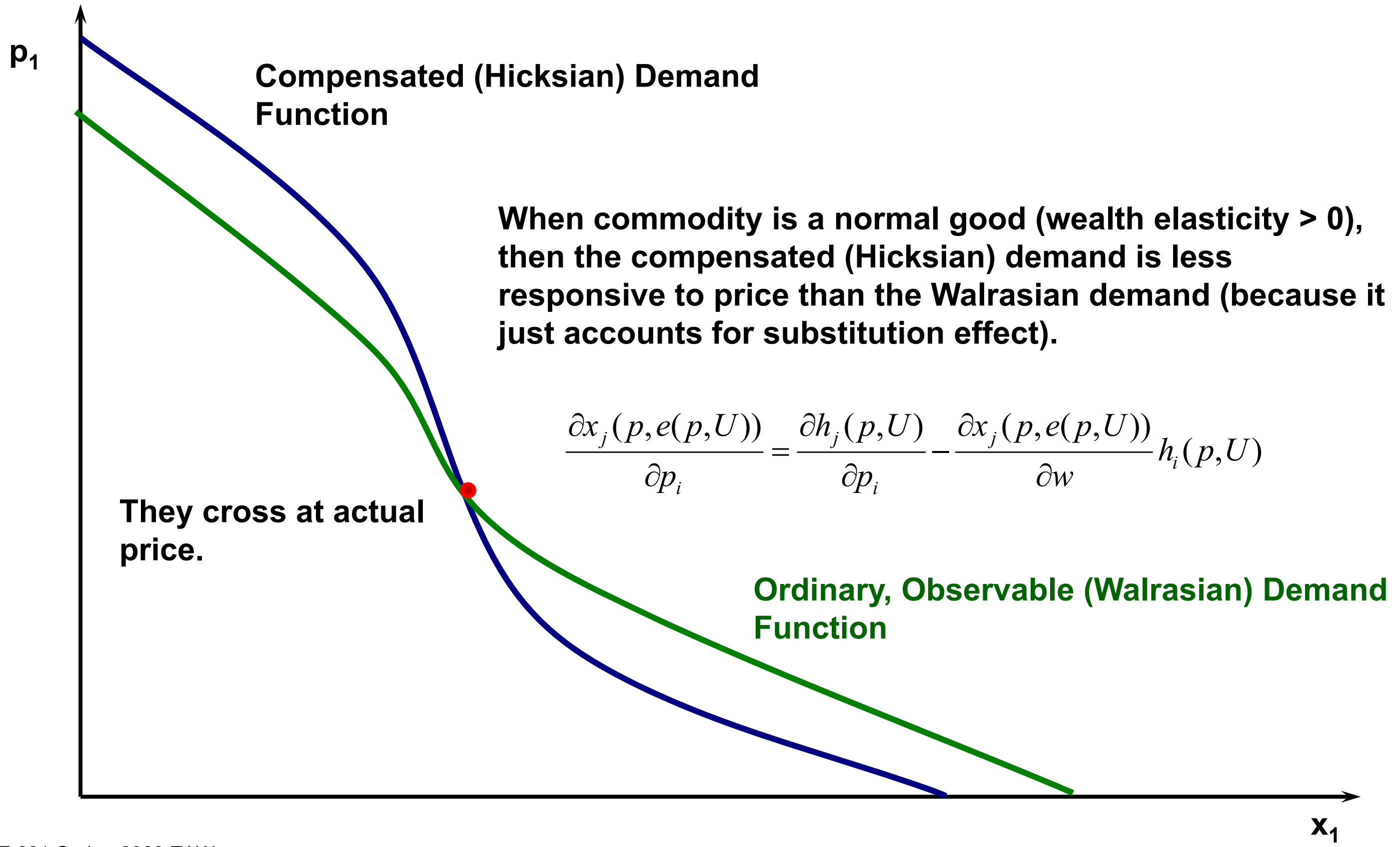
Expenditure at initial price minus expenditure at final price, evaluated at initial utility level

$$E(y) = e(p, v(\hat{p}, y)) - e(\hat{p}, v(\hat{p}, y))$$
$$= e(p, v(\hat{p}, y)) - y$$

**Equivalent Variation**

Expenditure at initial price minus expenditure at final price, evaluated at final utility level

# WALRASIAN DEMAND VS. COMPENSATED (HICKSIAN) DEMAND




When commodity is a normal good (wealth elasticity > 0), then the compensated (Hicksian) demand is less responsive to price than the Walrasian demand (because it just accounts for substitution effect).

$$\frac{\partial x_j(p,e(p,U))}{\partial p_i} = \frac{\partial h_j(p,U)}{\partial p_i} - \frac{\partial x_j(p,e(p,U))}{\partial w} h_i(p,U)$$



# COMPENSATED DEMAND FUNCTION

**Slope of the actual demand function, or Walrasian demand function is $\partial x_i/\partial p_i$**

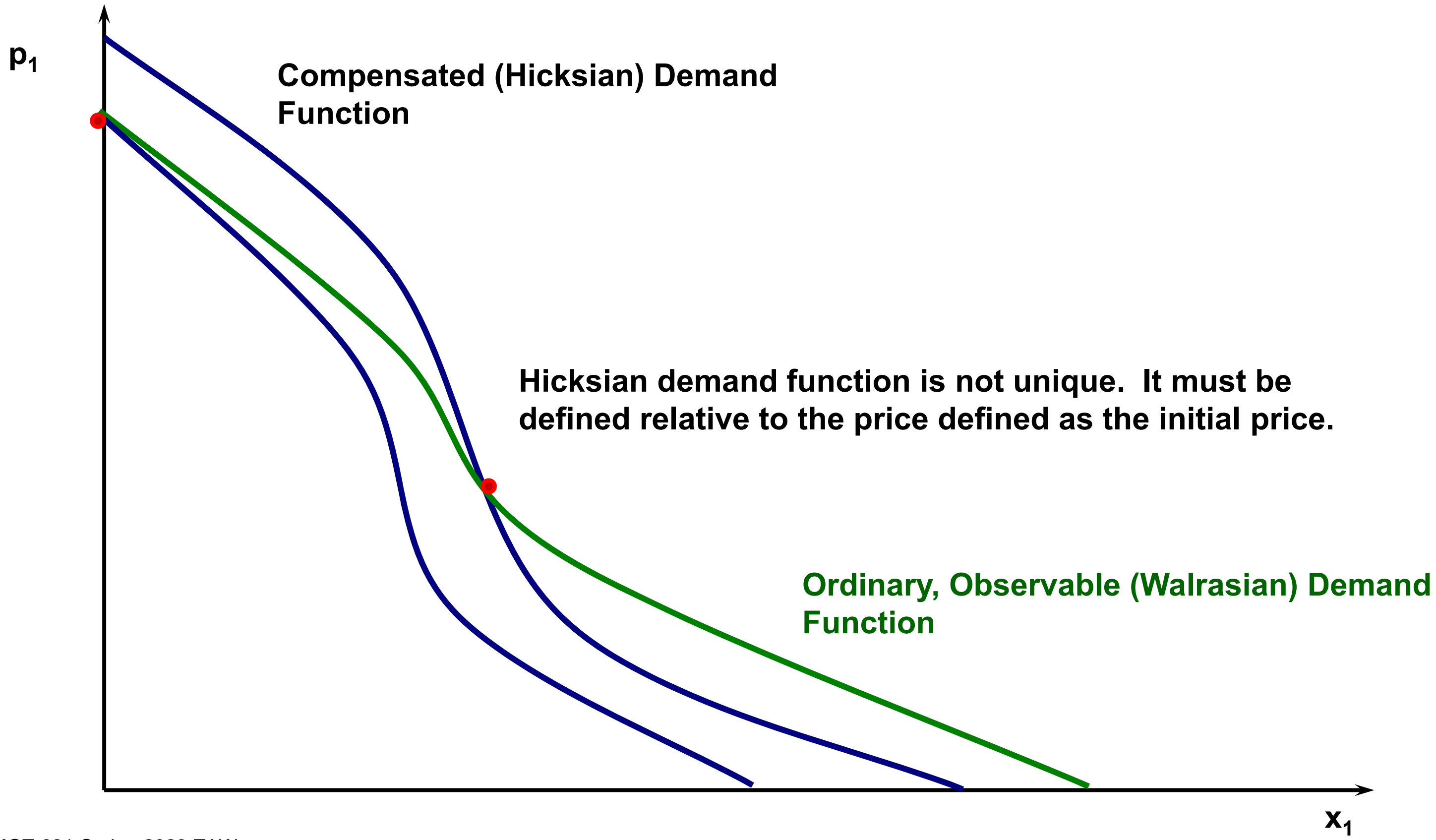
**Can use Slutsky equation to construct the "compensated demand function", also called the "Hicksian demand function"**

**Construct compensated demand function around some specific combination of $p_i$ and the resulting $x_i$ but with the slope $S_{ii}$**

$$S_{ii} = \partial x_i/\partial p_i + \partial x_i/\partial w \; x_i$$

**This is the slope of the artificial demand function, constructed as if at the same time the price is increasing, consumer is given exactly enough additional wealth to keep utility constant.**

# COMPENSATED (HICKSIAN) DEMAND FUNCTIONS FROM TWO DIFFERENT INITIAL PRICES






# SIGNIFICANCE

**Hicksian demand curve is used to create two conceptually correct measurements of welfare impacts of a price change (called "compensating variation" and "equivalent variation")**

- **Compensating Variation: Negative of dollar amount to compensate consumer for facing price change, so that utility remains unchanged.**
- **Equivalent Variation: Dollar amount consumer would accept in place of a price change, so utility change would be the same as it would be with the price change.**

**Compensating Variation and Equivalent variation are both positive for price decrease and negative for a price increase.**

- **A less conceptually correct measure, the change in consumer's surplus, will be approximately equal to compensating variation and to equivalent variation. Consumer's surplus is somewhat easier to calculate**

# WELFARE IMPACTS OF PRICE CHANGES

**If price of good i increases, what wealth increase would just compensate for the price increase if consumption of all goods adjusts optimally? What combination of price increase and wealth increase would leave the person's utility unchanged? Shown previously:**

$$\boxed{dw = x_i(p)\ dp_i}$$

**Discrete price changes: Integrate equation.**

$$\boxed{\begin{array}{l} \Delta w = \int dw = \int x_i(p)\ dp_i \\ \text{Limits on integrals: original price to final price.} \end{array}}$$

**But $x_i(p_i)$ is also a function of w. What w should we use? Actual w? Compensated w? Compensated from where? From original price? From final price? Should w change along with $p_i$? What if several prices are changing together?**



# CALCULATION OF WELFARE MEASURES FOR PRICE CHANGE

**Compensating Variation: Negative of dollar amount to compensate consumer for facing price change, so that utility remains unchanged.**

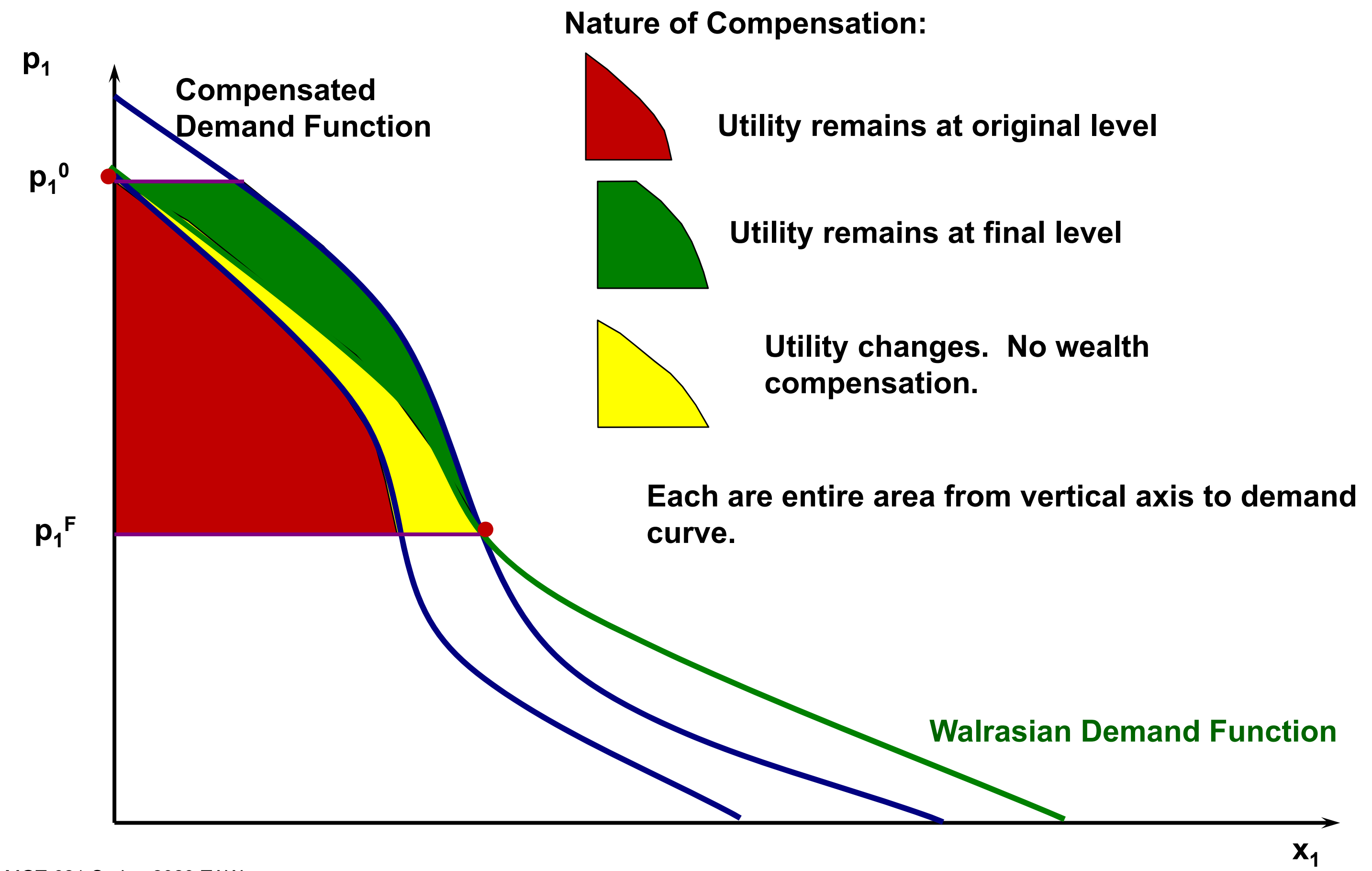
- **Integrate along Hicksian demand curve, crossing through the original price and quantity.**

**Equivalent Variation: Dollar amount consumer would accept in place of a price change, so utility change would be the same as it would be with the price change.**

- **Integrate along Hicksian demand curve, crossing through the final price and quantity.**

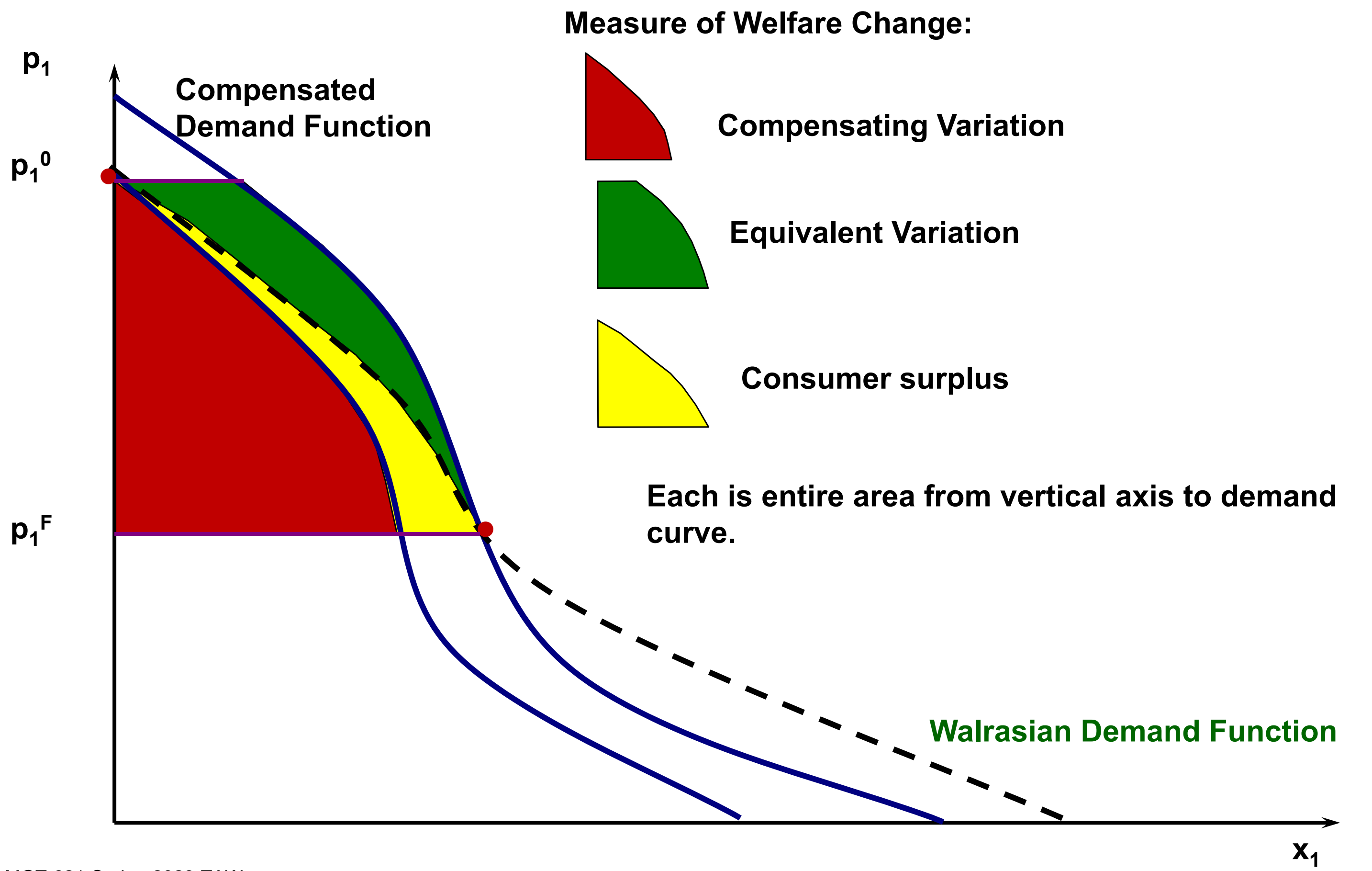
**Consumer Surplus: Integrate along ordinary (Walrasian) demand curve.**

# WELFARE IMPACTS OF PRICE REDUCTION






# WELFARE IMPACTS OF PRICE REDUCTION (Cont'd)

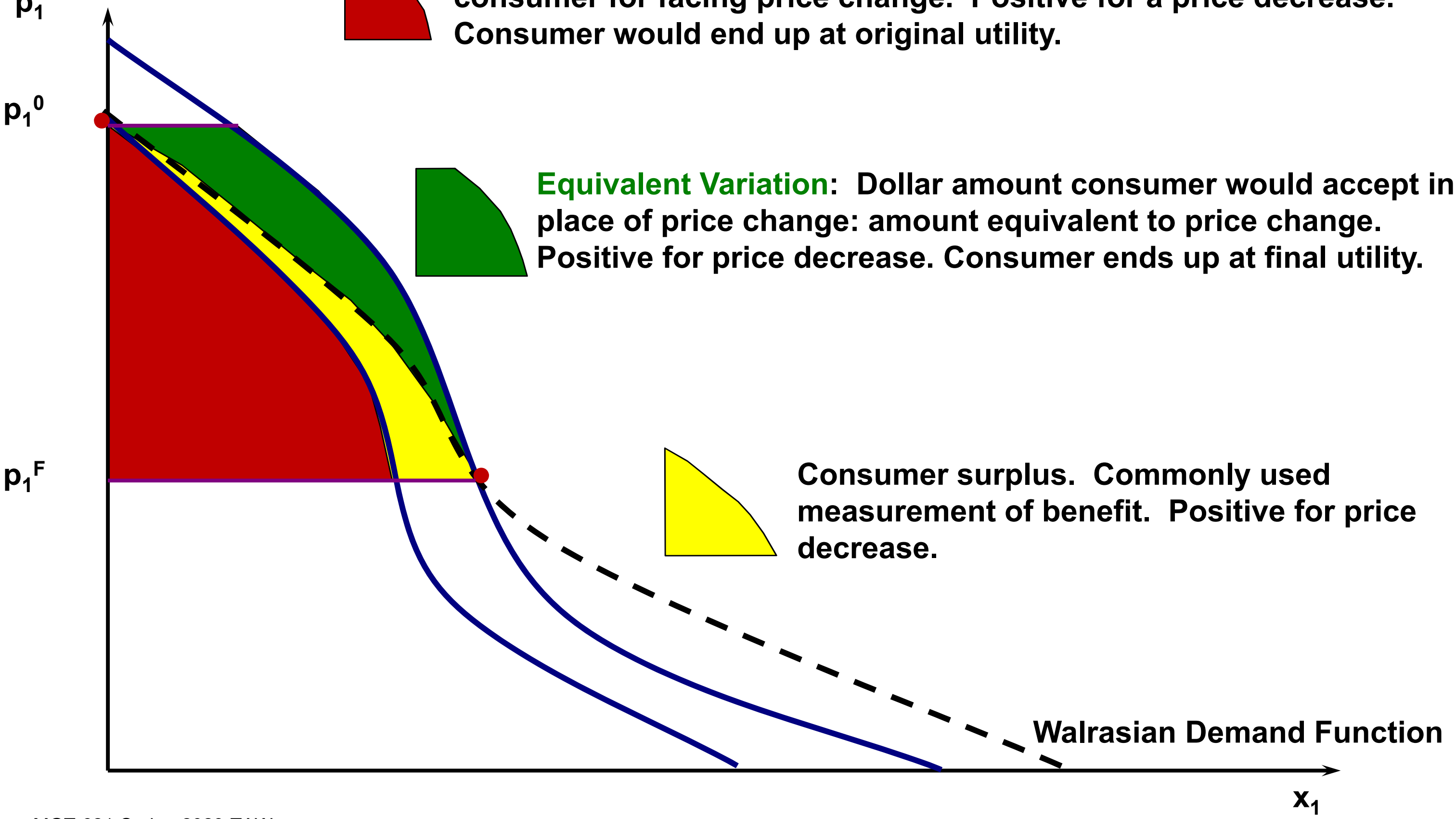


# WELFARE IMPACTS OF PRICE REDUCTION (Cont'd)

**Compensating Variation**: Negative of dollar amount to compensate consumer for facing price change. Positive for a price decrease. Consumer would end up at original utility.

**Equivalent Variation**: Dollar amount consumer would accept in place of price change: amount equivalent to price change. Positive for price decrease. Consumer ends up at final utility.

**Consumer surplus. Commonly used measurement of benefit. Positive for price decrease.**



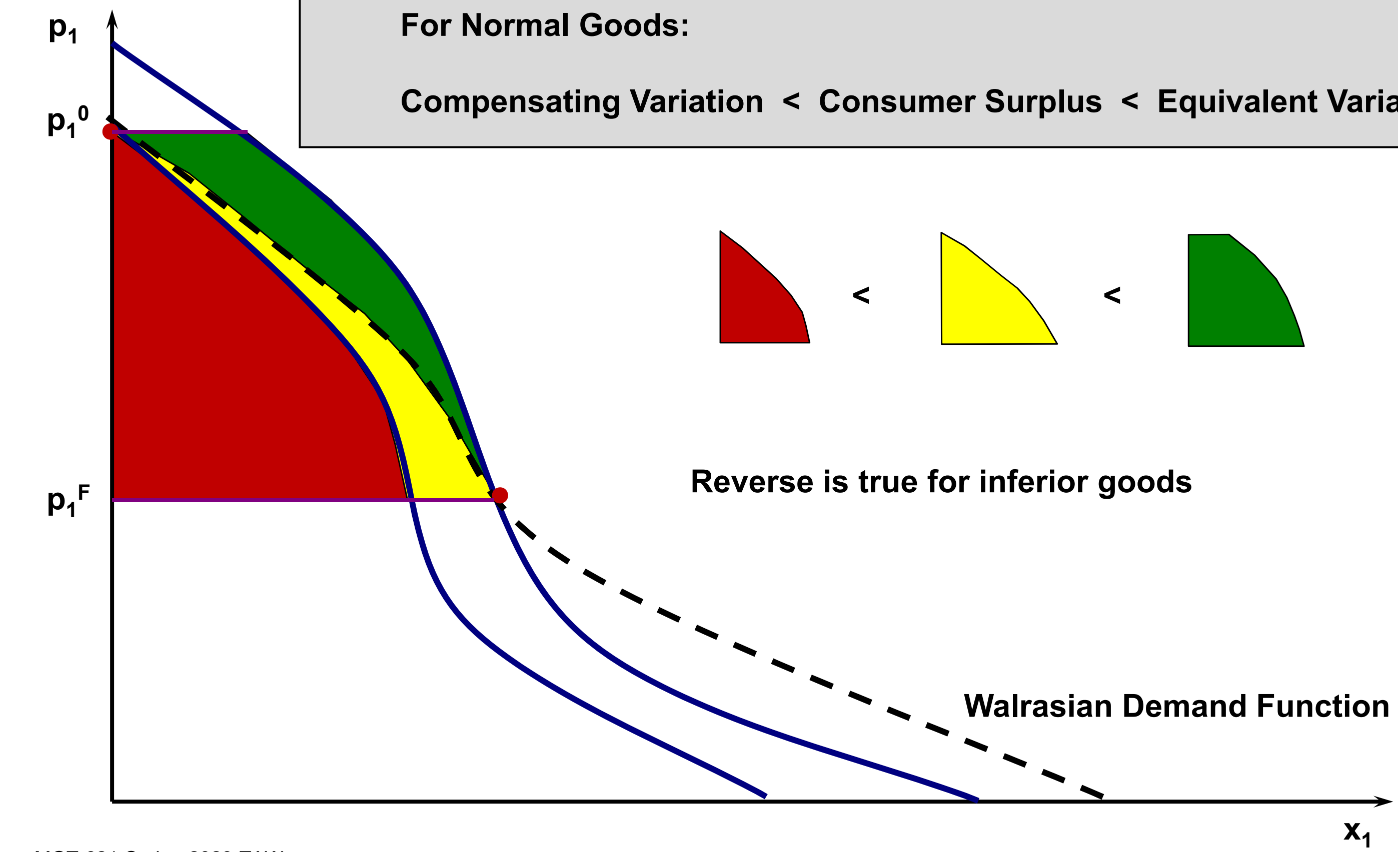



# WELFARE IMPACTS OF PRICE REDUCTION (Cont'd)

**For Normal Goods:**

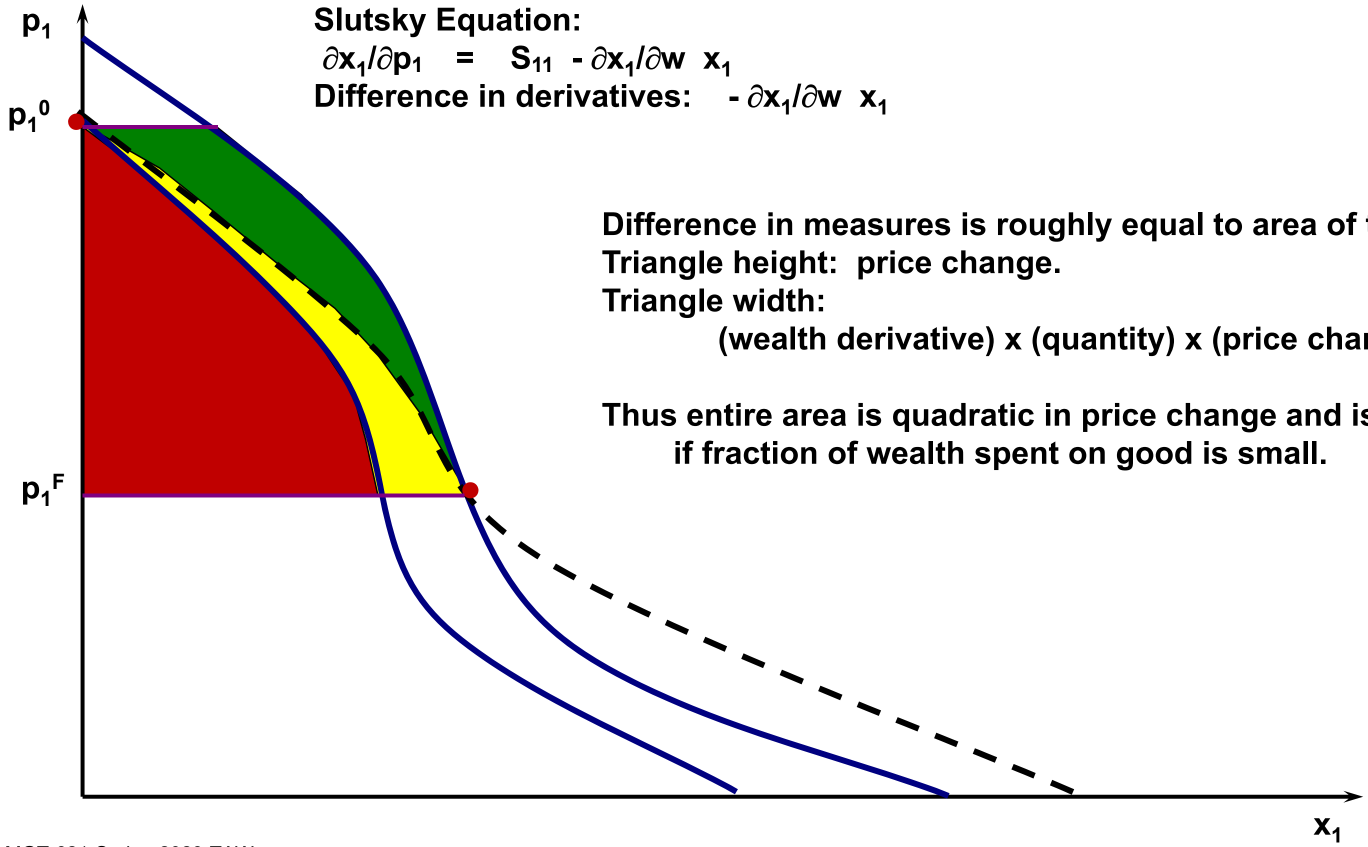
**Compensating Variation < Consumer Surplus < Equivalent Variation**

**Reverse is true for inferior goods**



# HOW DIFFERENT ARE THE THREE MEASURES?

**Slutsky Equation:**
**$\partial x_1/\partial p_1 \;=\; S_{11} - \partial x_1/\partial w \;\; x_1$**
**Difference in derivatives: $- \partial x_1/\partial w \;\; x_1$**




**Difference in measures is roughly equal to area of triangle.**
**Triangle height: price change.**
**Triangle width:**
**(wealth derivative) x (quantity) x (price change).**

**Thus entire area is quadratic in price change and is small if fraction of wealth spent on good is small.**



# EXAMPLE: CONSTANT ELASTICITY OF DEMAND
## Compute Hicksian Compensated Demand Function

$$x(p,w) = \begin{bmatrix} \alpha(w/p_1) \\ (1-\alpha)(w/p_2) \end{bmatrix}$$

**Walrasian Demand Function**
**corresponding to a Cobb-Douglas utility function $u(x) = Kx_1^{\alpha} x_2^{1-\alpha}$ in a two-good economy, where $\alpha \in (0,1),\, K > 0$ .**

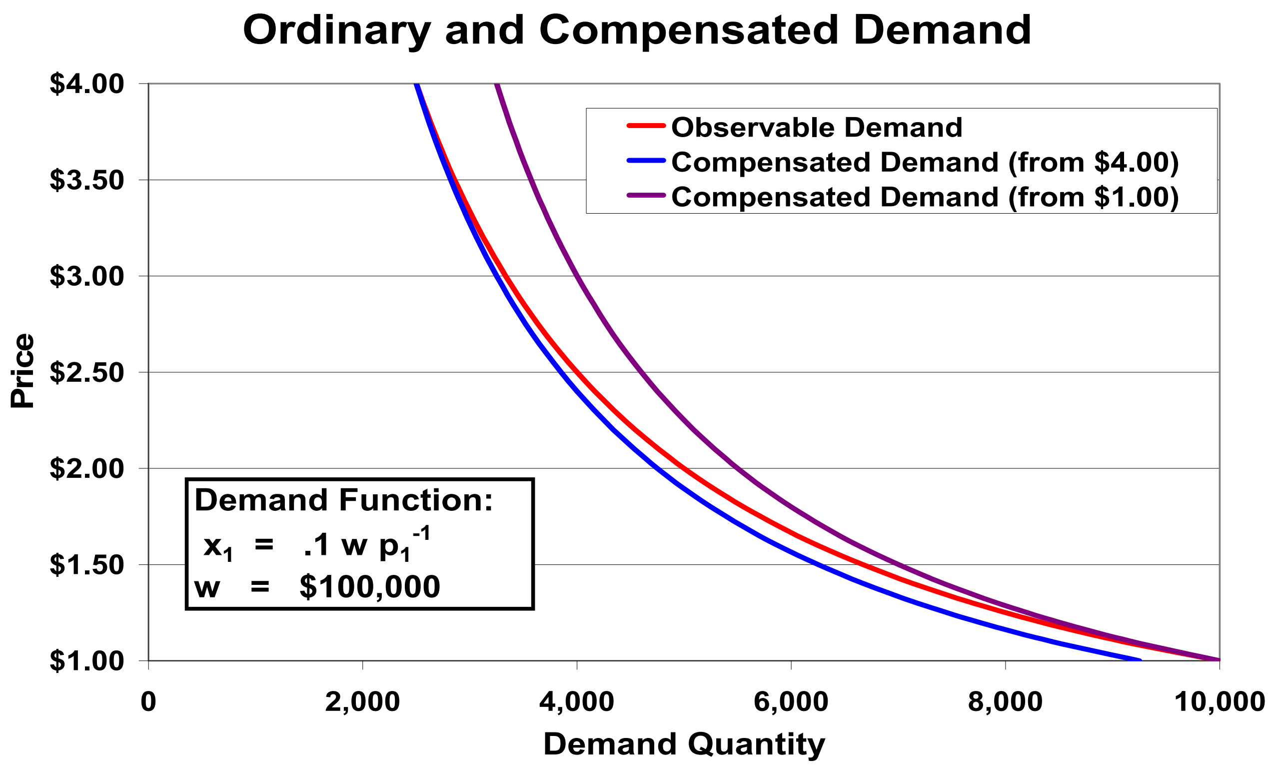
$$U = K(x_1(p,w))^{\alpha}(x_2(p,w))^{1-\alpha} = \frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}{p_1^{\alpha} p_2^{1-\alpha}} Kw \Rightarrow e(p,U) = \frac{p_1^{\alpha} p_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \frac{U}{K}$$

**Hicksian Compensated Demand Function**

$$h_1(p,U) = x_1(p, e(p,U)) = \left(\frac{\alpha\, p_2}{(1-\alpha)\, p_1}\right)^{1-\alpha} \left(\frac{U}{K}\right)$$

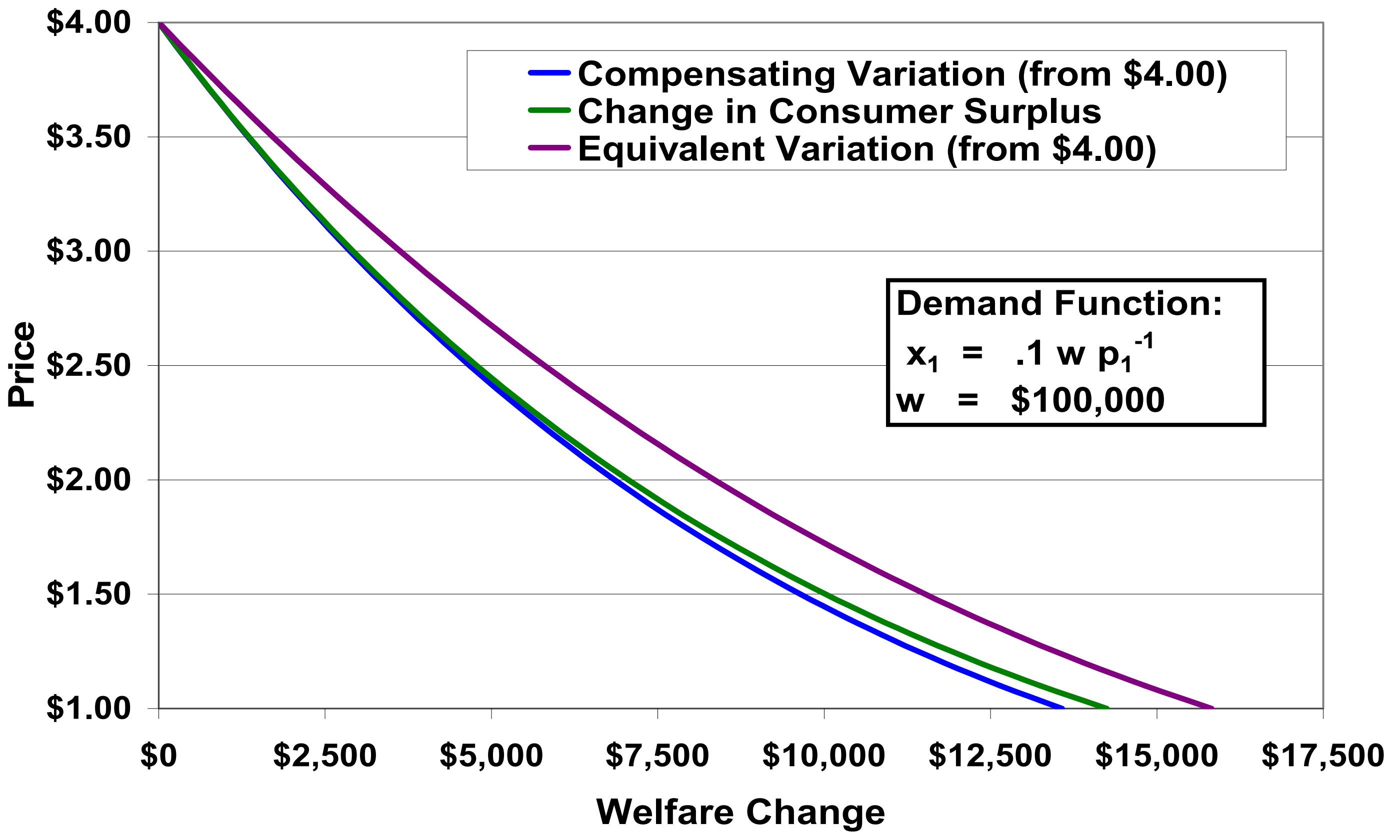
$$h_2(p,U) = x_2(p, e(p,U)) = \left(\frac{(1-\alpha)\, p_1}{\alpha\, p_2}\right)^{\alpha} \left(\frac{U}{K}\right)$$

## EXAMPLE: CONSTANT-ELASTICITY DEMAND






## EXAMPLE: CONSTANT-ELASTICITY DEMAND



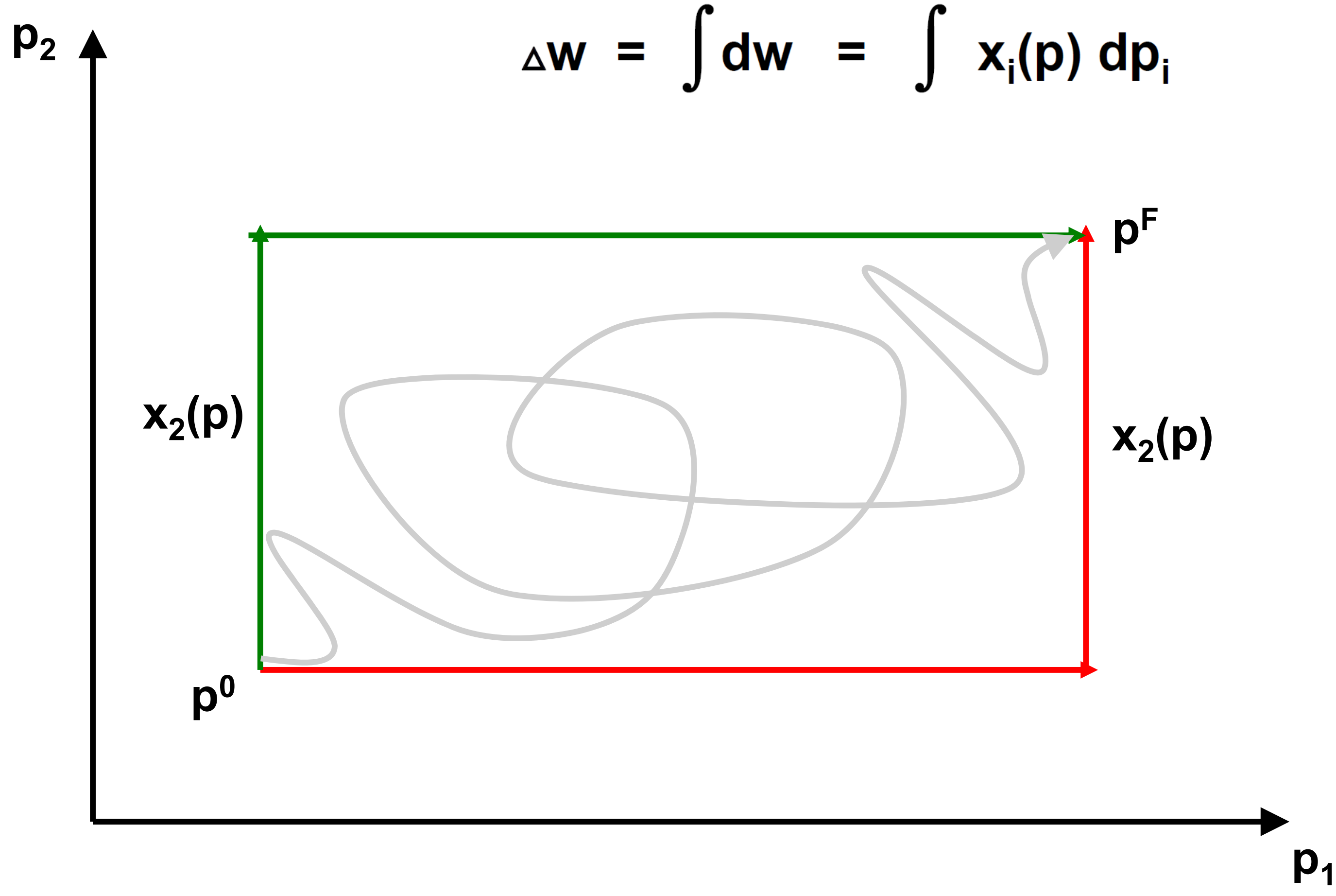
# INTEGRABILITY

- To calculate welfare effects we integrate x(p)dp.
- Integration is along some path of p from $p^0$ to $p^F$
- For measure to be unique the integral must not depend on the path of p from $p^0$ to $p^F$.
- Integrability tells us whether the measure is unique: whether the calculation depends on the path of p.
- For one price changing, all continuous functions are integrable.
- For multiple prices changing, function is integrable if and only if all cross derivatives are symmetric:

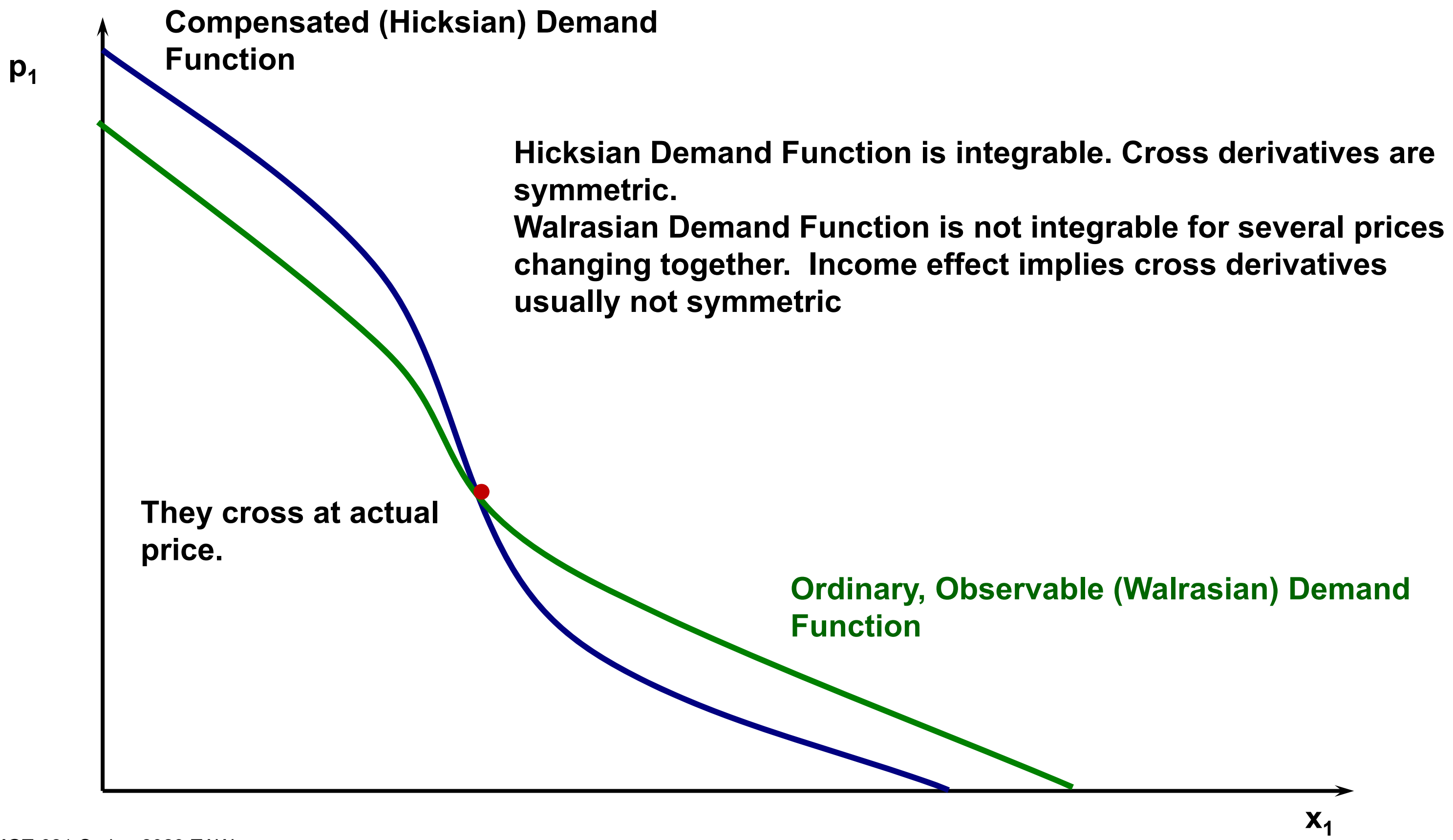
$$\frac{\partial x_i}{\partial p_j} = \frac{\partial x_j}{\partial p_i} \quad \text{for all i,j}$$



$$\Delta w = \int dw = \int x_i(p)\, dp_i$$



# WALRASIAN DEMAND VS. COMPENSATED (HICKSIAN) DEMAND






# SUMMARY OF WELFARE MEASURES

**Equivalent Variation and Compensating Variation**

- **are conceptually precise measurements**
- **are based on integrable demand functions**

**Consumer's Surplus**

- **is not based on conceptually precise concept (e.g., not based on integrable demand functions)**
- **is very easy to measure**
- **is the measure most often seen in calculations**

**All three measures are numerically almost the same UNLESS income effect is large.**

**Measures are routinely used in benefit-cost calculations**

# AGENDA

Some Special Utility Functions

Wealth Effects

Price Effects

Demand Aggregation

Standard Welfare Measures

Welfare Changes

**Key Concepts to Remember**



# KEY CONCEPTS TO REMEMBER

- Special Utility Functions (Cobb-Douglas, CES, Leontief)
- Envelope Theorem
- Monotone Comparative Statics
- Expenditure Minimization Problem
- Hicksian Demand
- Law of Compensated Demand
- Slutsky Compensation (Wealth Compensation)
- Indirect Utility (and Gorman Form)
- Roy's Identity
- Slutsky Equation
- Income Effect & Substitution Effect of a Price Change
- Aggregate Demand
- Bandwaggon Effect & Network Externalities & Fulfilled-Expectations Demand & Critical Mass
- Compensating Variation / Equivalent Variation / Consumer Surplus

## MGT 621 – MICROECONOMICS

**Thomas A. Weber**

*3. Choice Under Uncertainty*

**Autumn 2023**

**École Polytechnique Fédérale de Lausanne**
**College of Management of Technology**



## AGENDA

# CHOICE UNDER UNCERTAINTY

**So far in this course we have assumed that a consumer (decision maker) knows perfectly the consequences of choice.**

**However, in most practical economic decision situations there is uncertainty.**



# UNCERTAINTY IN CHOICE
## Some Examples

- **How much will my education help me in the job market?**
- **Will I be given valuable assignments if I accept this job offer?**
- **Is the used car I am buying a lemon?  Or will it be dependable?**
- **If I put effort into developing a proposal, will it be accepted?**
- **Will an R&D program be successful?**
- **How capable is the person I am considering hiring?**
- **Will my competitors introduce superior new products?**
- **Will potential customers purchase the product I offer?**
- **Will I enjoy the movie?**
- **What will the weather be like in the city I plan to visit?**
- **Will my home catch on fire in the next year and be destroyed?**
- **Will the price of the stock I purchase go up or down?**
- **Will prices for a commodity go up or down? Sign fixed-price contract?**
- **If I take a litigation to trial rather than settling, will I win?**
- **…**

# RISK VS. UNCERTAINTY

**Distinction between risk and uncertainty due to Frank Knight (1921):**

> **"The practical difference between the two categories, risk and uncertainty, is that *in the former the distribution of the outcome in a group of instances is known* (...), *while in the case of uncertainty this is not true*, the reason being in general that it is impossible to form a group of instances, because the situation dealt with is in a high degree unique" (p. 233)**

> **"We can also employ the terms "objective" and "subjective" probability to designate the risk and uncertainty respectively, as these expressions are already in general use with a signification akin to that proposed" (ibid.)**

- **Uncertainty: unknowable (e.g., the success probability of your new startup company, or the likelihood of an unforeseen contingency in your project)**
- **Risk: knowable (e.g., the outcome of a die roll)**

**In this course, *no explicit distinction between risk and uncertainty*, since in order to formally analyze optimal choice, need to introduce a probability space in either case.**



# SUBJECTIVE PROBABILITY
## Some Examples

- **All would not necessarily agree on the likelihood of events**
  - **Probability relevant for your decision is determined by all your knowledge**
  - **Most economic situations can best be described by subjective probabilities**
- **Will the used car be a lemon? Or will it be dependable?**
  - **Seller knows, i.e., for seller probability of being lemon is 0 or 1**
  - **Buyer does not know, i.e., buyer must assign a probability (= "belief")**
- **What will the weather be like in the city I plan to visit?**
  - **Probability assessment may change after you read weather forecast**
- **How capable is the person I am considering for a job?**
  - **Potential employee has more information about work habits**

# SUBJECTIVE PROBABILITY (Cont'd)

New information (e.g., observing other agents' actions) can change beliefs

- What does this imply? (value of information, … disinformation)

Can different people have very different probability assessments given the same choice, the same events, and the same information?

- Agreeing to disagree …

Most of the time we assume that probabilities are subjective



# DISCRETE RANDOM VARIABLES

**Definition**: A **discrete random variable** (or **lottery**) $X = [p_1, x_1; p_2, x_2; \ldots; p_n, x_n]$ is a variable x that can take on one of the values

- $x_1, x_2, x_3, \ldots, x_n$

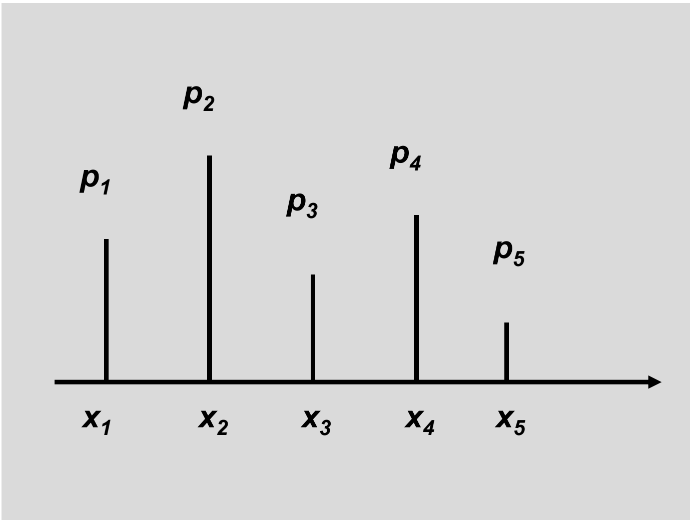
with the respective probabilities

- $p_1, p_2, p_3, \ldots, p_n$

where each $p_i \geq 0$ and

$$\sum_{i=1}^{n} p_i = 1 \qquad \text{(normalization)}$$

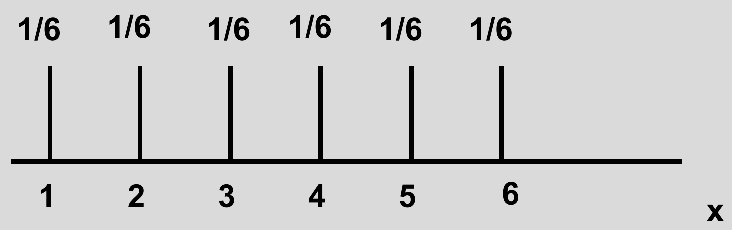
Using the **probability mass function p(.)** we can write the probabilities $p_i$ as a function of $x_i$ : $p_i = p(x_i)$.



# EXAMPLE: ROLL OF A DIE

$$
x = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases} \text{ w/ probability } \begin{matrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{matrix}
$$

| $x_i$ | | $p_i$ |
|---|---|---|
| 1 | | 1/6 |
| 2 | | 1/6 |
| 3 | | 1/6 |
| 4 | w/ probability | 1/6 |
| 5 | | 1/6 |
| 6 | | 1/6 |






# EXPECTATION

**The expectation of a (discrete) random variable X is defined as:**

$$\overline{X} = E[X] = \sum_{i=1}^{n} p_i x_i$$

**Key Property:**

| | |
|---|---|
| **Linearity** | **If X and Y are random variables, and a and b are constants.** **E[aX+bY] = a E[X] + b E[Y]** |

**Other Properties:**

| | |
|---|---|
| **Sign Preservation** | **If X can take on only positive values, then E[X]>0.** |
| **Certain Value** | **If X is perfectly known (and equal to x), then E[X]=x.** |

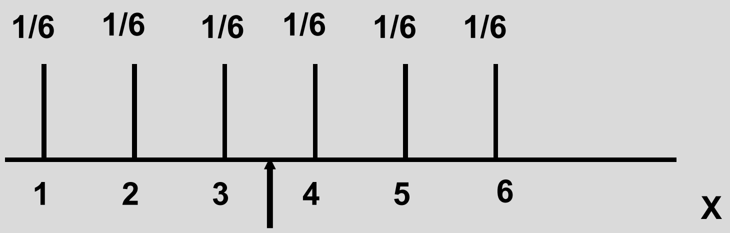
# EXAMPLE (Cont'd)

$$X = \begin{cases} 1 & \\ 2 & \\ 3 & \\ 4 & \text{w/ probability} \\ 5 & \\ 6 & \end{cases} \quad \begin{matrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{matrix}$$

$$E[X] = \sum_{i=1}^{6} p_i x_i$$

$$= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \ldots + \frac{1}{6} \cdot 6$$

$$= \sum_{i=1}^{6} \frac{1}{6} \cdot i = 3.5$$




**The expectation is where this picture would balance on your finger**



# VARIANCE AND STANDARD DEVIATION

**The variance is a measure of the spread of a random variable around its mean.**

$$V[X] = E[(X - \overline{X})^2] \ = E[(X^2 - 2X\overline{X} + \overline{X}^2)]$$

$$= E[X^2] - 2\overline{X}E[X] + \overline{X}^2 \quad = E[X^2] - \overline{X}^2$$

**The standard deviation is the square root of the variance.**

$$\sigma_X = \sqrt{V[X]}$$

**It has the same units as X.**

# EXAMPLE (Cont'd)

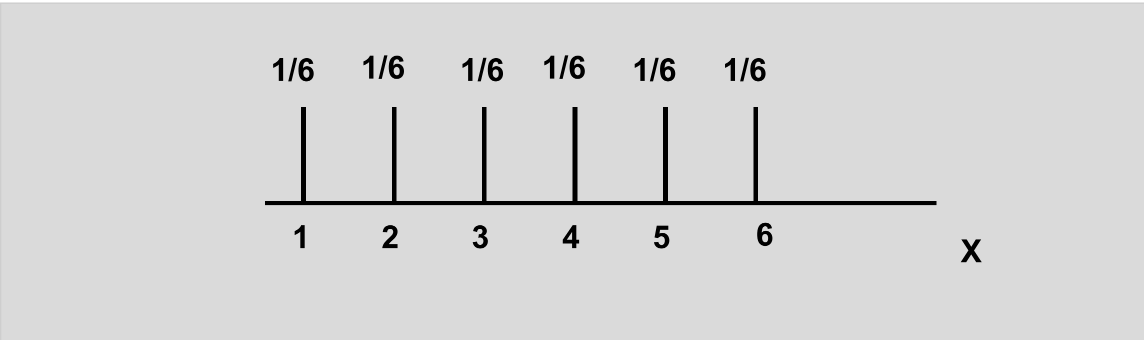
$$X = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases} \quad \text{w/ probability} \quad \begin{matrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{matrix}$$

$$V[X] = E[(X-\overline{X})^2]$$

$$= \frac{1}{6}(1-3.5)^2 + \frac{1}{6}(2-3.5)^2 + \ldots$$

$$+\frac{1}{6}(6-3.5)^2$$

$$= 2.92$$






# CONDITIONAL PROBABILITY

**Definition: The conditional probability P(A|B) of event A conditional on event B having realized is <u>defined</u> as**

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

**Question. Given a six-sided die what is the probability of rolling a 3 conditional on rolling a 1, 2 or 3?**

$$P(\{3\} \mid \{1,2,3\}) = \frac{P(\{3\} \cap \{1,2,3\})}{P(\{1,2,3\})} \qquad = \frac{P(\{3\})}{P(\{1,2,3\})}$$

$$= \frac{1/6}{1/2} = 1/3$$

# INDEPENDENCE

**Definition: A and B are independent events, if**

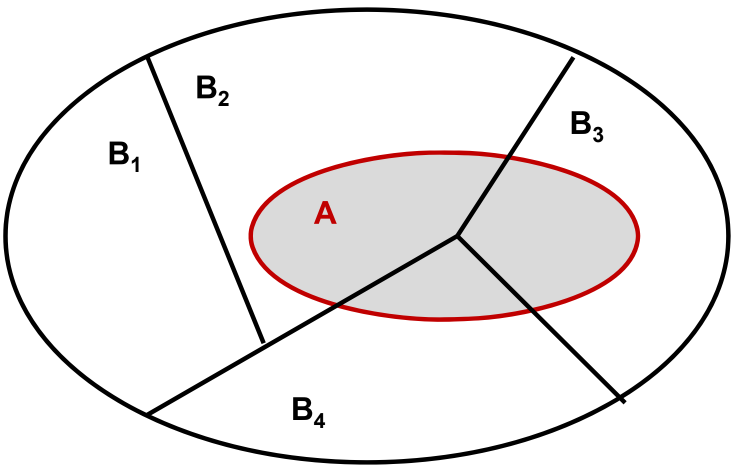
- **P(A|B) = P(A)**
- **P(B|A) = P(B)**

$$P(A \cap B) = P(A \mid B)P(B)$$



# THE LAW OF TOTAL PROBABILITY

$$P(A) = \sum_{n=1}^{N} P(A \cap B_n) = \sum_{n=1}^{N} P(A \mid B_n)P(B_n)$$



# AGENDA

**Elements of Probability**

**Choice Under Uncertainty**

**Expected Utility Theory**

**Risk Aversion and Decision Biases**

**Key Concepts to Remember**



# LOTTERIES ARE (DISCRETE) RANDOM VARIABLES

**Let X be a random variable with possible outcomes in the set $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ (the "outcome space"). Each outcome $x_i$ occurs with probability $p_i$.**

**The random variable X is sometimes also called a lottery and denoted**

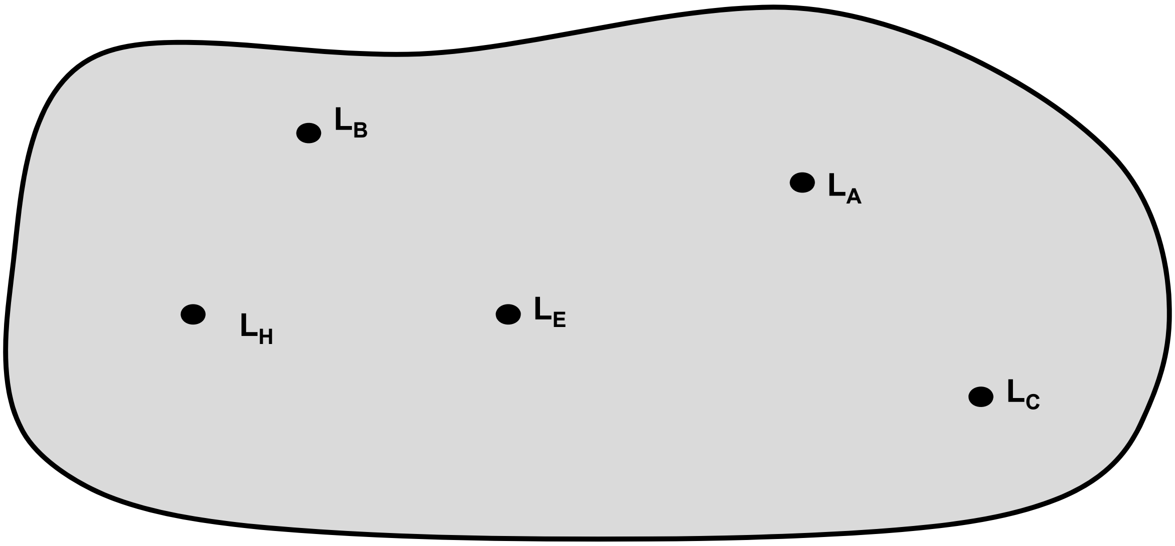
$$X = [p_1, x_1; p_2, x_2; \dots; p_n, x_n]$$

**If all outcomes $x_i$ are real and measured in dollars (or any other currency), then X is commonly referred to as a "money lottery."**

**The set of all lotteries with outcomes in $\mathcal{X}$ is the "lottery space" $L(\mathcal{X})$.**

**Example: A coin-flip lottery X (with an unbiased coin) pays \$1 if heads and zero if tails. Then X = [0.5,\$1; 0.5, \$0].**

# THEORY OF CHOICE UNDER UNCERTAINTY




**The choice set contains all simple lotteries over the various outcomes.**



# PREFERENCES OVER LOTTERIES

**For a decision maker (DM), *choosing between actions corresponds generally to choosing between lotteries* (given a decision d in the set of possible decisions D, the outcome $x_i$ in *X* occurs with probability $P(x_i|d)$).**

**Example: wearing a helmet on a motorcycle changes the probability of injury. In a decision tree, each decision node represents a lottery.**

**Hence, the DM needs to be able to order lotteries according to his preference implying a complete preference (pre-)ordering[(1)] over elements in the lottery space L(*X*).**
**More specifically, if A and B are elements of L(*X*), then**

- **A > B   means DM prefers A to B**
- **A $\geq$ B   means DM does not prefer B to A or DM "weakly" prefers A to B**
- **A ~ B   means  DM is indifferent between A and B, i.e., DM will take either one and would play a 50-50 lottery to choose between them.**

(1) Formally, a pre-ordering R is a *binary relation* (i.e., it takes two inputs) that is *reflexive* (i.e., xRx) and *transitive* (i.e., xRy and yRz implies xRz). The pre-ordering is *complete* if for any elements x and y in *X* it is either xRy or yRx (or both).

# PREFERENCES OVER LOTTERIES: EXAMPLE

**Question**. Jane likes to play ping pong and she wonders about how to respond to an opponent's serve.

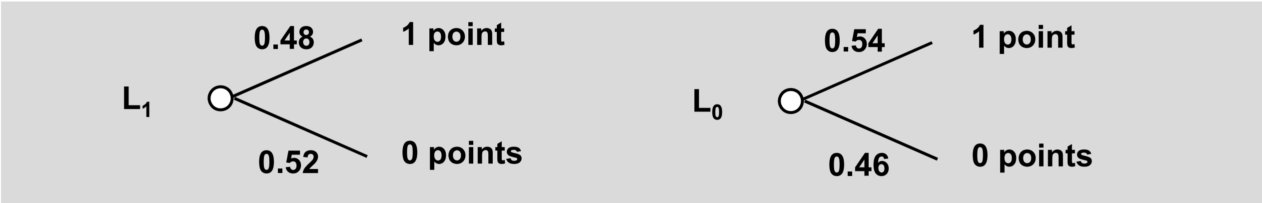
- If she hits a top spin (decision $d_1$), the ball is going to be on the table with probability 0.6 and given that it is, she is going to score with probability 0.8.
- If she does not play a top spin (decision $d_0$), the ball is going to land on the table with probability 0.9, but she is only going to score with probability 0.6.

What decision should she take?

Solution:

Jane needs to choose between the following two lotteries:

- $L_1$ = [P(score|$d_1$), 1 point; P(don't score|$d_1$), 0 points]
- $L_0$ = [P(score|$d_0$), 1 point; P(don't score|$d_0$), 0 points]




Thus, she should prefer $L_0$ (i.e., $L_0 \geq L_1$) which implies "don't play top spin" as her decision.



# PREFERENCES OVER OUTCOMES
## Utility Representation (Reminder)

Preferences over lotteries imply preferences over particular (certain) outcomes in *X*. Indeed, for any x and y in *X* one could just consider the lotteries X and Y that produce the outcomes x and y with probability one respectively.

Thus, $x \geq y$ if the outcome x is (weakly) preferred to the outcome y.

> Definition: A real-valued function u with domain *X* represents the DM's preferences over outcomes in *X*, if for any x,y in *X*:
>
> $$x \geq y \text{ if and only if } u(x) \geq u(y).$$
>
> The function u is called the DM's utility function.

For some preferences no utility function representation exists (e.g., lexicographic preferences). We typically take a utility function as an input for a decision model.(1) A utility function always exists for finite sets of outcomes.

A *utility representation of a DM's preferences is generally* not unique: given any utility function u and a strictly increasing function $\phi$ (from real numbers to real numbers), the function $v = \phi(u)$ is an equivalent utility representation.

(1) For more details on utility theory see Kreps, D.M. (1988) *Notes on the Theory of Choice*, Westview Press, Boulder, CO, or Fishburn, P.C. (1970) *Utility Theory for Decision Making*, Wiley, New York, NY.

# UTILITY REPRESENTATION

**Mark has the utility function $u(x) = x^{½}$ for any nonnegative amount of money x (in dollars).**

**Thus, he prefers \$100 to \$36, since u(100) = 10 > u(36) = 6.**

**Similarly, for $\phi(y) = y^2$, we have that $\phi(u(100)) = 100 > \phi(u(36)) = 36$, and Mark would have the same preferences for outcomes (amounts of money) for any other $\phi$, as long as $\phi$ is strictly increasing.**

- **$x \geq y$ if and only if $u(x) \geq u(y)$ if and only if $\phi(u(x)) \geq \phi(u(y))$**

**Thus, in the absence of uncertainty Mark can just maximize x instead of u(x).**



# AGENDA

**Elements of Probability**

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**Risk Aversion**

**Key Concepts to Remember**

# EXPECTED UTILITY MAXIMIZATION

**Given a utility representation u of the DM's preferences over outcomes, we would like to infer his preferences over lotteries of outcomes (random variables), which corresponds to his preferences over actual decisions (e.g., at which speed to drive a car).**

**Under certain axioms (= assumptions on the DM's preferences over lotteries, typically one uses the Von Neumann-Morgenstern axioms), the DM's expected utility of a particular decision d in D which induces a lottery X(d) with probability distribution P(.|d) is**

$$EU(X(d)) = \sum_{x \in X} P(x \mid d) u(x)$$

**Thus, under uncertainty the DM maximizes expected utility, i.e., he solves**

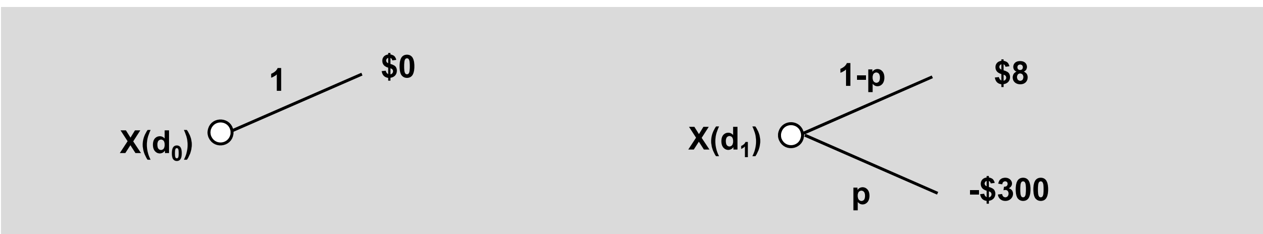
$$\boxed{d^* \in \arg\max_{d \in D} EU(X(d))}$$



# EXPECTED UTILITY MAXIMIZATION: EXAMPLE

**Question. Joe needs to decide how fast to drive on highway E25 from Lausanne to Geneva. Any minute saved he values at \$1. At 120 km/h it takes about 40 min, and at 140 km/h about 32 min. However, if he drives 140 km/h there is a chance p that he gets pulled over and has to pay a ticket worth \$280 plus a delay of 20 min (over the 120 km/h time).**

**Let $d_0$ : drive 120 km/h, $d_1$ : drive 140 km/h. He thus needs to choose between the lotteries $X(d_0)$ = [1,\$0] and $X(d_1)$= [p, -\$300; 1-p, \$8].**




**If Joe's utility function for money is u(x)=-exp(-x/1000), at what detection probability p would he be indifferent between $d_0$ and $d_1$?**

**Answer:** $EU(X(d_0)) = u(\$0) = EU(X(d_1)) = pu(-\$300) + (1-p)u(\$8)$

$$\Rightarrow \quad p = \frac{u(\$8) - u(\$0)}{u(\$8) - u(-\$300)} = 2.2\%$$

**(i.e., for p>2.2%, Joe drives 65 mph!)**

# VON NEUMANN-MORGENSTERN AXIOMS IMPLY EXPECTED UTILITY REPRESENTATION

## Fundamental Justification for Expected Utility Maximization

**Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of outcomes and L($X$) the corresponding lottery space. Consider arbitrary elements A,B,C of L($X$) and x,y,z in $X$.(1)**

1. **Completeness: A $\geq$ B or B $\geq$ A (can compare any two lotteries)**
2. **Reflexivity: A $\geq$ A**
3. **Transitivity: A $\geq$ B and B $\geq$ C implies A $\geq$ C (o/w construct a "money pump")**

4. **Continuity: If x > y > z, then there exists p in (0,1) such that one can achieve indifference between the lottery A = [1, y] and the lottery B(p) = [p, x; 1-p, z], i.e., there is a p such that A ~ B(p)**

- **Example: What happens if z is "death"?**

5. **Independence (of Irrelevant Alternatives): If x > y, then for any z:**
   **[p, x; (1-p), z] > [p, y; (1-p), z]**

(1) Note that if x is an outcome in $X$, then [1, x] is a lottery in L($X$). Thus, any outcome can be viewed as a (degenerate) lottery.



# KEY IMPLICATION OF THE VNM AXIOMS

**Theorem (Von Neumann-Morgenstern, 1944): If a DM's preferences "$\geq$" on L($X$) satisfy the Von Neumann-Morgenstern axioms, then there exists an expected utility function EU(.) for that DM which represents his preferences in the sense that for any two lotteries A, B in L($X$):**

**A $\geq$ B if and only if EU(A) $\geq$ EU(B)**

Proof: See notes on "Risk and Uncertainty" posted on the course website.

**In more detail, suppose:**

**A = $[p_1,x_1; p_2,x_2; \dots ; p_n,x_n]$, where $p_1 + \dots + p_n = 1$**

**B = $[q_1,x_1; q_2,x_2; \dots ; q_n,x_n]$, where $q_1 + \dots + q_n = 1$**

**If DM's preferences satisfy the VNM Axioms, then the DM prefers A to B (A $\geq$ B) if and only if**

$$EU(A) = \sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{n} q_i u(x_i) = EU(B)$$

**Note: Two expected utility functions EU(x) and EV(x) represent the same preferences over lotteries if and only if EU(x) = $\alpha$EV(x) + $\beta$, where $\alpha > 0$ and $\beta$ is any real number. In the same vein, a "positive affine transformation" of the DM's utility function u (to v = $\alpha$ u + $\beta$, for $\alpha > 0$) does not change the DM's preferences over lotteries.**

The original reference for the Von Neumann-Morgenstern result is their immensely readable text Von Neumann, J., Morgenstern, O. (1944) *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ.

# KEY IMPLICATION OF THE VNM AXIOMS (Cont'd)

**Example**: Consider the utility functions $u(x) = 1 - 2e^{-\rho x}$ and $v(x) = 3 - 5e^{-\rho x}$, where $\rho>0$ is some constant.

**Since $u(x) = .4v(x) - .2 = \alpha v(x) + \beta$ (with $\alpha > 0$), one can check that for any two lotteries A and B in L($X$):**

$$EU(A) \geq EU(B) \quad \text{if and only if} \quad EV(A) \geq EV(B)$$

**More generally, invariance with respect to positive affine transformations implies that we can fix any two values of a DM's utility function without disturbing his preference ordering over lotteries.**



# ALLAIS PARADOX

**Imagine the following two decision situations—each involving a pair of gambles.**

**SITUATION I**

| | P(Winning) | AMOUNT TO WIN |
|---|---|---|
| Lottery A | 100% | $1,000,000 |
| Lottery B | 10% | $5,000,000 |
| | 89% | $1,000,000 |
| | 1% | -0- |

**SITUATION II**

| | P(Winning) | AMOUNT TO WIN |
|---|---|---|
| Lottery C | 11% | $1,000,000 |
| | 89% | -0- |
| Lottery D | 10% | $5,000,000 |
| | 90% | -0- |

# ALLAIS PARADOX
## Implies Critique of ‘Independence of Irrelevant Alternatives’

| | Ticket Numbers | | |
|---|---|---|---|
| Lottery | 1 | 2-11 | 12-100 |
| A | $1 million | $1 million | $1 million |
| B | $0 million | $5 million | $1 million |
| C | $1 million | $1 million | $0 million |
| D | $0 million | $5 million | $0 million |

**NOTE: Ticket numbers 12-100 are the same for A and B so they are irrelevant for choices between them; the same for C and D**

**If you eliminate these ticket numbers then**
**A and C are identical and so are B and D**



# AGENDA

**Elements of Probability**

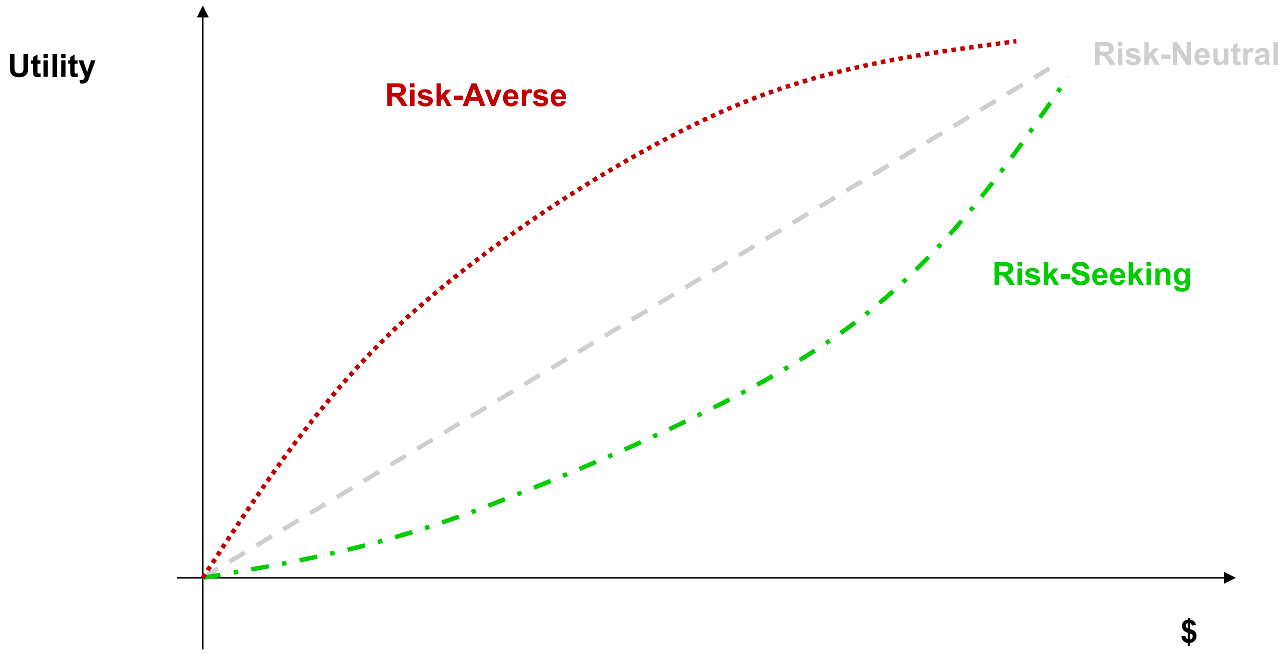
**Choice Under Uncertainty**

**Expected Utility Theory**

**Risk Aversion and Decision Biases**

**Key Concepts to Remember**

# UTILITY FUNCTIONS: SOME COMMON SHAPES






# RISK AVERSION: EXAMPLES

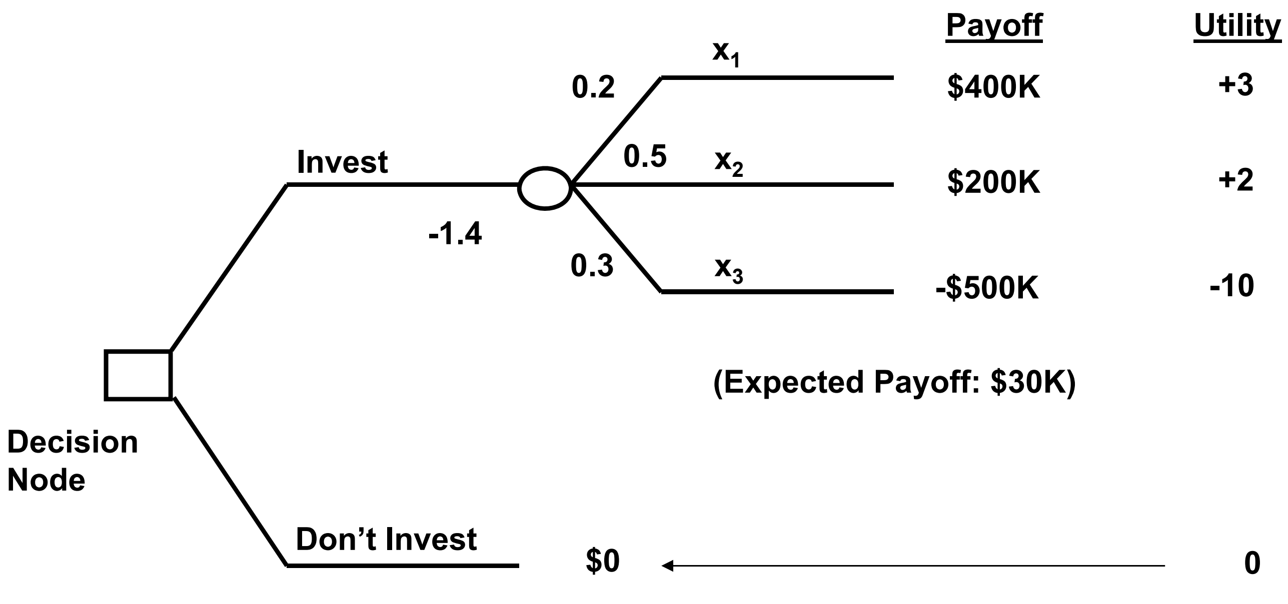
**People will buy insurance, even though the expected value of the payment from insurance is smaller than its price. That is the result of risk aversion.**

**People purchase a portfolio of stocks and bonds, rather than only one. Such diversification reduces risk and is consistent with risk aversion.**

**People will incur costs to purchase hedges, assets that reduce the risk of the overall portfolio.**

**Typically the larger the monetary lottery, the greater the degree of risk aversion people exhibit.**

# A SIMPLE DECISION: RISK-AVERSE DM

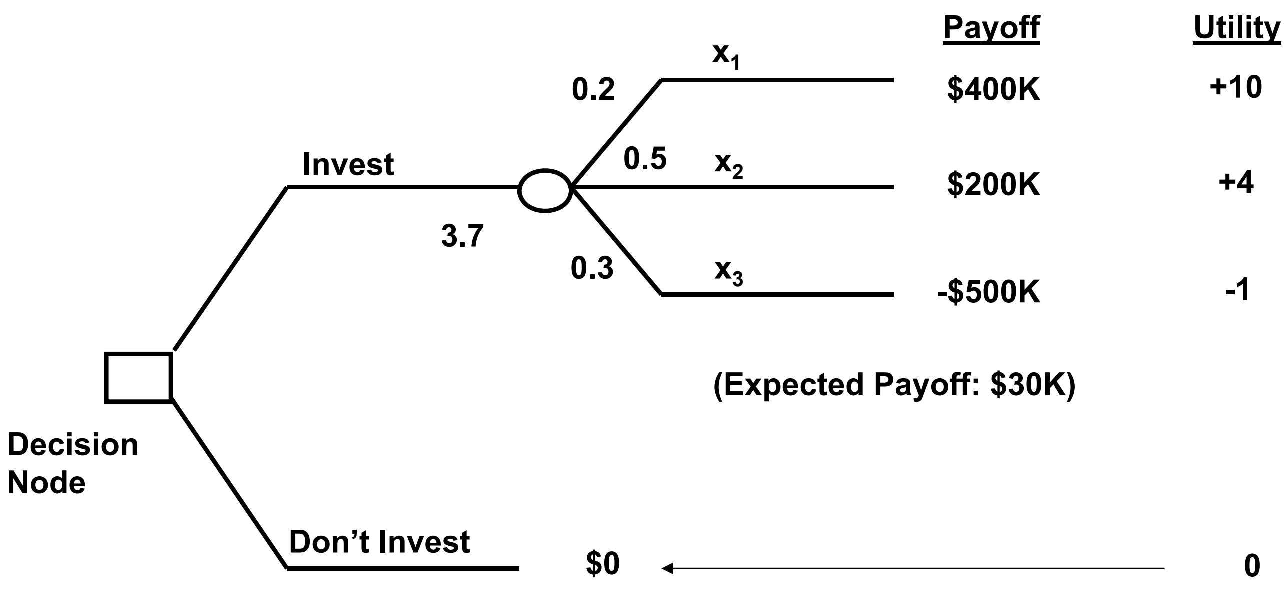



**Decision Criterion: Maximize Expected Utility**

**EU = (0.2)(3)+(0.5)(2)+(0.3)(-10) = -1.4 < 0** → **Don't Invest!**



# A SIMPLE DECISION TREE: RISK-SEEKING DM




**Decision Criterion: Maximize Expected Utility**

**EU = (0.2)(10)+(0.5)(4)+(0.3)(-1) = 3.7.** **Invest!**

## ABSOLUTE AND RELATIVE RISK AVERSION DESCRIBE A DM'S RISK ATTITUTE

**The level of risk aversion may be measured by the (Arrow-Pratt) absolute-risk-aversion coefficient,**

$$R(x) = -\frac{u''(x)}{u'(x)}$$

**or the relative-risk-aversion coefficient,**

$$r(x) = xR(x)$$

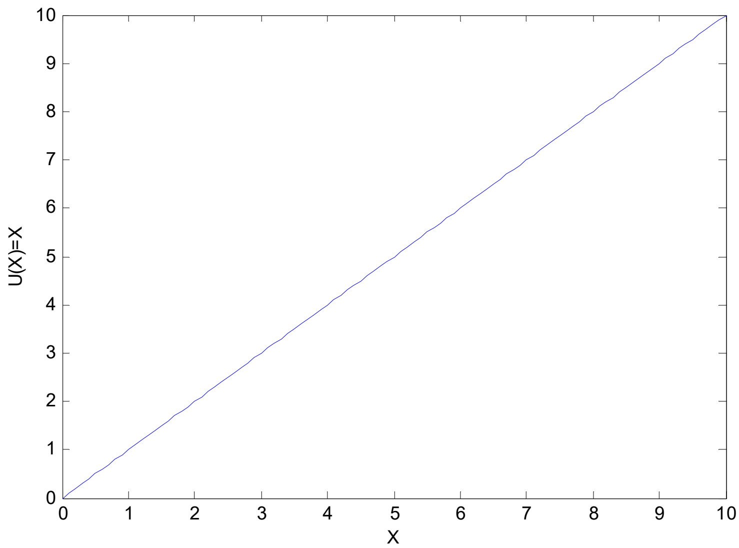
**If R(x) > 0, the DM is risk-averse. Similarly, if R(x) < 0, the DM is risk-seeking, while R(x)=0 for a risk-neutral DM.**

**Both absolute and relative risk aversion are local properties: they can vary for different outcomes.**



## RISK NEUTRALITY: LINEAR UTILITY

$$u(x) = \alpha\, x \qquad u'(x) = \alpha \qquad u''(x) = 0$$




$$(\alpha > 0)$$

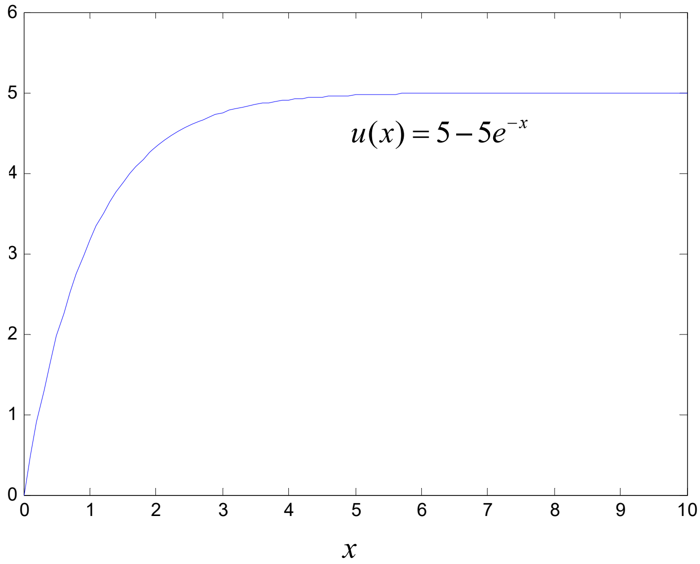
$$R(x) = -\frac{u''(x)}{u'(x)} = 0$$

# CONSTANT ABSOLUTE RISK AVERSION

**Exponential utility:** $u(x) = \alpha - \beta e^{-\rho x}$

$$u'(x) = \rho\beta e^{-\rho x}$$

$$u''(x) = -\rho^2 \beta e^{-\rho x}$$




$$R(x) = -\frac{u''(x)}{u'(x)} = \rho$$

**Exponential utility functions exhibit constant absolute risk aversion (CARA).[1]**

(1) CARA utility functions are often used in financial modeling, since it allows obtaining conclusions free from wealth effects (adding a constant w to an individual's wealth just amounts to a positive linear transformation and thus leads to the same decisions, since the expected utility representation of the individual's preferences does not change).



# CERTAINTY EQUIVALENT

**The certainty equivalent of a lottery is a single certain outcome for which the DM is indifferent between receiving the outcome for sure and participating in the lottery.**

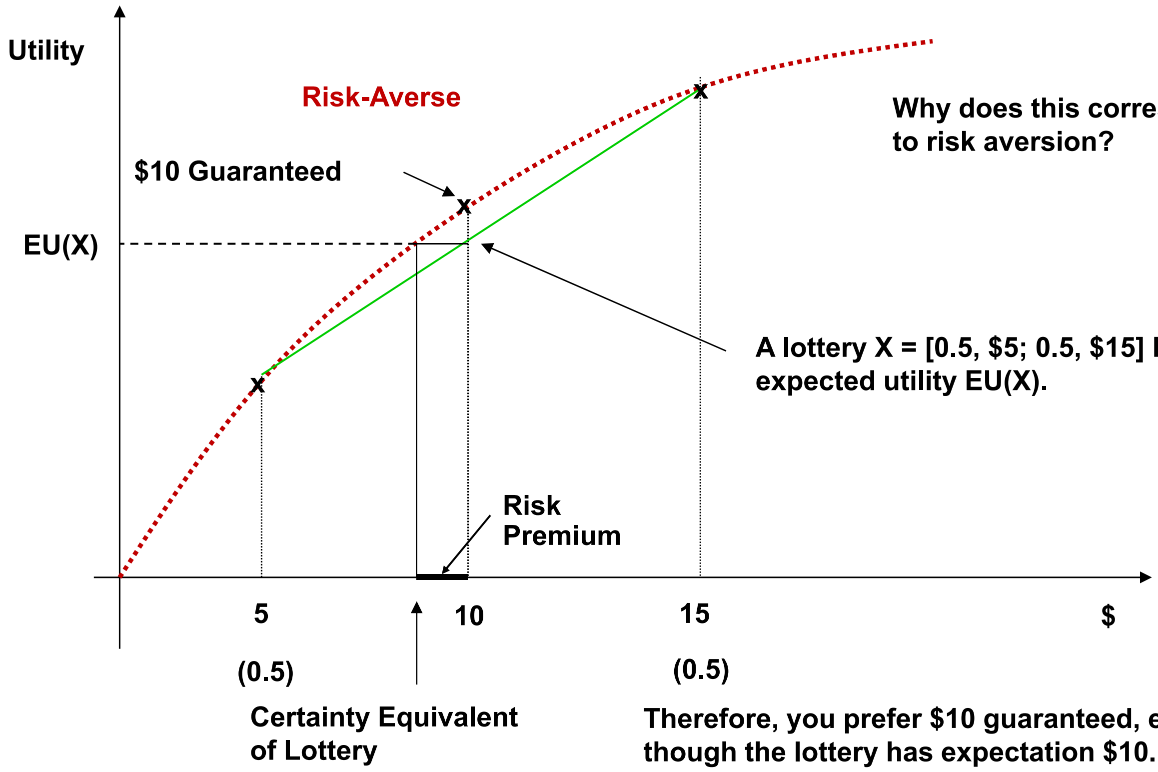
**It represents the "selling price" of the lottery.**

**Denote the certainty equivalent of a lottery X by** $CE(X)$

**Then:** $u(CE(X)) = EU(X)$ **(DM is indifferent)**

$$\Rightarrow CE(X) = u^{-1}(EU(X))$$

# CERTAINTY EQUIVALENT: EXAMPLE




Why does this correspond to risk aversion?

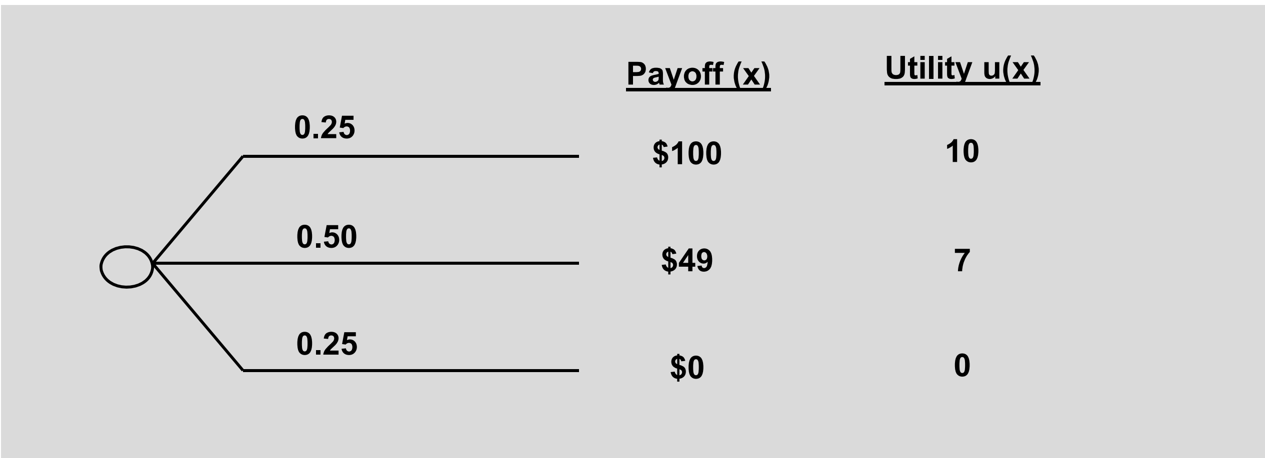
A lottery X = [0.5, \$5; 0.5, \$15] has expected utility EU(X).

Therefore, you prefer \$10 guaranteed, even though the lottery has expectation \$10.



# CERTAINTY EQUIVALENT: ANOTHER EXAMPLE

**Consider the lottery X = [.25, \$100; .5, \$49; .25, \$0] and the utility function** $u(x) = \sqrt{x}$




**Expected utility: EU(X) = (0.25)(10) + (0.5)(7) + (0.25)(0) = 6**

$$EU(X) = u(CE) = \sqrt{CE} \quad \Longrightarrow \quad \sqrt{CE} = 6 \quad \Longrightarrow \quad CE = 36$$

# EXAMPLE: CONSTRUCTING A UTILITY FUNCTION FOR MONEY

**Arbitrarily assign utilities to two real-valued outcomes, $x_1$ and $x_2$ (say, measured in dollars). For example, $x_1$ = -\$128 and $x_2$ = \$128, and**

$$u(-\$128) = -100 \quad \text{and} \quad u(\$128) = 100.$$

**Use continuity axiom to specify other utilities.**

**<u>Certainty Equivalence Method</u>: Fix p and two outcomes $x_1$ and $x_2$. Then find an outcome y which makes you indifferent between having y for certain or taking the lottery $[p, x_1; 1-p, x_2]$.(1)**

**The value y is commonly referred to as the certainty equivalent (CE) of the lottery $[p,x_1;1-p,x_2]$: y = CE,**

$$u(CE) = p\, u(x_1) + (1-p)\, u(x_2).$$

(1) In a set of discrete outcomes such an element y might not be available. Then one needs to adjust the probability p accordingly, which by the continuity axiom can always achieve indifference.



# A MARKET FOR COIN FLIPS

**Consider the following game. You flip a fair coin.**

- **If the first flip is heads (H) you win \$2 and you flip the coin again. If it is tails (T), the you win \$0 and the game is over.**
- **If the second flip is H you win \$4 and you flip the coin again. If it is T, then you keep the \$2 you won on the first flip and the game is over.**
- **If the n-th flip is H you win $\$2^n$ and you flip the coin again. If it is T, then you keep the $\$2^{n-1}$ you won on the (n-1)st flip and the game is over.**

**The following table summarizes the outcome (we restrict the length to $n \leq 7$ to avoid bankruptcy of players).**

| Number of Heads in a Row (n) | Total Winnings |
|---|---|
| 1 | \$2 |
| 2 | \$4 |
| 3 | \$8 |
| 4 | \$16 |
| 5 | \$32 |
| 6 | \$64 |
| 7 | \$128 |

# A MARKET FOR COIN FLIPS (Cont'd)

**Please answer the following question (depending on your role):**

- **Bankers: if you are a banker (i.e., act as a bank in this game), how much would you need to be paid for sure to run the game? The person with the lowest amount will serve as the banker and play the game for real.**

- **Players: if you are a player (i.e., you get to potentially win in this game), how much would you be willing to pay to participate in the game? The person with the highest amount will play the game for real.**



# MARKET FOR COIN FLIPS: ACTUARIAL VALUE

**Let us compute the actuarial value of the coin-flip game X:**

$$E[X] = (½)\ \$0 + (¼)\ \$2 + (1/8)\ \$4 + \ldots + (1/2^n)\ \$2^{n-1} + (1/2^n)\ \$2^n$$

$$= (n-1)\ (\$0.5) + \$1 \underset{n = 7}{=} \$4.$$

**The certainty equivalent CE(X) of the coin-flip lottery X *for a player*,(1) given a utility function u, satisfies therefore:**

$$u(CE(X)) = (½)\ u(\$0) + (¼)\ u(\$2) + (1/8)\ u(\$4) + \ldots + (1/2^n)\ u(\$2^{n-1}) + (1/2^n)\ u(\$2^n) = EU(X),$$

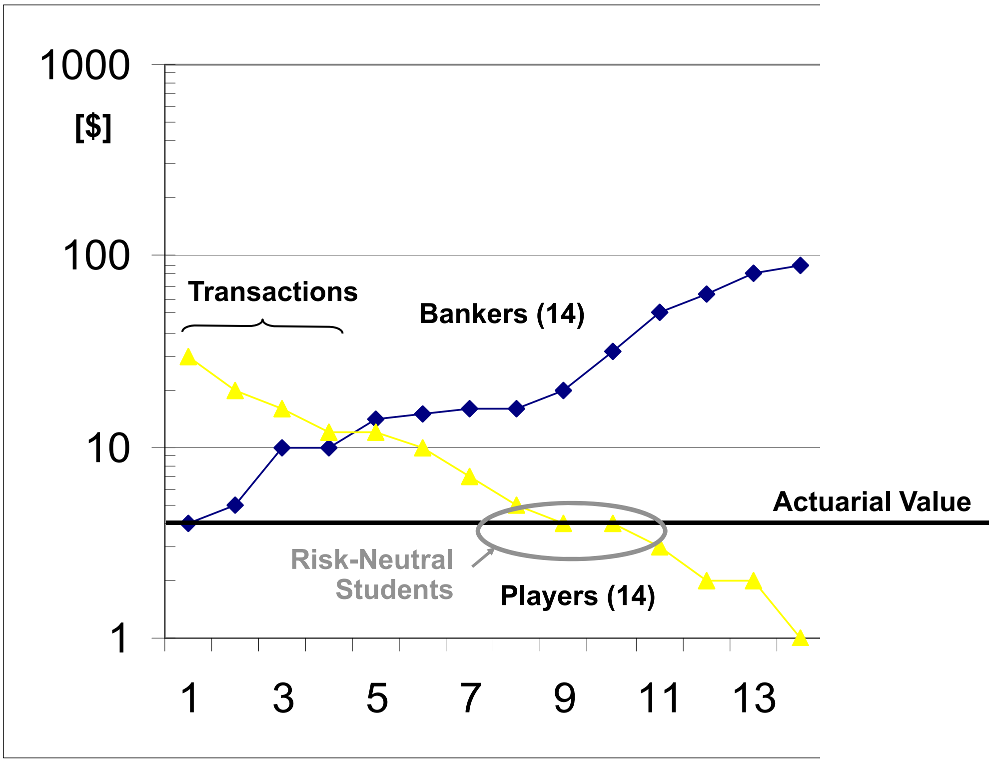
**so that**

$$CE = u^{-1}(EU(X))$$

**You can read the utilities off your utility function constructed a couple of slides ago (for n = 7)**

(1) For a *banker* one can compute the CE in a similar way, just by taking the outcomes in terms of losses instead of gains.

## MARKET FOR COIN FLIPS: CLASS RESULTS




**The plotted values correspond to the certainty equivalents of bankers and players respectively. Why are they not the same?**



## INDIVIDUALS ARE RISK-AVERSE IN GAINS AND RISK-SEEKING IN LOSSES (SENSITIVITY TO REFERENCE POINT)

### We already did this experiment!

**1. You have been given $200 and have a choice between the following two options**

**A: Win $150 with certainty**

**B: Win $300 with probability .5**

**Win $0 with probability .5**

- **Do you prefer A or B ?**

**2. You have been given $500 and have a choice between the following two options**

**C: Lose $150 with certainty**

**D: Lose $300 with probability .5**

**Lose $0 with probability .5**

- **Do you prefer C or D?**

# SENSITIVITY TO REFERENCE POINT: CLASS RESULTS

| | | Gamble C | Loss Averse<br>Gamble D |
|---|---|---|---|
| Risk Averse | Gamble A | 35 | 28 |
| | Gamble B | 7 | 8 |

Rational choices (satisfying VNM axioms)

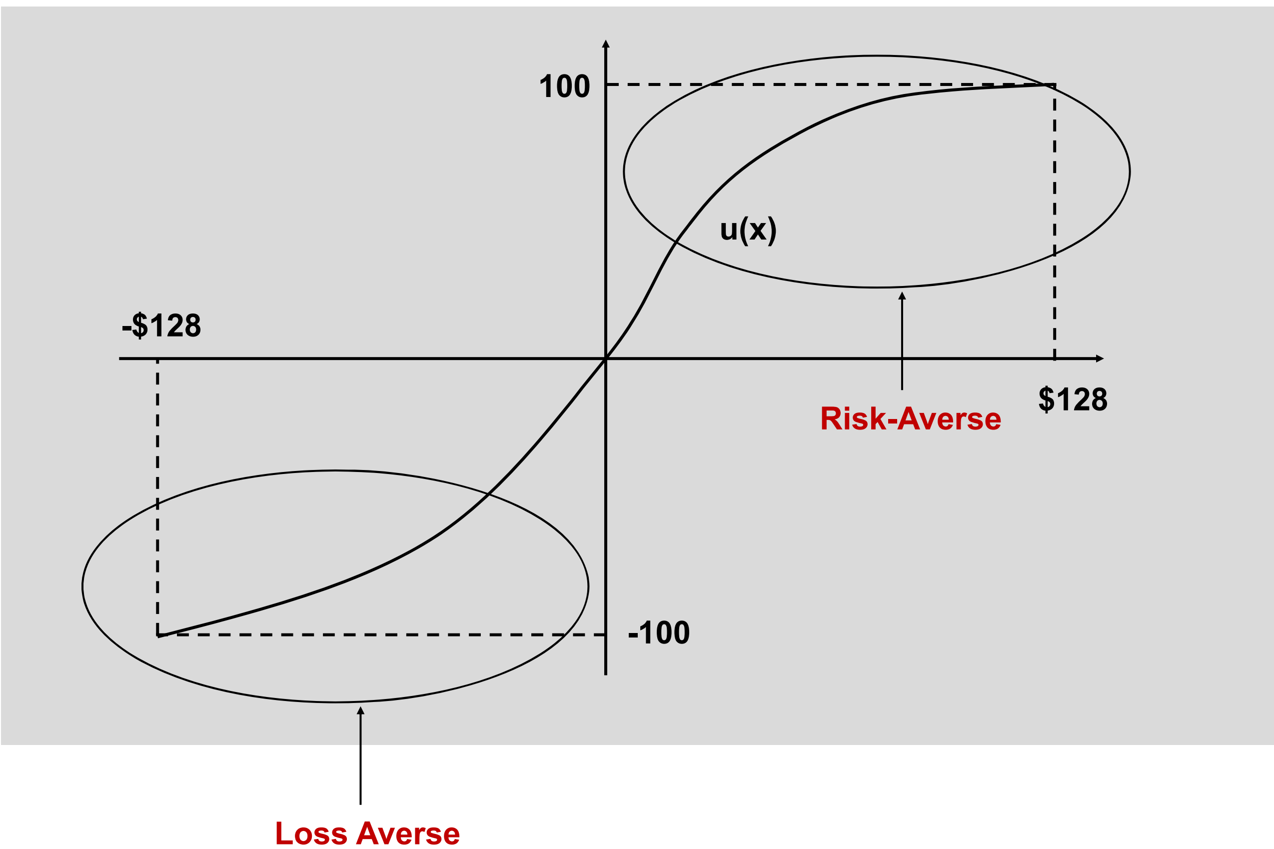
**According to utility theory A = C and B = D; so if A is preferred to B then C should be preferred to D and vice versa.**

**The "modal choices" are (i.e., "most people prefer") A and D to avoid losses.**

Remark: This sensitivity to the framing of the lottery choice in terms of either gains or losses is sometimes also referred to as *reflection effect*: preferences tend reverse if the lottery is "reflected" from the domain of gains (with respect to the *status quo*) to the domain of losses. For details, see Kahneman, D., Tversky, A. (1979) "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica*, Vol. 47, No. 2, pp. 263—291.



# "REAL" UTILITY FUNCTIONS OFTEN LOOK LIKE THIS



# COMPARISON OF RISK AVERSION

**Theorem (Pratt, 1964).** *Assume that agents $U$ and $V$ have the same initial wealth $w$ and suppose that their utility functions $u$ and $v$ are twice differentiable. Then the following statements are equivalent:*

*(i) Agent $U$ is more risk averse than agent $V$.*

*(ii) There is a strictly increasing concave function $\varphi$ such that $u = \varphi \circ v$.*

*(iii) Agent $U$'s absolute risk aversion is larger than agent $V$'s absolute risk aversion, i.e., $\rho_A(u; w) \geq \rho_A(v; w)$ for all $w$.*

*(iv) The risk premium that agent $U$ is willing to pay exceeds the risk premium that agent $V$ is willing to pay, i.e., $\pi(u; w) \geq \pi(v; w)$ for all $w$.*

**Proof: See notes on "Risk and Uncertainty" posted on the course website.**



# WHEN IS ONE LOTTERY PREFERRED TO ANOTHER FOR A *CLASS* OF UTILITY FUNCTIONS?

$$Eu(\tilde{x}) \leq Eu(\tilde{y}) \quad \forall u \in \mathcal{U}. \qquad (**)$$

**Definition.** *If* **(**)** *holds, the risk $\tilde{y}$ is said to* stochastically dominate *$\tilde{x}$ with respect to $\mathcal{U}$, denoted by $\tilde{x} \preceq_{\mathcal{U}} \tilde{y}$. Necessary and sufficient conditions on $\tilde{x}$ and $\tilde{y}$ for* **(**)** *to hold are called a* stochastic dominance order (representation) relative to $\mathcal{U}$.

**Class of Utility Functions (= Class of Agents)**

**Answer: Construct a stochastic dominance order**

- **First-order stochastic dominance: all agents with increasing utility**
- **Second-order stochastic dominance: all agents with increasing concave utility**

## STOCHASTIC DOMINANCE: DEFINITIONS

Let $X = [p_1, x_1; p_2, x_2; \ldots; p_n, x_n]$ and $Y = [q_1, y_1; q_2, y_2; \ldots; q_m, y_m]$ be two given *discrete* random variables, each with a finite number of realizations.

Without any loss of generality we can assume that m = n and that $x_i = y_i$ for all i in {1, … , n}, and that $x_1 < x_2 < \ldots < x_n$. This situation can always be achieved by extending the discrete variables X and Y to all events in the union of $\{x_1, \ldots , x_n\}$ and $\{y_1, \ldots , y_m\}$ assigning zero probabilities if necessary and subsequent relabeling.

**Definition**. Let m = n and $x_i = y_i$ for all i in {1, … , n}. Y first-order stochastically dominates X if

$$\sum_{i=1}^{k} q_i \leq \sum_{i=1}^{k} p_i$$

for all k in {1, …, n}.

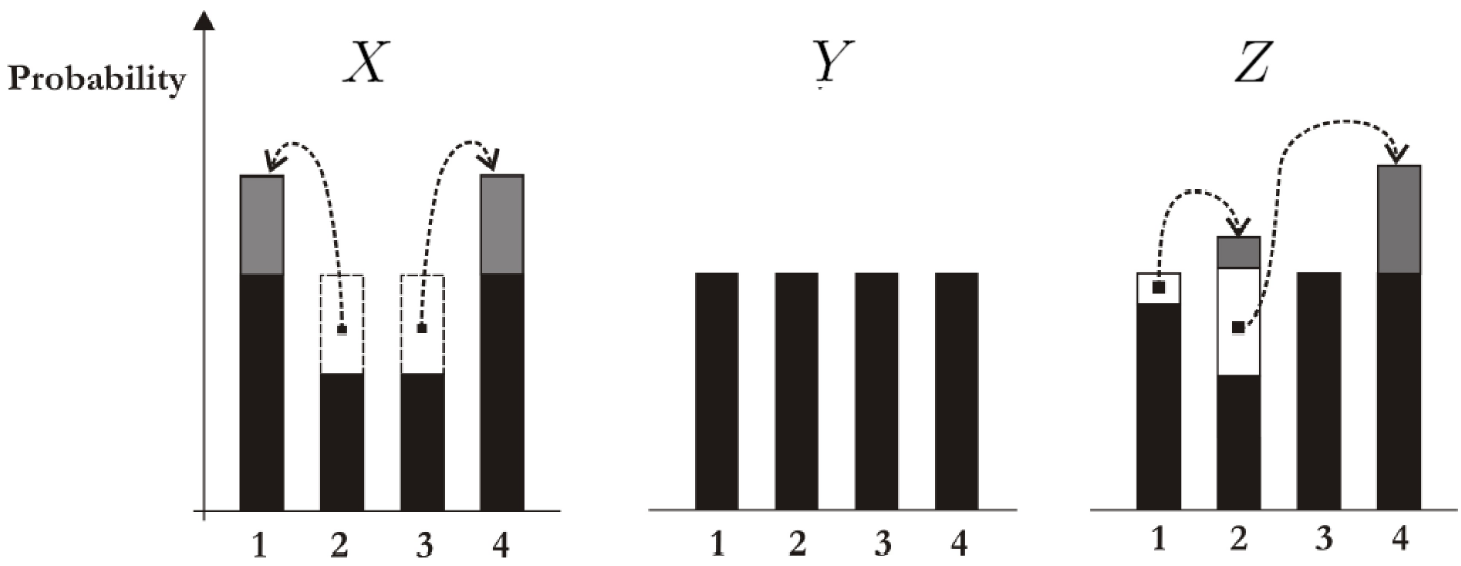
**Definition**. Let m = n and $x_i = y_i$ for all i in {1, … , n}. Y second-order stochastically dominates X if

$$\sum_{j=1}^{k} \sum_{i=1}^{j} q_i \leq \sum_{j=1}^{k} \sum_{i=1}^{j} p_i$$

for all k in {1, …, n}.



## FIRST- AND SECOND-ORDER STOCHASTIC DOMINANCE



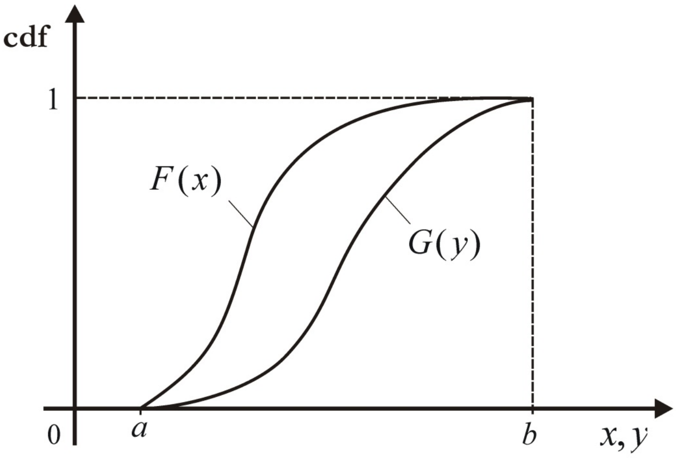

SOSD-Shift — Original Distribution — FOSD-Shift

$X \preceq_{\text{SOSD}} Y$

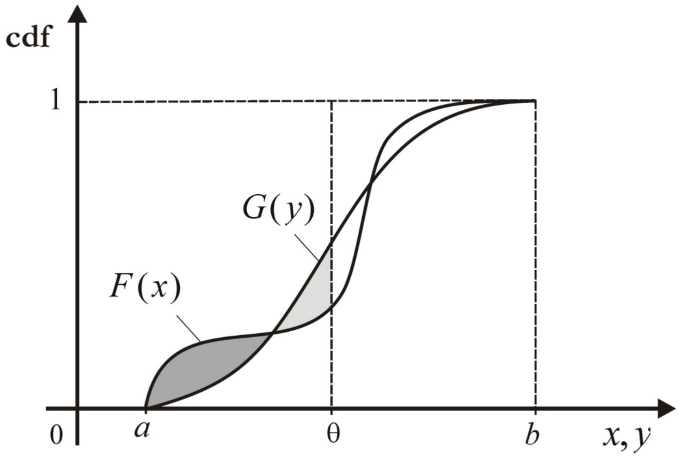
$Y \preceq_{\text{FOSD}} Z$

# FIRST- AND SECOND-ORDER STOCHASTIC DOMINANCE
## General Case

**Let F, G be two cumulative distribution functions (measures) for random variables X and Y, respectively, distributed on the set [a,b]. When does Y FOSD/SOSD-dominate X ?**




**FOSD**




**SOSD**

$$X \preceq_{\mathrm{FOSD}} Y \quad \Leftrightarrow \quad G(z) \leq F(z) \quad \forall\, z \in [a, b]$$

$$X \preceq_{\mathrm{SOSD}} Y \quad \Leftrightarrow \quad \int_a^z G(y)dy \leq \int_a^z F(x)dx \quad \forall\, z \in [a, b]$$



# AGENDA

**Elements of Probability**

**Choice Under Uncertainty**

**Expected Utility Theory**

**Risk Aversion and Decision Biases**

**Key Concepts to Remember**

# KEY CONCEPTS TO REMEMBER

- **Risk and Uncertainty**
- **Objective and Subjective Probability**
- **Discrete Random Variable**
- **Lottery**
- **Von Neumann-Morgenstern Axioms → Expected Utility Representation**
- **Allais Paradox**
- **Expected Utility Maximization**
- **Risk Aversion (Absolute & Relative)**
- **Sensitivity to Reference Point (Reflection Effect)**
- **Stochastic Dominance (First-Order & Second-Order)**

# MGT 621 – MICROECONOMICS

**Thomas A. Weber**

***4. Theory of the Firm***

**Autumn 2023**

**École Polytechnique Fédérale de Lausanne**
**College of Management of Technology**



# AGENDA

**Introduction**

**Production Sets**

**Profit Maximization: Some Intuition**

**The Firm's Cost Function**

**Profit Maximization**

**Key Concepts to Remember**

# THEORY OF THE FIRM

## Production Side of the Economy

**Including …**

- **Corporations** **(… General Motors, Microsoft, Virgin Atlantic)**
- **Public utilities** **(… Pacific Gas and Electric Co., Metropolitan Water District)**
- **Partnerships** **(… law firm, McKinsey, start-up firm)**
- **Small businesses** **(… retail store, individual consultant, restaurant, internet radio station)**
- **Home production** **(… home improvement, prepare meals)**
- **Educational institutions** **(… UC Berkeley, Stanford U)**
- **Non-profit organizations** **(… community hospital, YMCA)**



# THE FIRM AS AN OPTIMIZER

**Possible Objectives**

- **Maximize (expected) profit**
- **Maximize (expected) utility (which takes profits as one argument)**
- **Minimize cost (e.g., given fixed outputs)**

**Feasible Actions**

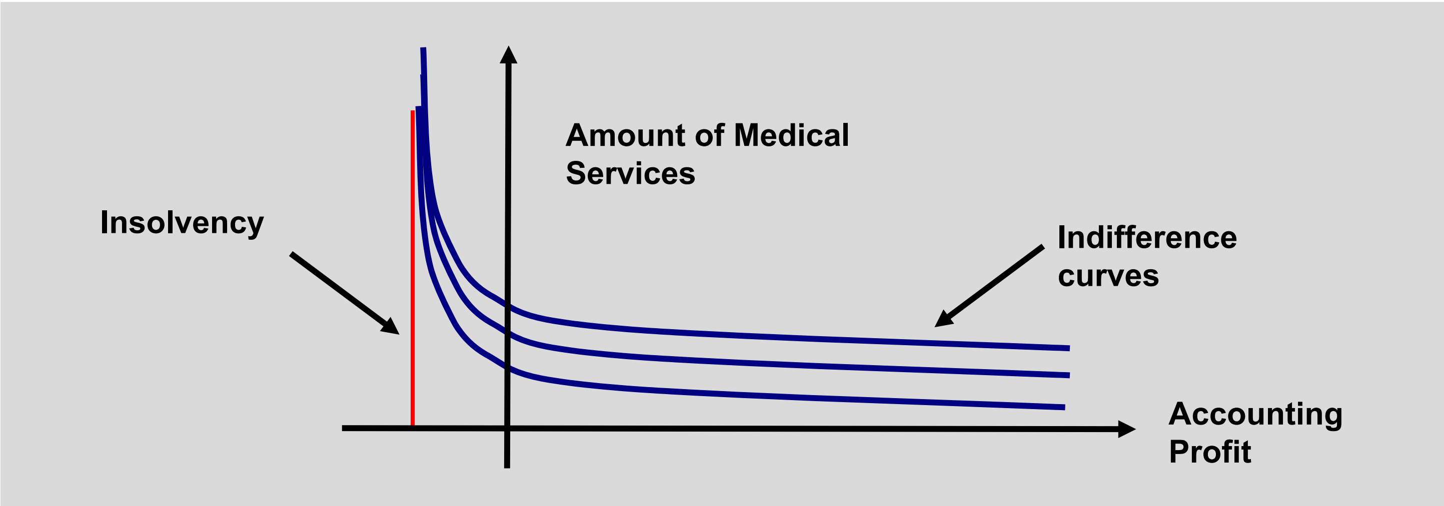
- **Production possibilities**
- **Legal constraints**
- **Competitive necessities**

**Firm chooses a most preferred action from the set of all feasible actions.**

# A FIRM'S POSSIBLE OBJECTIVES: EXAMPLE

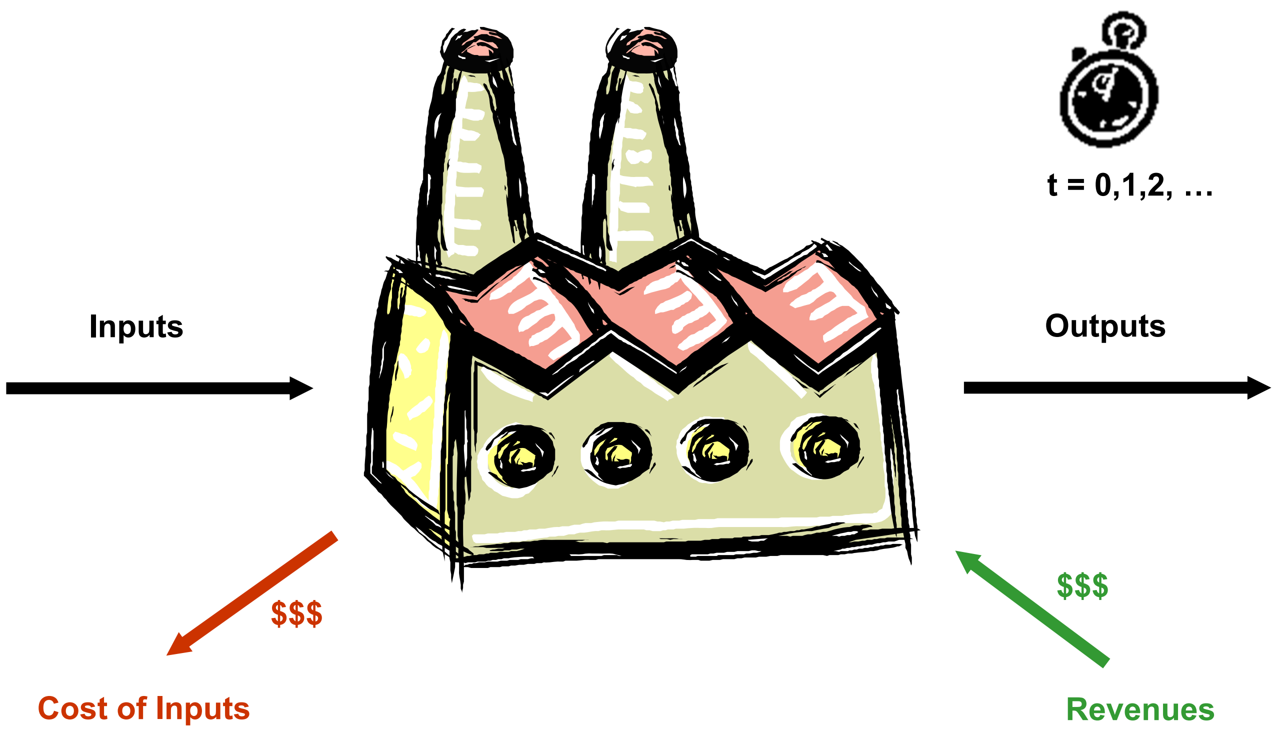
**Maximization of (Expected) "Utility"** (Objective Function)

- Could include accounting profits as only one variable among several others (such as a measure of output)
  - Example: a community hospital may put a high value on accounting profits if it is losing money (negative profit), and a high value on the provision of medical services otherwise

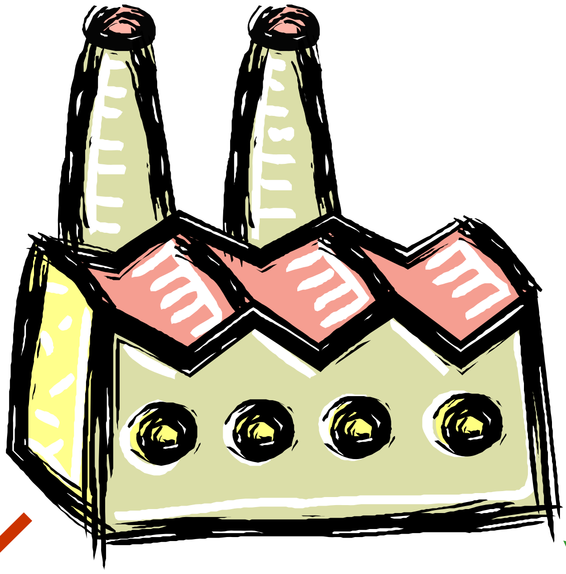





# THE FIRM AS A BLACK BOX
## Compare Inflows and Outflows



## THE FIRM AS A BLACK BOX (Cont'd)
### Converts Inputs to Outputs



$t = 0,1,2, \ldots$

Inputs $\underline{z}$ → 

Stream
$\underline{z} = (z^0, z^1, z^2, \ldots)$

Outputs $\underline{q} = F(\underline{z})$ →

Stream
$\underline{q} = (q^0, q^1, q^2, \ldots)$

$q^t = F^t(\underline{z})$

Production Function F

Costs: $C^t(\underline{q})$ ← **Profit at time t:** $\Pi^t(\underline{q}) = R^t(\underline{q}) - C^t(\underline{q})$ ← Revenues: $R^t(\underline{q})$

!!!



## PRODUCTION FUNCTION

**The production function at time t might depend on current and/or (anticipated) future inputs,**

$$q^t = F^t(\underline{z}) = F^t(z^0, z^1, z^2, \ldots)$$

**Question: Why?**

**Answer: There are many real-world phenomena which may produce dependencies such as**

- **Learning-Curve Effects**[(1)]
- **Demand Effects (e.g., Saturation)**

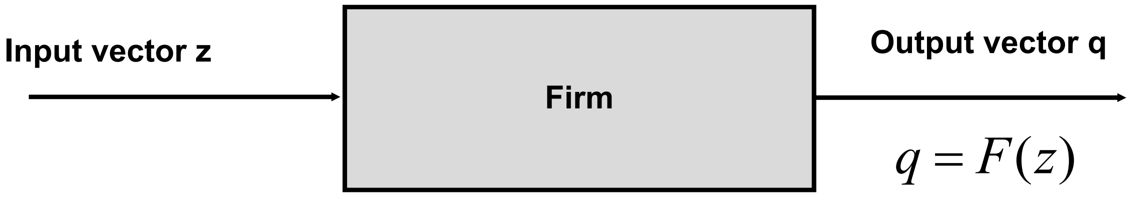
**In some practical applications current output depends only on current input, i.e.,**

$$q^t = F^t(\underline{z}) = F^t(z^t)$$

(1) The commander of Wright-Patterson Air Force Base in Ohio observed in 1925 that the required direct labor hours for the assembly of a plane decreased in the number of planes built. First academic observations include Hirschman, W.B. (1964) "Profit from the Learning Curve," Harvard Business Review, Jan/Feb, and Arrow, K.J. (1962) "Economic Welfare and the Allocation of Resources to Invention," in: Nelson, R. (ed.) *The Rate and Direction of Inventive Activity: Economic and Social Factors*, Princeton University Press, Princeton, NJ.

# A FIRM CONVERTS INPUTS TO OUTPUTS
## Simplification: Static Conversion → Omit Time Index

**Production function F summarizes conversion**




**Sometimes it is difficult to decide what exactly is input and what is output, so that it is useful to consider the "general" output (or "production possibility")**

**y = (-z,q)**

**(Net) inputs have negative components and (net) outputs are represented by nonnegative components of y.**



# GENERAL (STATIC) MODEL OF THE FIRM

**Firm chooses vector z $\geq$ 0 of inputs to produce vector q $\geq$ 0 of outputs = Firm chooses production possibility vector y = (-z,q).**

## Feasible Set

- **Production Set (or Production Possibility Set) Y**

## Objective Function

- **For simplicity, first assume that profit maximization is the objective, given a price p(y) for outputs**
  **(it turns out that cost minimization for inputs is a necessary condition)**

## → Profit-Maximization Problem

- **Choose y to maximize the product $\Pi$(y) = p(y) y subject to y being feasible, i.e., solve**

$$\max_{y \in Y} \ \Pi(y)$$

# AGENDA

Introduction

**Production Sets**

Profit Maximization: Some Intuition

The Firm's Cost Function

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Key Concepts to Remember



# PRODUCTION SET Y

The firm's production set Y contains all input-output bundles y it might choose.

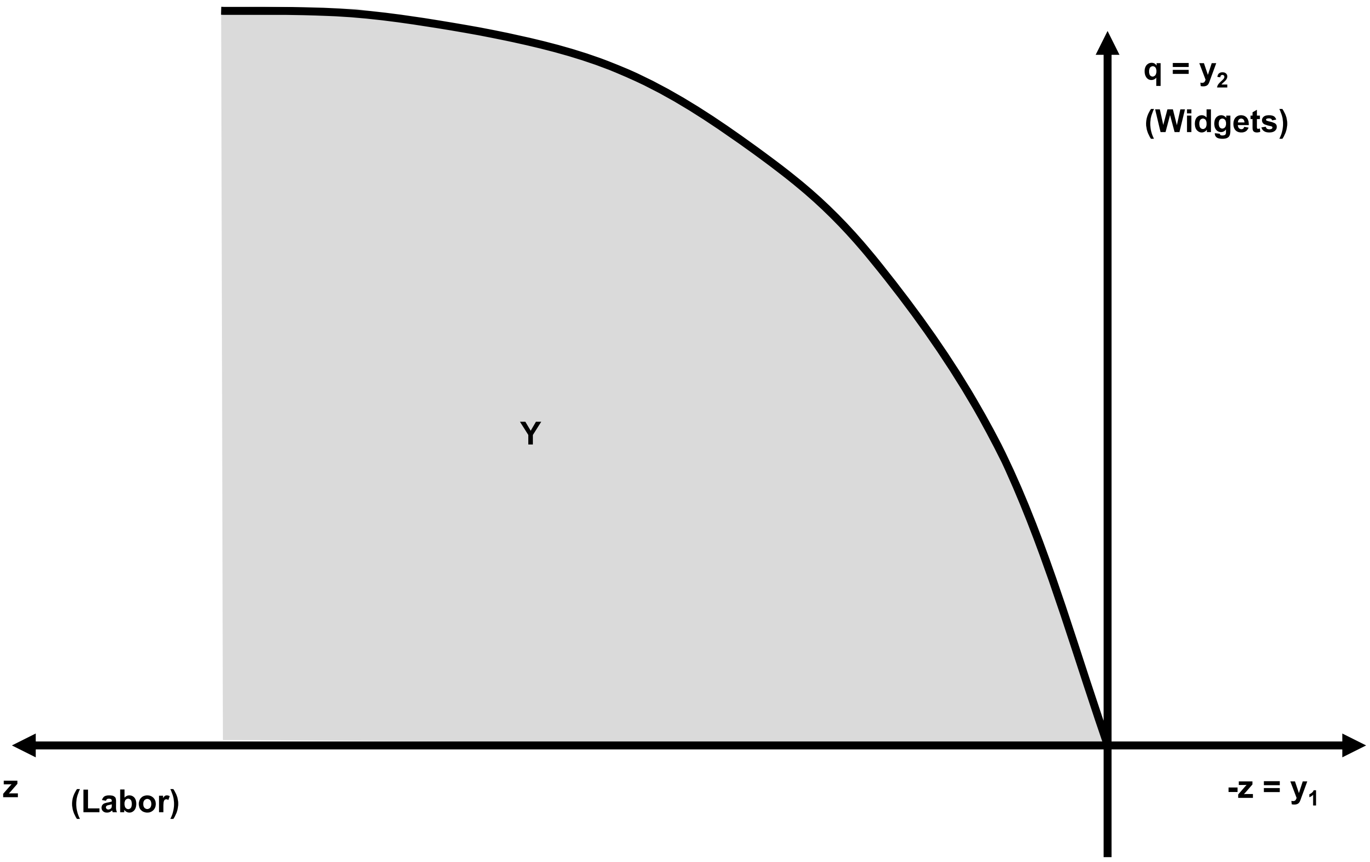
**Possible inputs:** labor, land, buildings, raw materials, capital equipment, services, intermediate goods (produced by other firms), etc.

**Possible outputs:** finished goods, services, knowledge, intermediate goods (to be used by other firms), etc.

Shape of Y implied by

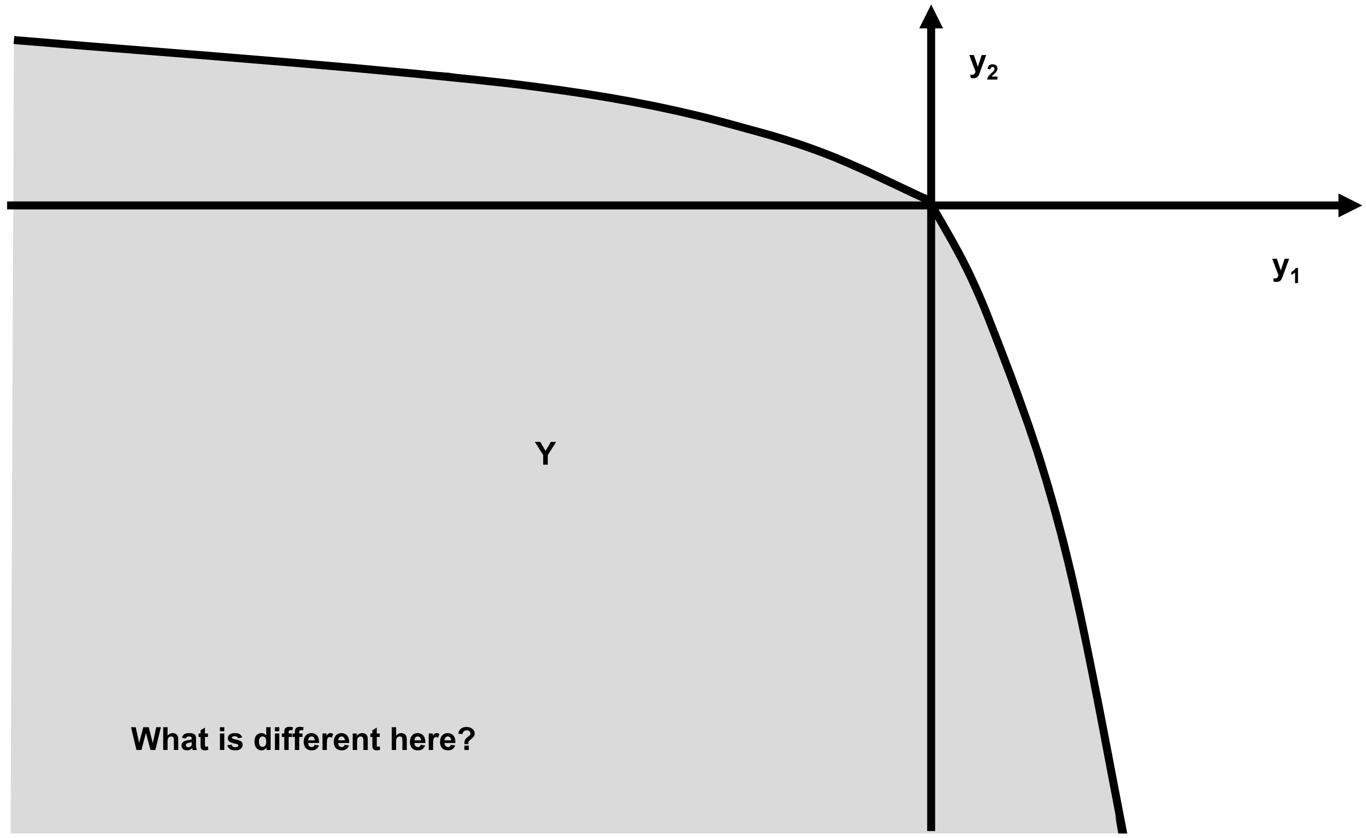
- technological possibilities
- legal constraints

## PRODUCTION SET: EXAMPLE






## PRODUCTION SET: ANOTHER EXAMPLE



## PROPERTIES OF PRODUCTION SETS

**We now discuss standard assumptions on a firm's production set $Y \subset \Re^L_+$ .**

**A1 (Non-Emptiness). The set Y contains at least one element.**

**A2 (Closedness). It is not possible to construct a sequence of elements of Y with limit outside Y.**

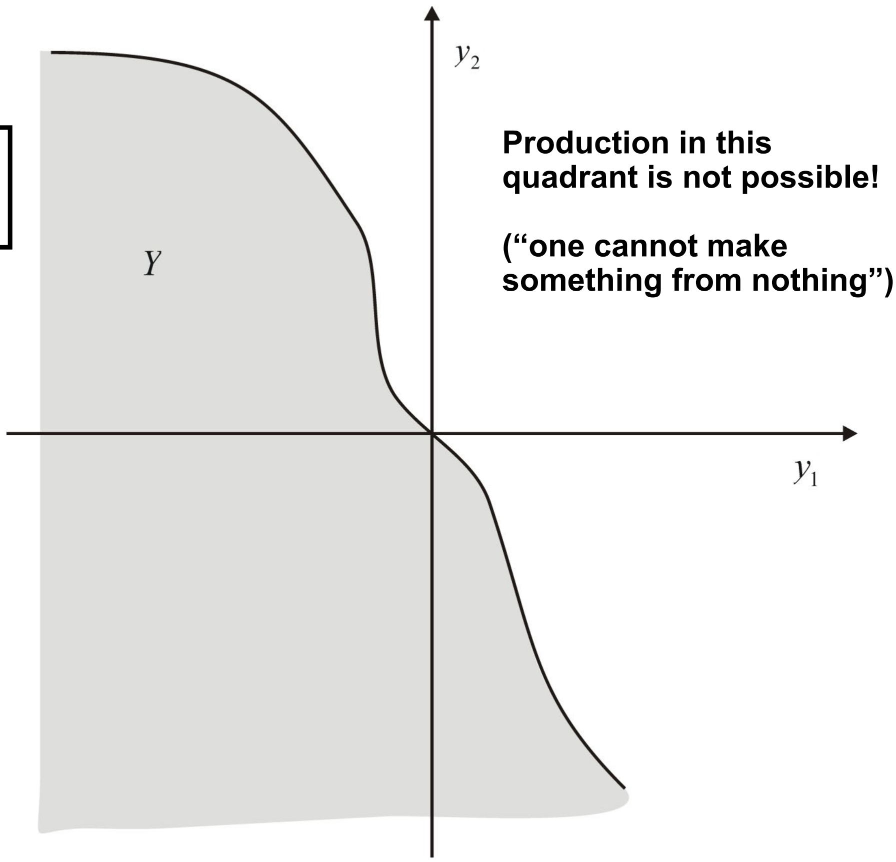
**A3 (No Free Lunch). $Y \cap \Re^L_+ \subseteq \{0\}$ : it is not possible to have a positive net output y > 0 (i.e., one cannot produce at least one positive net output and not use any positive net inputs).**

| |
|---|
| **A1 – A3 are fundamental requirements which we assume are always satisfied.** |



## NO FREE LUNCH

$$\boxed{Y \cap \Re^L_+ \subseteq \{0\}}$$



# PROPERTIES OF PRODUCTION SETS (Cont'd)

**A4 (Possibility of Inaction).** $0 \in Y$ **, i.e., it is possible to do nothing.**

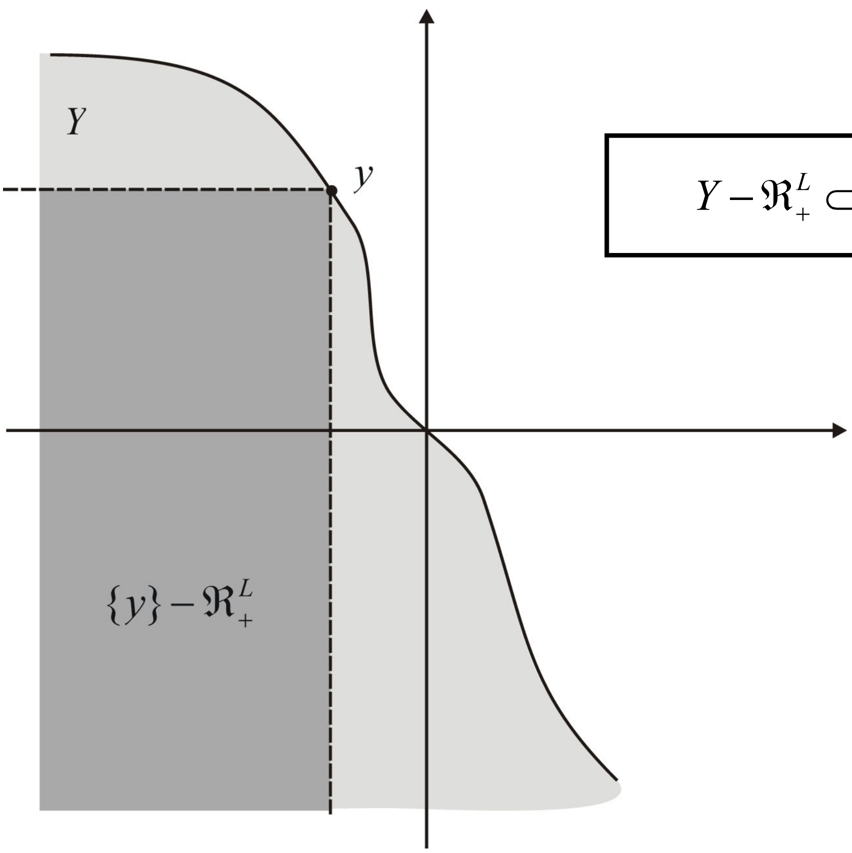
**A5 (Free Disposal).** $Y - \Re^L_+ \subset Y$ **, i.e., it is always possible to use more net inputs for the same net output.**

**A6 (Irreversibility).** $y \in Y \Rightarrow -y \notin Y$ **: it is not possible to obtain the inputs back once the outputs have been created.**

**A4 – A6 are basic properties of production sets that may sometimes be violated.**

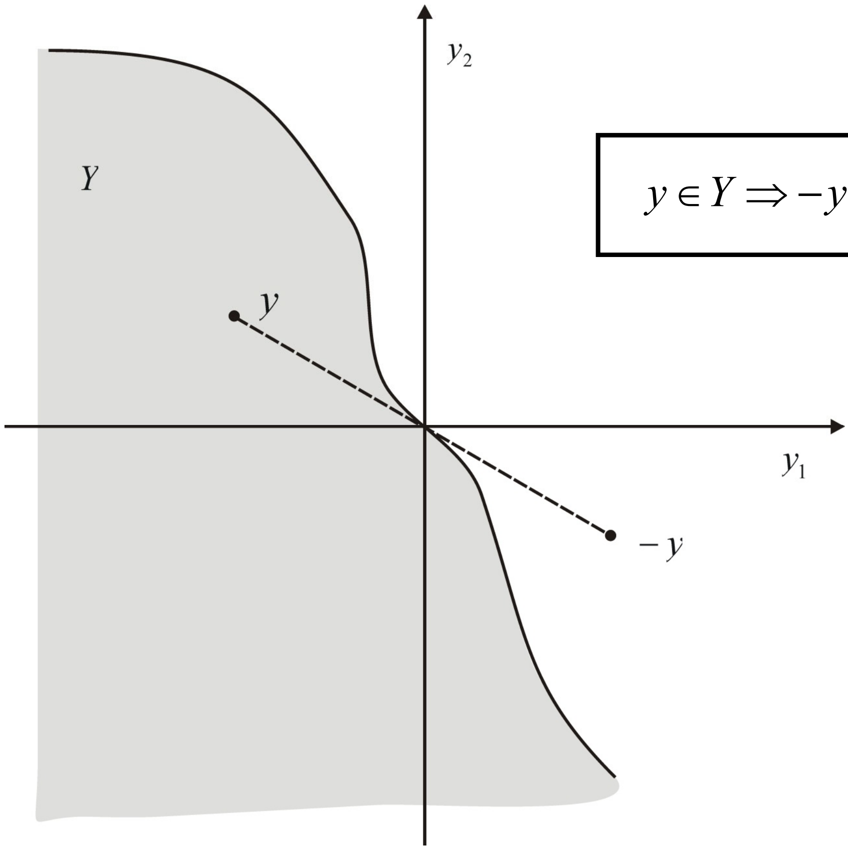


# FREE DISPOSAL




$$Y - \Re^L_+ \subset Y$$

# IRREVERSIBILITY




$$y \in Y \Rightarrow -y \notin Y$$



# PROPERTIES OF PRODUCTION SETS (Cont'd)

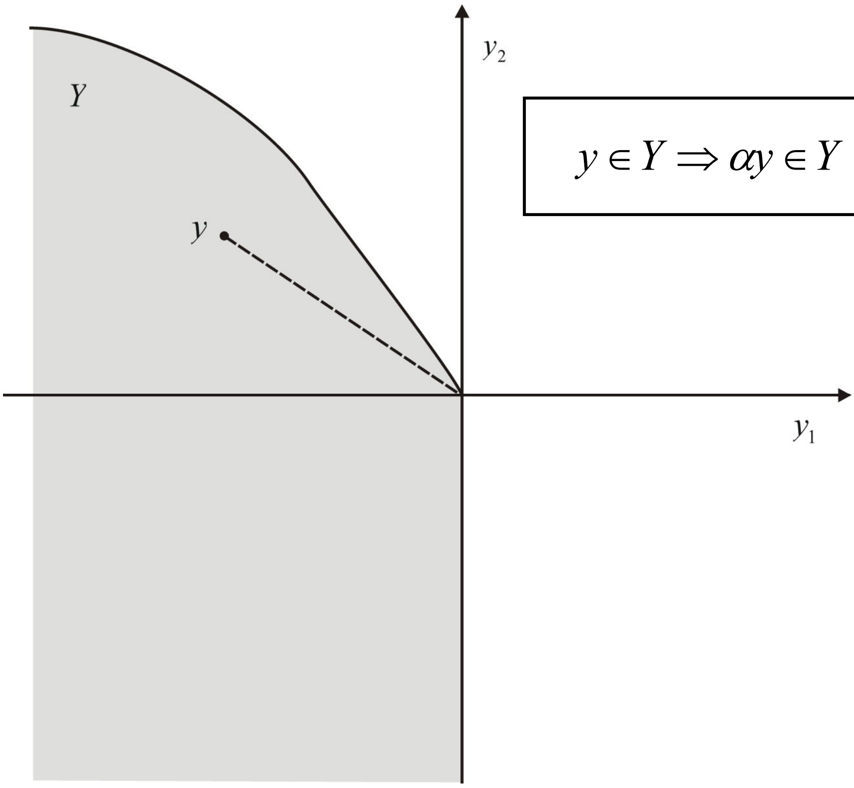
**A7 (Nonincreasing Returns to Scale).** $y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \in [0,1]$ **, i.e., it is possible to scale down any feasible production vector. Mathematically, this means that Y is "star-shaped" with respect to the origin.**

**A8 (Nondecreasing Returns to Scale).** $y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \geq 1$ **, i.e., it is always possible to scale up any feasible production vector.**

**A9 (Constant Returns to Scale).** $y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \geq 0$ **, i.e., any feasible production vector is completely scalable. Mathematically, this means that Y is a cone.**

**A7 – A9 are useful properties of production sets that may or may not be satisfied.**

# NONINCREASING RETURNS TO SCALE

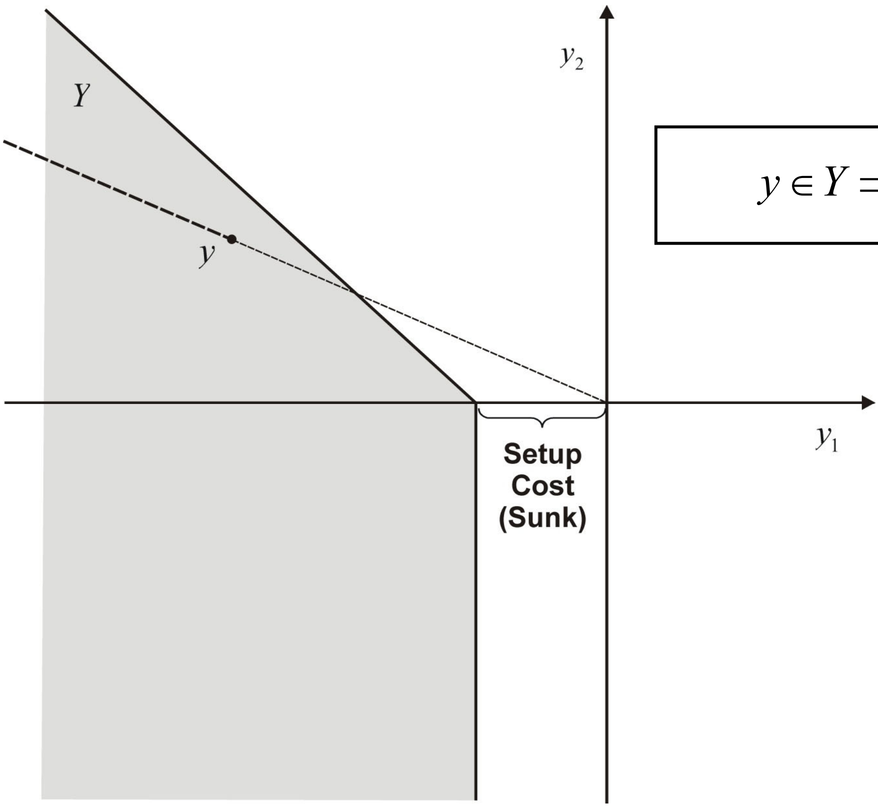



$$y \in Y \Rightarrow \alpha y \in Y \ \ \forall \alpha \in [0,1]$$

Remark: The set Y exhibits *decreasing* returns to scale if $y \in Y \Rightarrow \alpha y \in \text{int}\, Y \ \ \forall \alpha \in (0,1)$



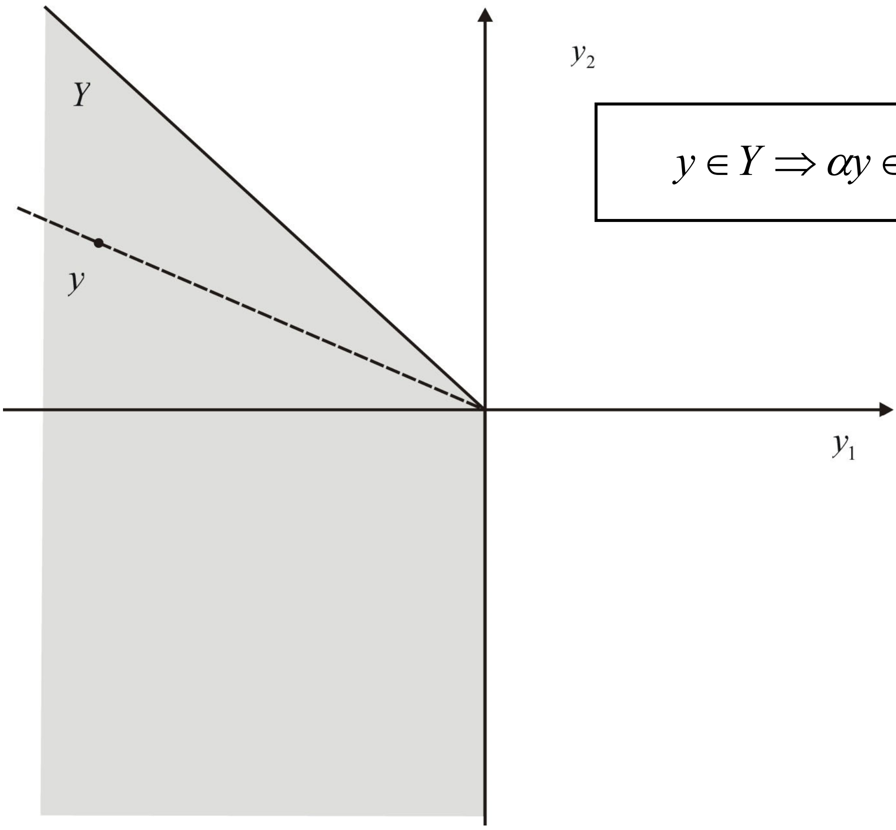
# NONDECREASING RETURNS TO SCALE




$$y \in Y \Rightarrow \alpha y \in Y \ \ \forall \alpha \geq 1$$

Remark: The set Y exhibits *increasing* returns to scale if $y \in Y \Rightarrow \alpha y \in \text{int}\, Y \ \ \forall \alpha > 1$ .

# CONSTANT RETURNS TO SCALE




$$y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \geq 0$$



# PROPERTIES OF PRODUCTION SETS (Cont'd)

**A10 (Additivity/Free Entry).** $y, \hat{y} \in Y \Rightarrow y + \hat{y} \in Y$ **, i.e., it is possible to combine any feasible production vectors. For an economy this means that the aggregate production possibilities are obtained by summing up the firms' individual production possibilities, provided each firm is free to contribute or not (= free entry).**

**A11 (Convexity).** $y, \hat{y} \in Y \Rightarrow \alpha y + (1-\alpha)\hat{y} \in Y \quad \forall \alpha \in (0,1)$ **, i.e., any convex combination of feasible production vectors is feasible.**

**A12 (Convex Cone Property).** $y, \hat{y} \in Y \Rightarrow \alpha y + \beta \hat{y} \in Y \quad \forall \alpha, \beta \geq 0$ **, i.e., any feasible production vectors can be combined and scaled. The convex cone property is equivalent to the combination A9 and A11.**

**A10 – A12 are structural properties of production sets that affect the optimization methods when looking for optimal production vectors.**

# AGENDA





# GENERAL (STATIC) MODEL OF THE FIRM

Firm chooses vector $z \geq 0$ of inputs to produce vector $q \geq 0$ of outputs = Firm chooses production possibility vector $y = (-z,q)$.

**Feasible Set**

- Production Set (or Production Possibility Set) Y

**Objective Function**

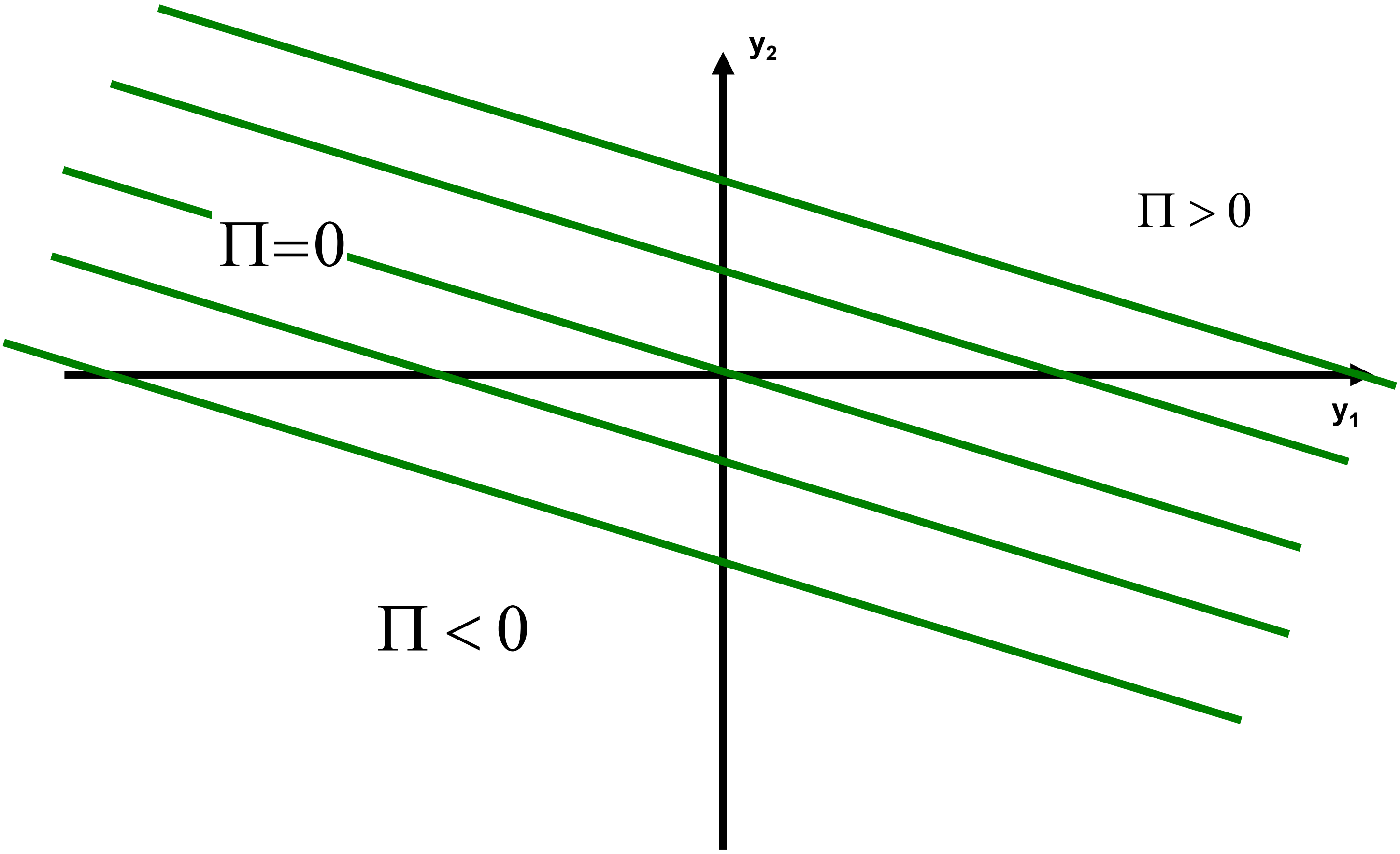
- For simplicity, first assume that profit maximization is the objective, given a price p(y) for outputs
(it turns out that cost minimization for inputs is a necessary condition)

**→ Profit-Maximization Problem**

- Choose y to maximize the product $\Pi(y) = p(y)\, y$ subject to y being feasible, i.e., solve

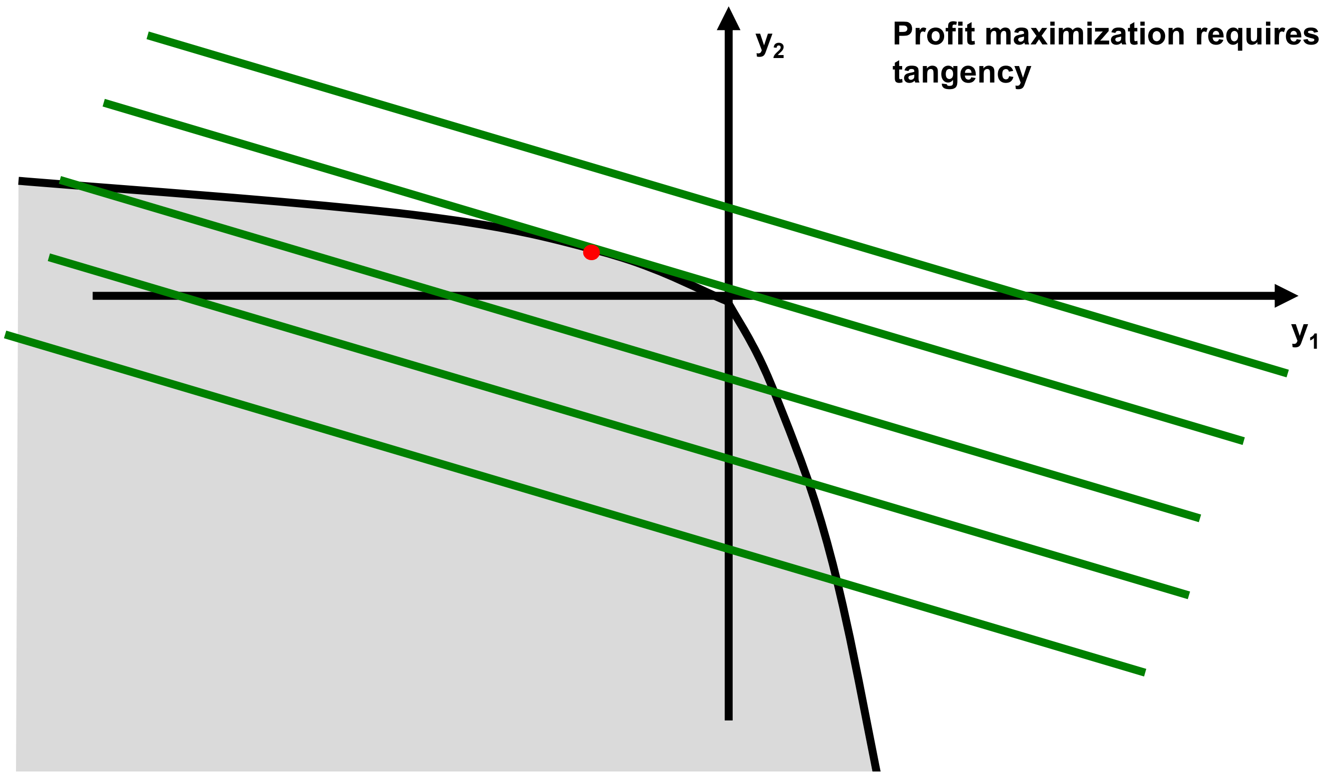
$$\max_{y \in Y} \ \Pi(y)$$

## ASSUMPTION THAT p(y)=p (I.E., LINEAR PRICING) … … YIELDS CONSTANT ISO-PROFIT CURVES

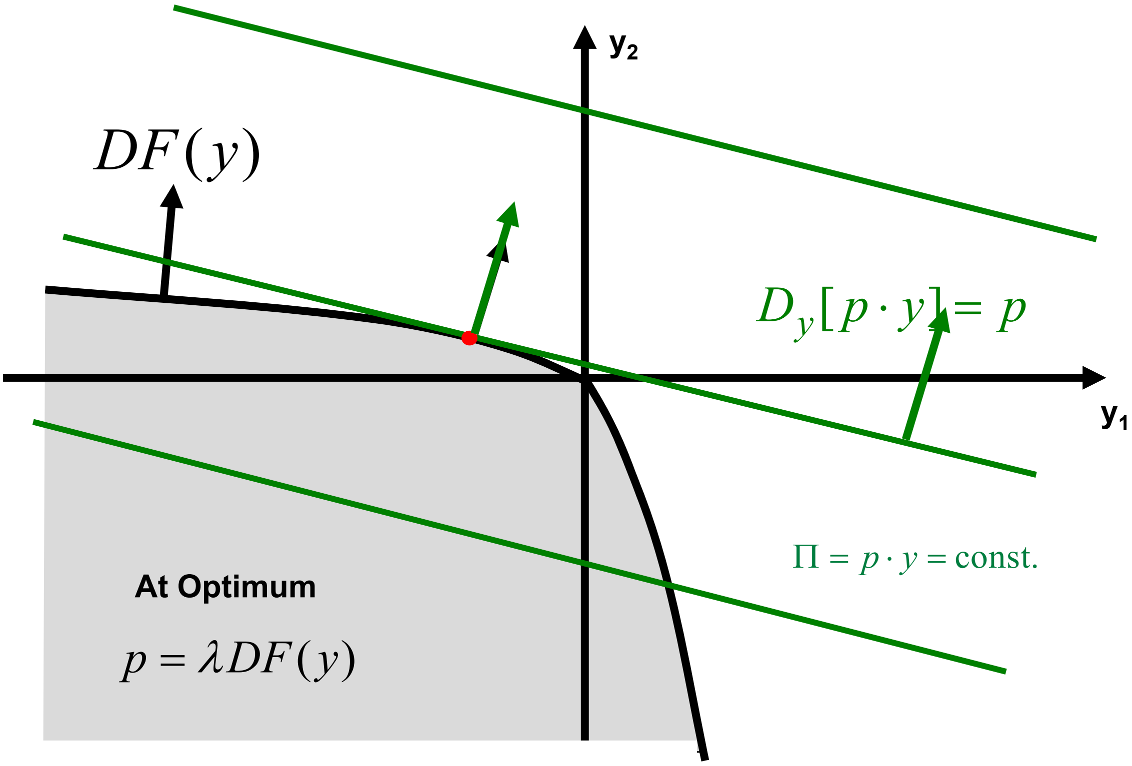





## PROFIT MAXIMIZATION OVER PRODUCTION SET

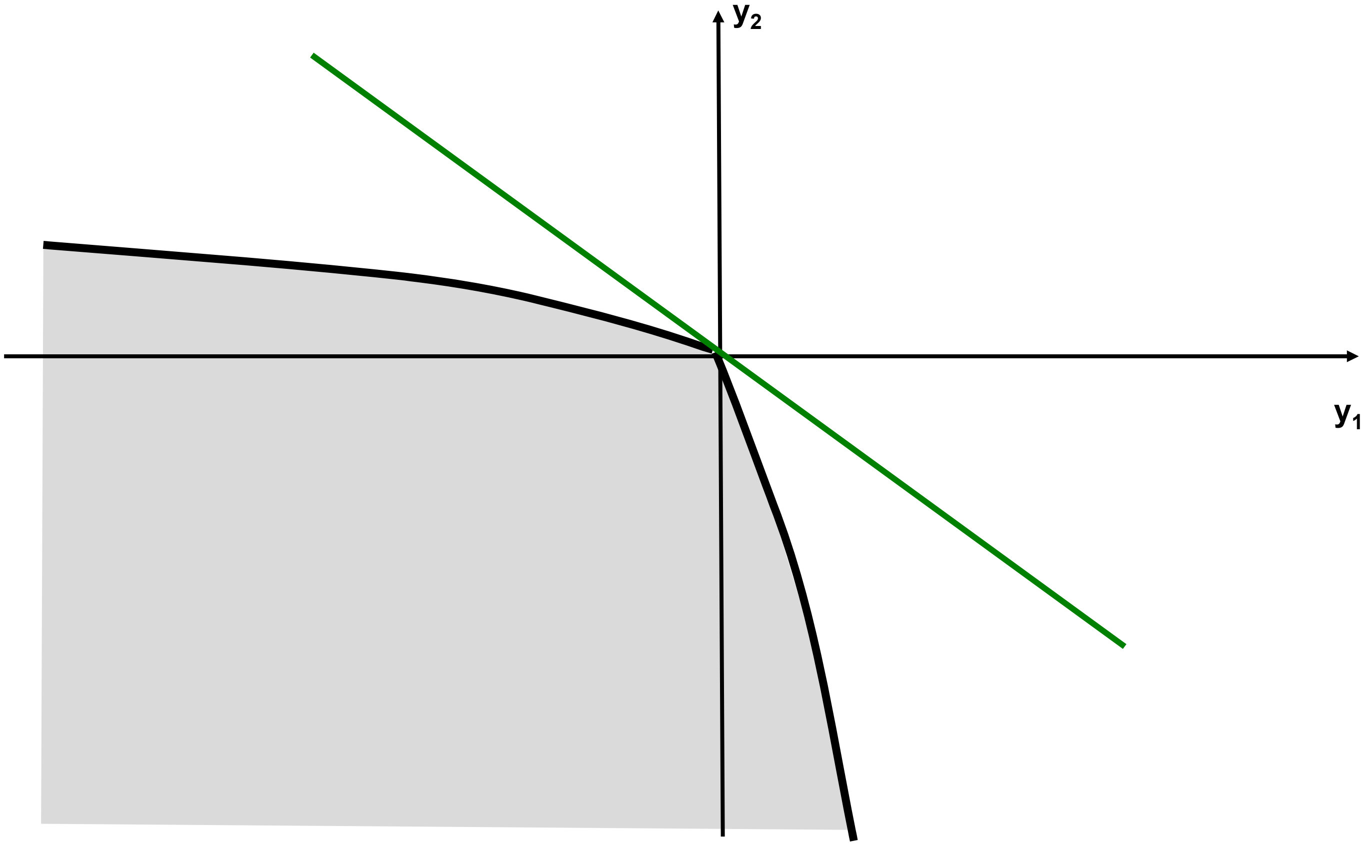


# PROFIT MAXIMIZATION OVER PRODUCTION SET (Cont'd)

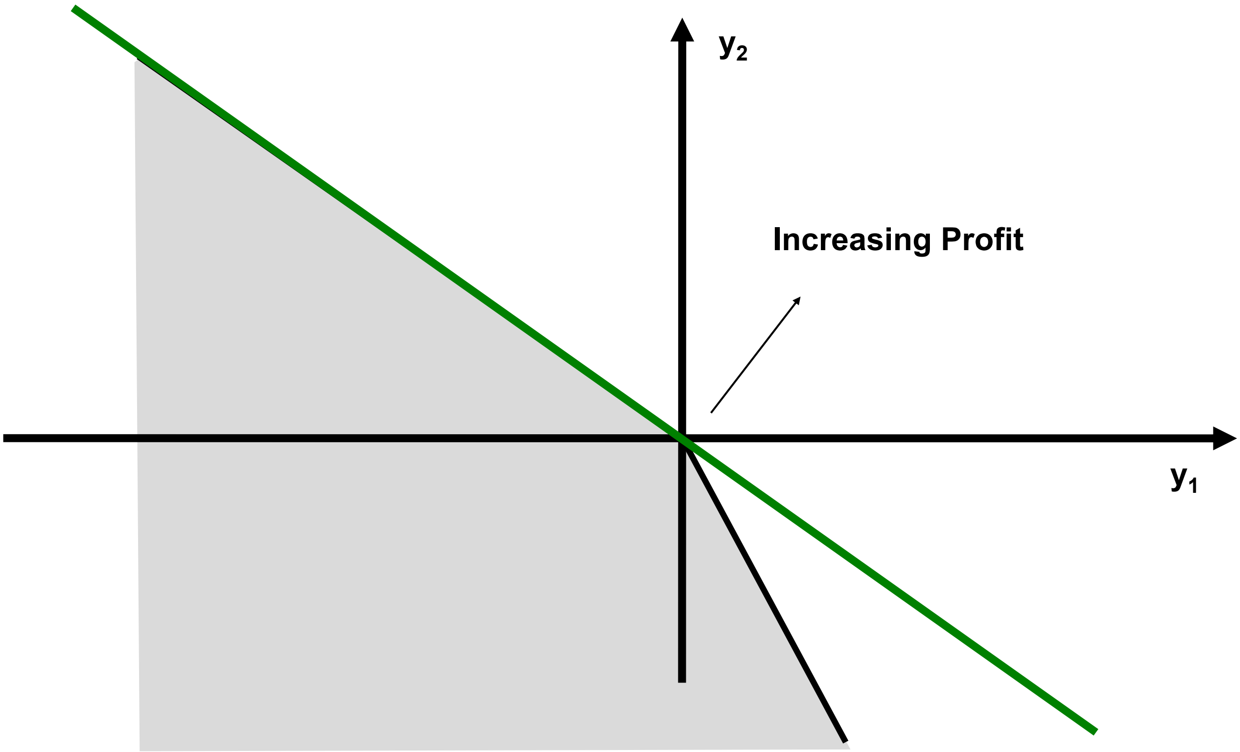





# WHAT HAPPENS WHEN THERE ARE KINKS IN PRODUCTION SET?

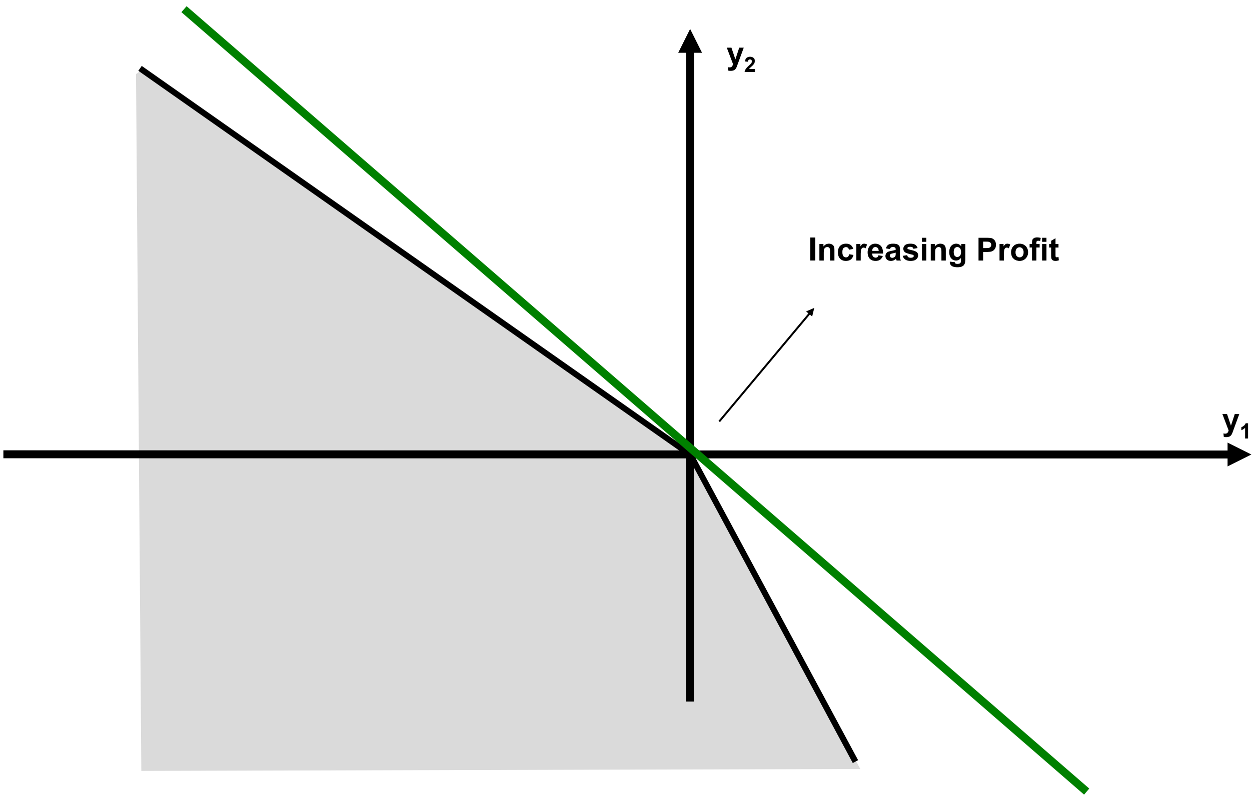


# PROFIT-MAXIMIZING FIRM WITH CONSTANT RETURNS TO SCALE

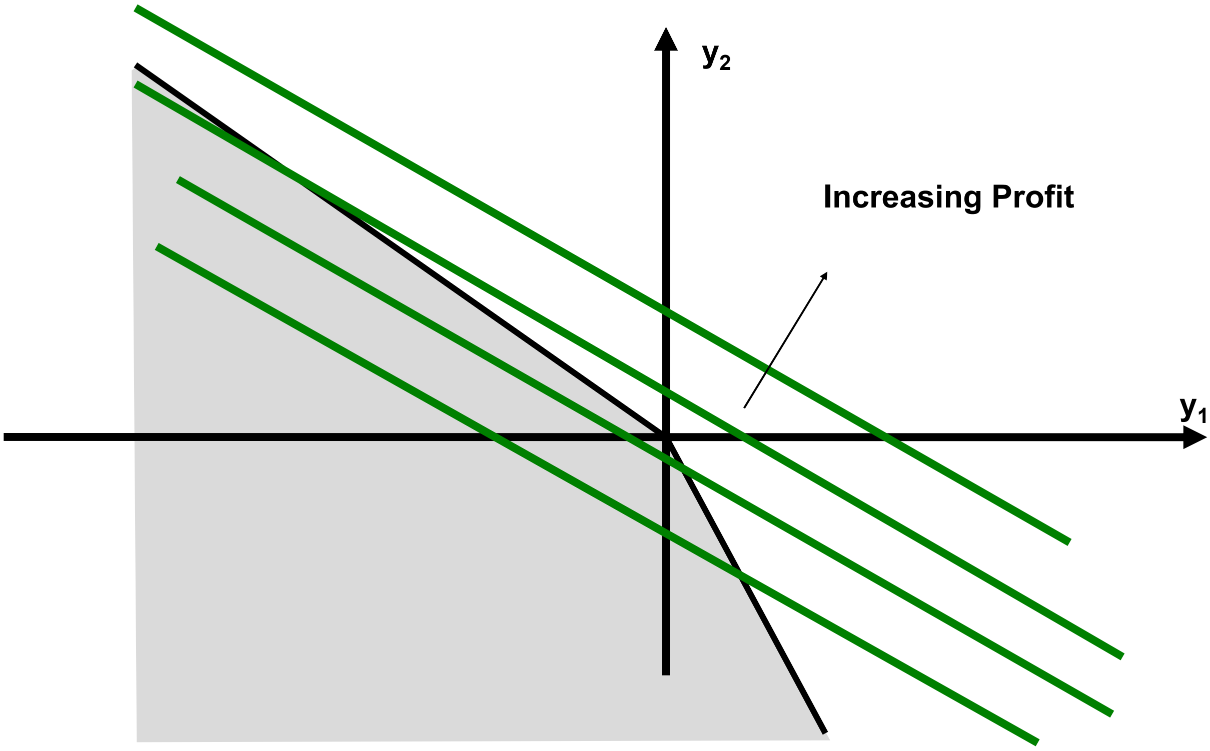





# PROFIT MAXIMIZATION WITH CONSTANT RETURNS TO SCALE

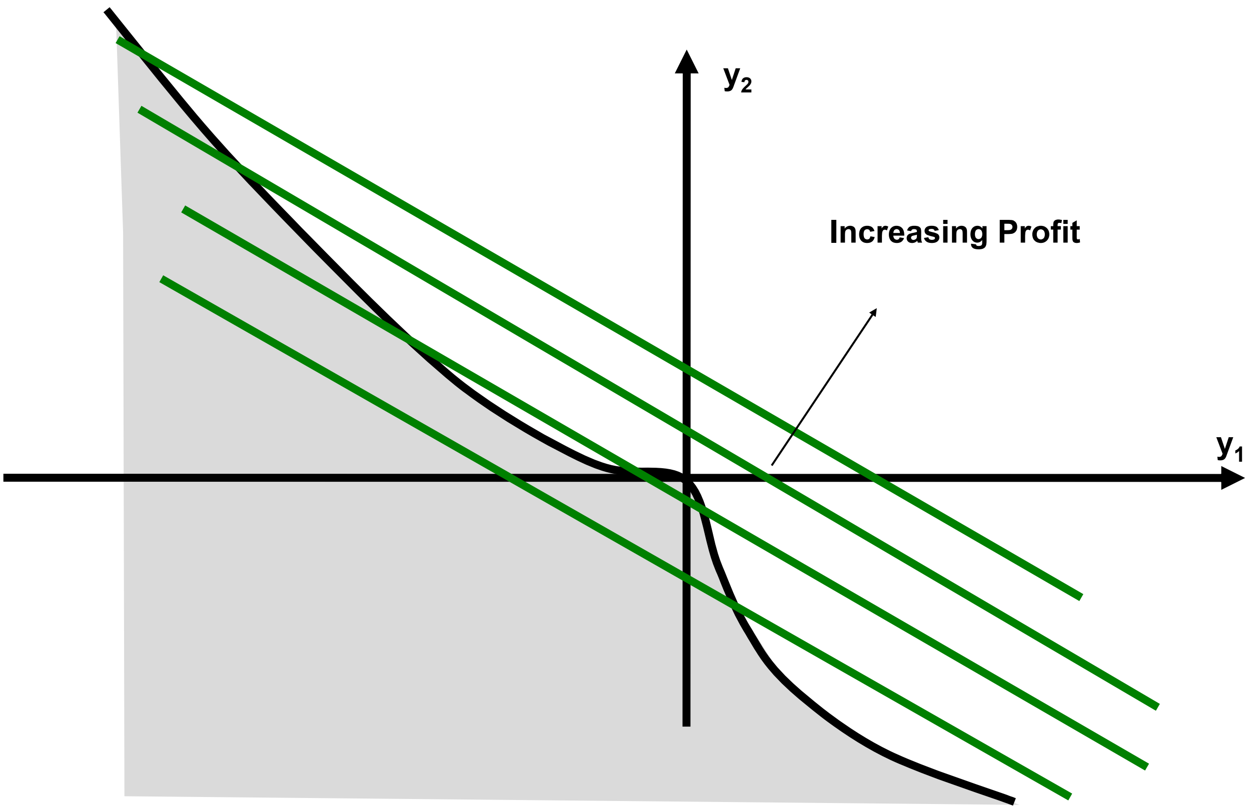


# PROFIT MAXIMIZATION WITH CONSTANT RETURNS TO SCALE?

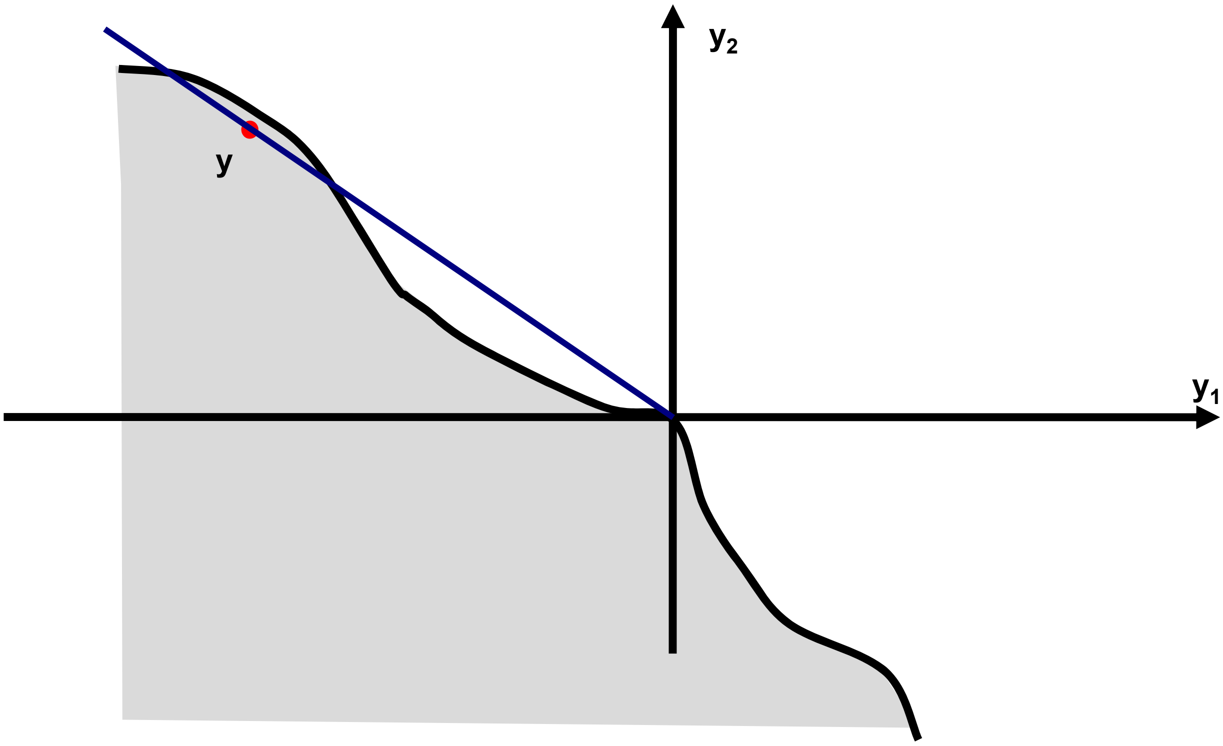





# PROFIT-MAXIMIZATION WITH INCREASING RETURNS TO SCALE?



# NEITHER INCREASING NOR DECREASING RETURNS TO SCALE






# AGENDA

Introduction

Production Sets

Profit Maximization: Some Intuition

**The Firm's Cost Function**

Profit Maximization

Key Concepts to Remember

# PRODUCTION FUNCTION

**Assume that a firm can clearly distinguish between inputs and outputs, i.e., any y in the production set Y can be written in the form y = (-z,q) where z $\geq$ 0 is the vector of inputs and q $\geq$ 0 is the vector of outputs.(1)**

**Then it is possible to represent Y in the form**

$$Y = \{(-z, q) \in \Re^L : q \leq F(z)\}$$

**where F(z) is referred to called the firm's production function.**

(1) This holds when the firm's production is "irreversible," i.e., Y satisfies Assumption A6.



# COST FUNCTION

**Question. Given an increasing production function F(z), determine the firm's cost function C(q), i.e., the firm's minimum cost to produce a (feasible) output vector q $\geq$ 0.**

**Answer: Given a feasible vector q of outputs, the firm solves the *expenditure minimization problem* (or *'cost minimization problem'* in this context)**

$$\min_{y=(-z,q)\in Y} \{w(z) \cdot z\} = \min_{z:F(z)\geq q} \{w(z) \cdot z\}$$

**where w(z) is the vector of (positive) input prices. The firm's cost function C(q) is its minimal expenditure,**

$$\boxed{C(q) = \min_{z:F(z)\geq q} \{w(z) \cdot z\} = \min\{w(z) \cdot z : F(z) \geq q\}}$$

# COST FUNCTION: EXAMPLE

**Problem Set. Find the cost function C(q) implied by the production set**

$$Y = \left\{ (-z_1, -z_2, q) : (z_1, z_2, q) \in \mathbb{R}^3_+, \, z_1^\alpha z_2^\beta \geq q \right\}$$

**where $\alpha$ and $\beta$ are positive constants with $\alpha + \beta < 1$.**



# ECONOMIES/DISECONOMIES OF SCALE

**A cost function C(q) with a scalar output q > 0 exhibits economies of scale if the average cost decreases in q > 0, i.e.,**

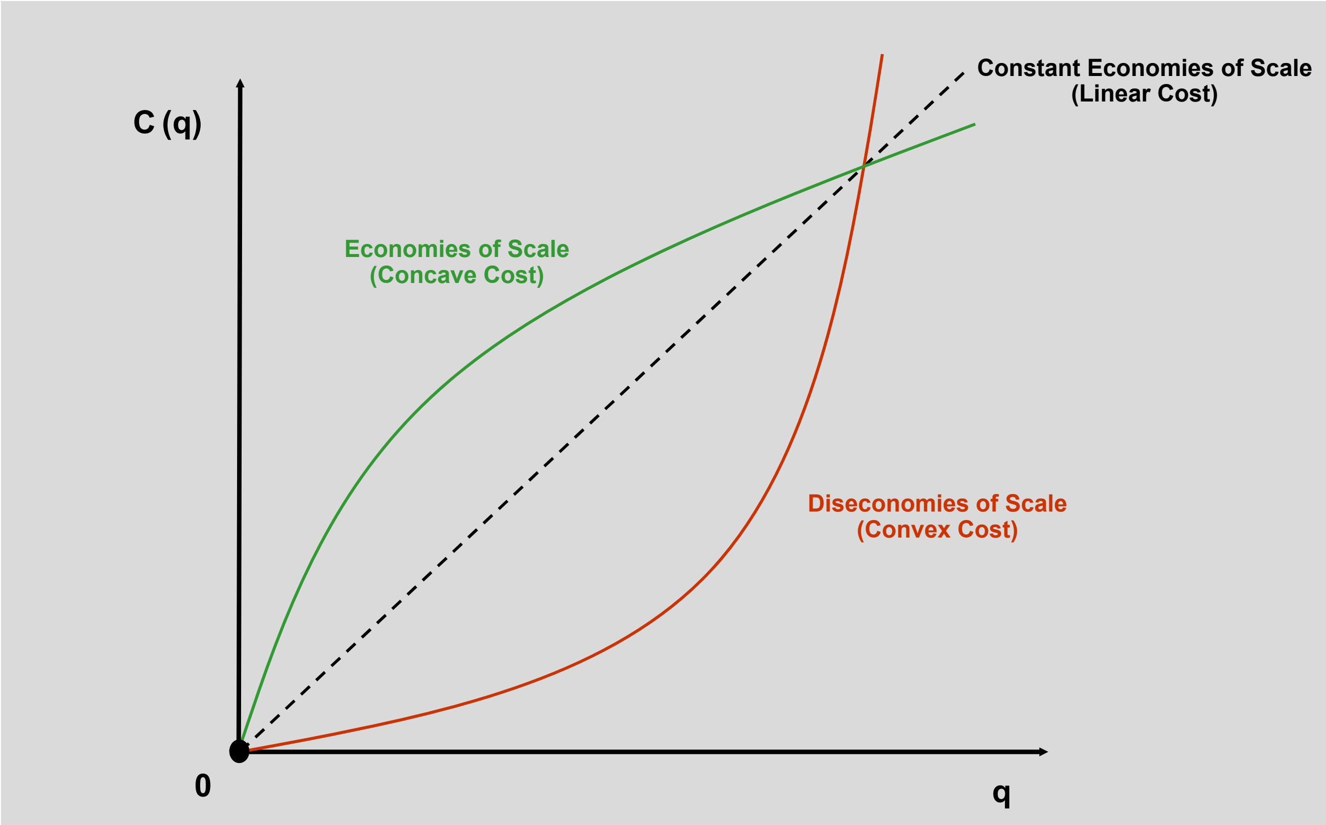
$$AC(q) = \frac{C(q)}{q}$$ **goes down, as q goes up.**

**If average costs increase in q, then C(.) exhibits diseconomies of scale; if average costs stay constant, then C(.) exhibits constant economies of scale.**

**[Remark. Marginal cost is the cost "at the margin," corresponding to the slope of C(q) at q, i.e., MC(q) = C'(q).]**

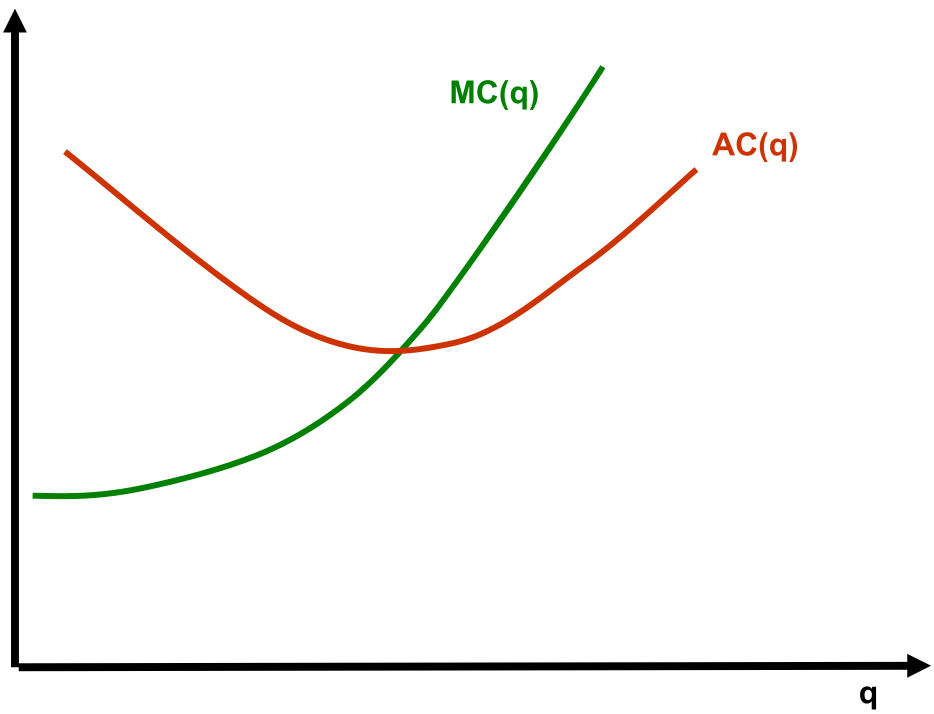
**Similarly, a production function F(z) with a scalar input z exhibits economies of scale (*diseconomies of scale/constant economies of scale*) if the conversion rate F(z)/z increases (decreases/stays constant) in z > 0.**

# ECONOMIES/DISECONOMIES OF SCALE






# AVERAGE COST AND MARGINAL COST




MC(q) = AC(q) at the minimizer of the average cost (= "minimum efficient scale")
You should be able to prove this relation yourself

# PROFIT

**Difference between value to firm of outputs and cost to the firm of inputs.**

**For firm that sells outputs:**

- **Profit $\Pi$(q) is difference between total dollar revenues R(q) received by the firm and total dollar costs C(q) the firm incurs, as a function of its output q**

$$\Pi(q) = R(q) - C(q)$$

**Revenues include**

- **Money for selling the outputs, service fees, royalties, license fees, etc.**

**Costs include**

- **Expenditures for purchased goods, wages paid for labor, taxes**
- **Normal rate of return on invested capital**
- **Value of firm-owned resources allocated to alternative uses**
- **Value of time of the firm owner, allocated to firm activities**



# AGENDA

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**The Firm's Cost Function**

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**Key Concepts to Remember**

# PROFIT MAXIMIZATION

$$\Pi(q) = R(q) - C(q)$$

**First-order necessary optimality conditions for local unconstrained (or "interior") maximum.**

$$\frac{\partial \Pi}{\partial q_i} = \frac{\partial R(q)}{\partial q_i} - \frac{\partial C(q)}{\partial q_i} = 0 \iff MR_i(q) = \frac{\partial R(q)}{\partial q_i} = \frac{\partial C(q)}{\partial q_i} = MC_i(q)$$

**Second-order necessary optimality condition for local unconstrained maximum:**

$$\left[\frac{\partial^2 \Pi}{\partial q_i \partial q_i}\right]_{i,j=1}^{n} \leq 0$$

**That is, the Hessian** $D^2\Pi(q)$ **must be a negative semi-definite matrix at the local maximizer.**

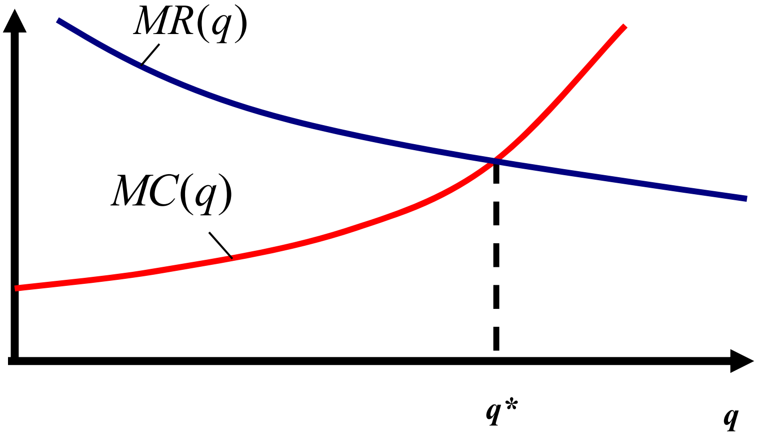


# PROFIT MAXIMIZATION (Cont'd)

**Necessary optimality conditions in the one-dimensional case**

$$\frac{\partial R(q)}{\partial q} = \frac{\partial C(q)}{\partial q}$$

$$R''(q) - C''(q) \leq 0$$



## PROFIT MAXIMIZATION (Cont'd)

**Necessary optimality conditions for profit-maximization (one-dimensional case)**

$$MR(q) = MC(q) \qquad\qquad R''(q) - C''(q) \leq 0$$

**Different Perspectives**

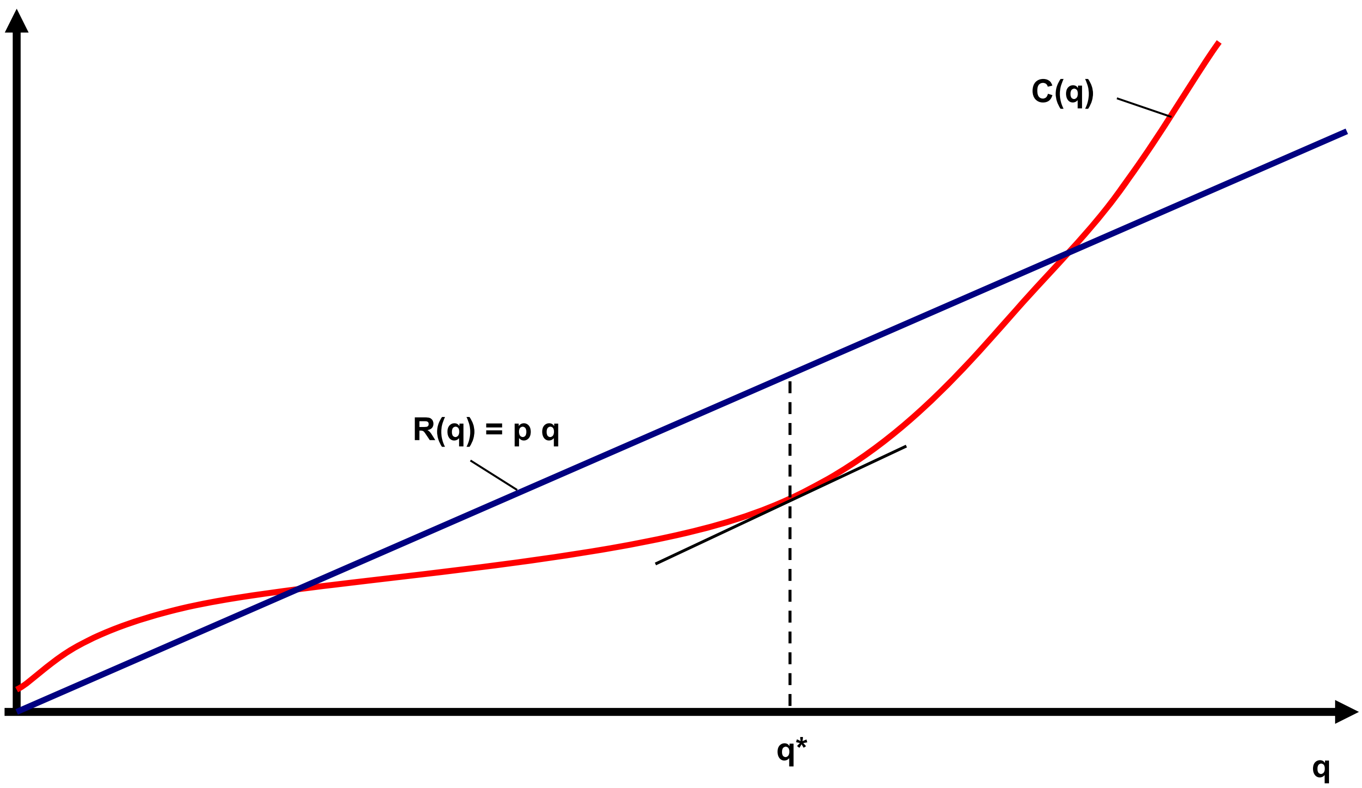
- **External Analyst**
  - **- Assumes marginal revenue = marginal cost**
  - **- Uses to predict activities of firms**

- **Internal Analyst**
  - **- Tries to see if the marginal revenue is equal to marginal cost, and adjusts levels of activities to bring two into equality.**



## TOTAL COST AND REVENUE

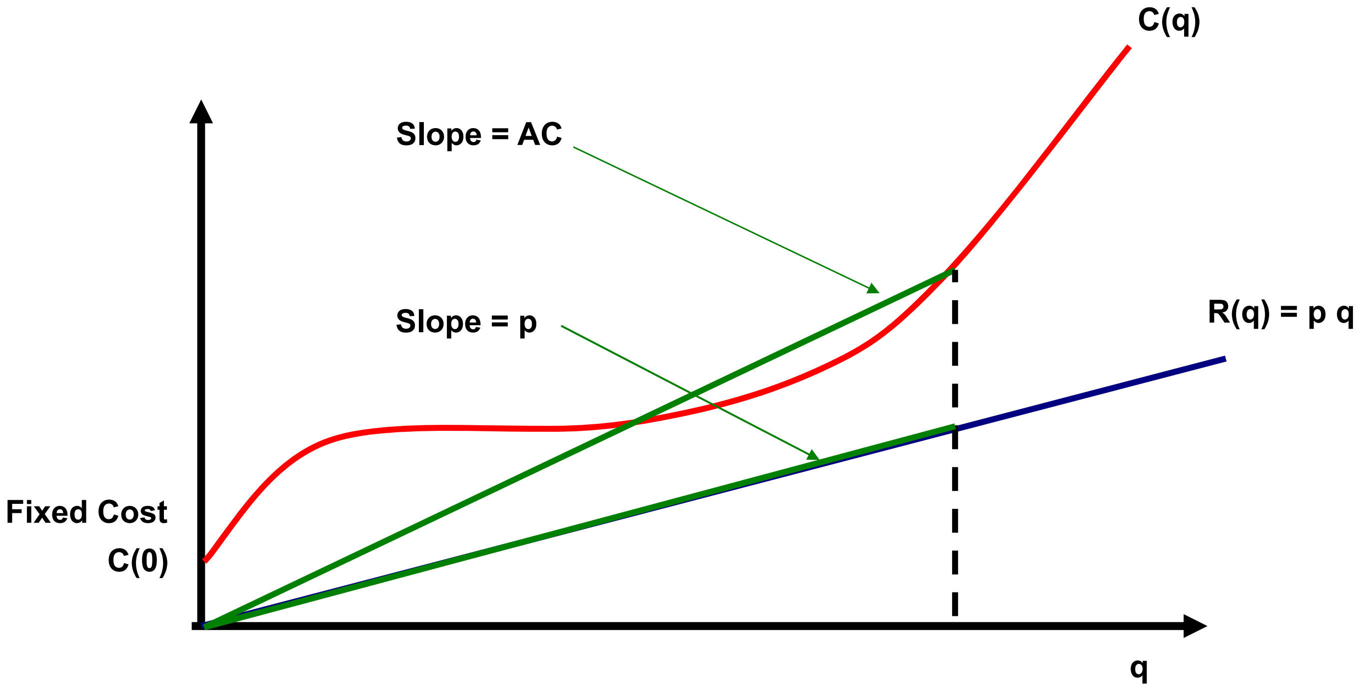
**Consider firm in a competitive market, where price p is a given constant**



## AVERAGE COST & AVERAGE REVENUE

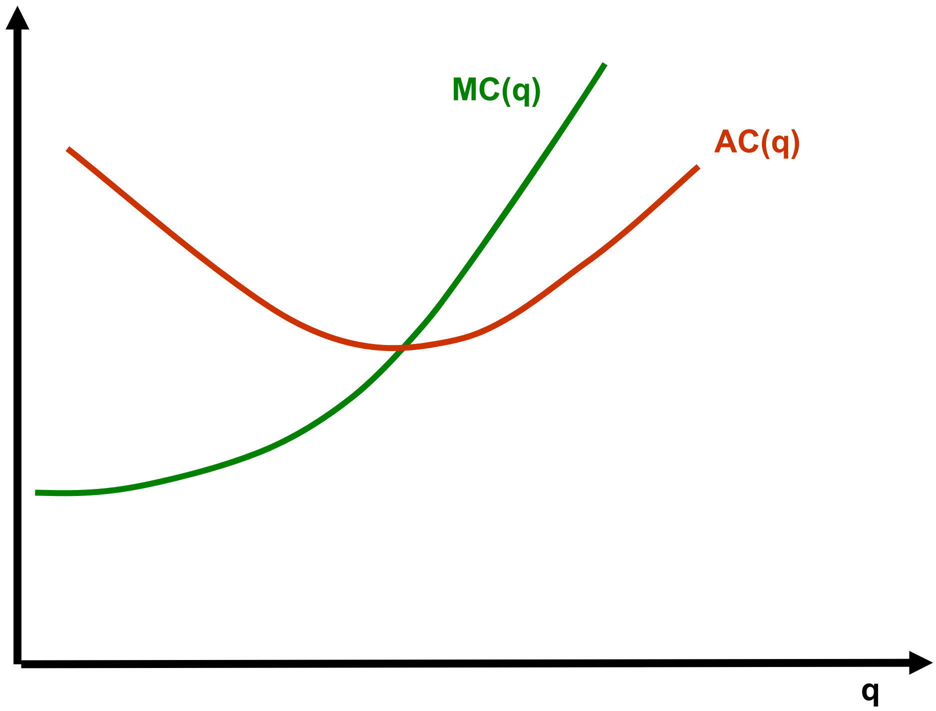
**Average Cost:** AC(q) = C(q)/q

**Average Revenue:** AR(q) = R(q)/q = p





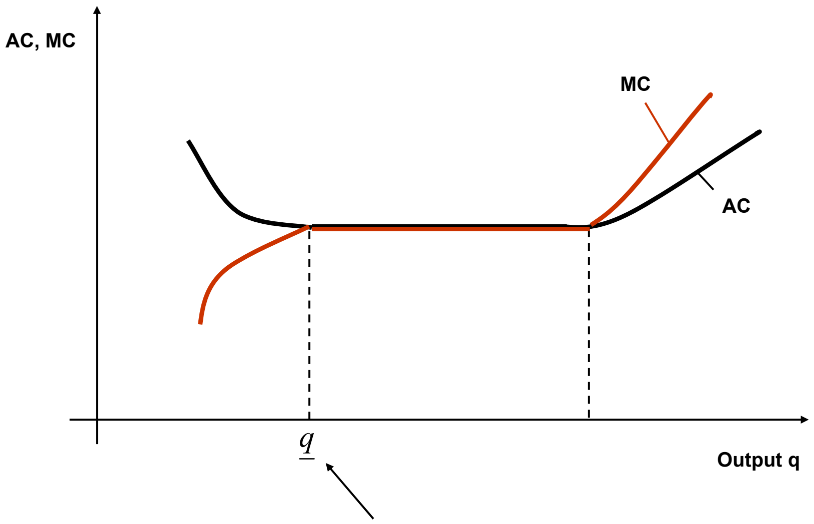

## AVERAGE COST AND MARGINAL COST




**MC(q) = AC(q) at the minimizer of the average cost (= "minimum efficient scale")**
**You should be able to prove this relation yourself**

# MINIMUM EFFICIENT SCALE

**Let $C(q)$ be a firm's increasing, smooth, and convex cost function**




$$AC(q) = C(q)/q$$

$$MC(q) = C'(q)$$

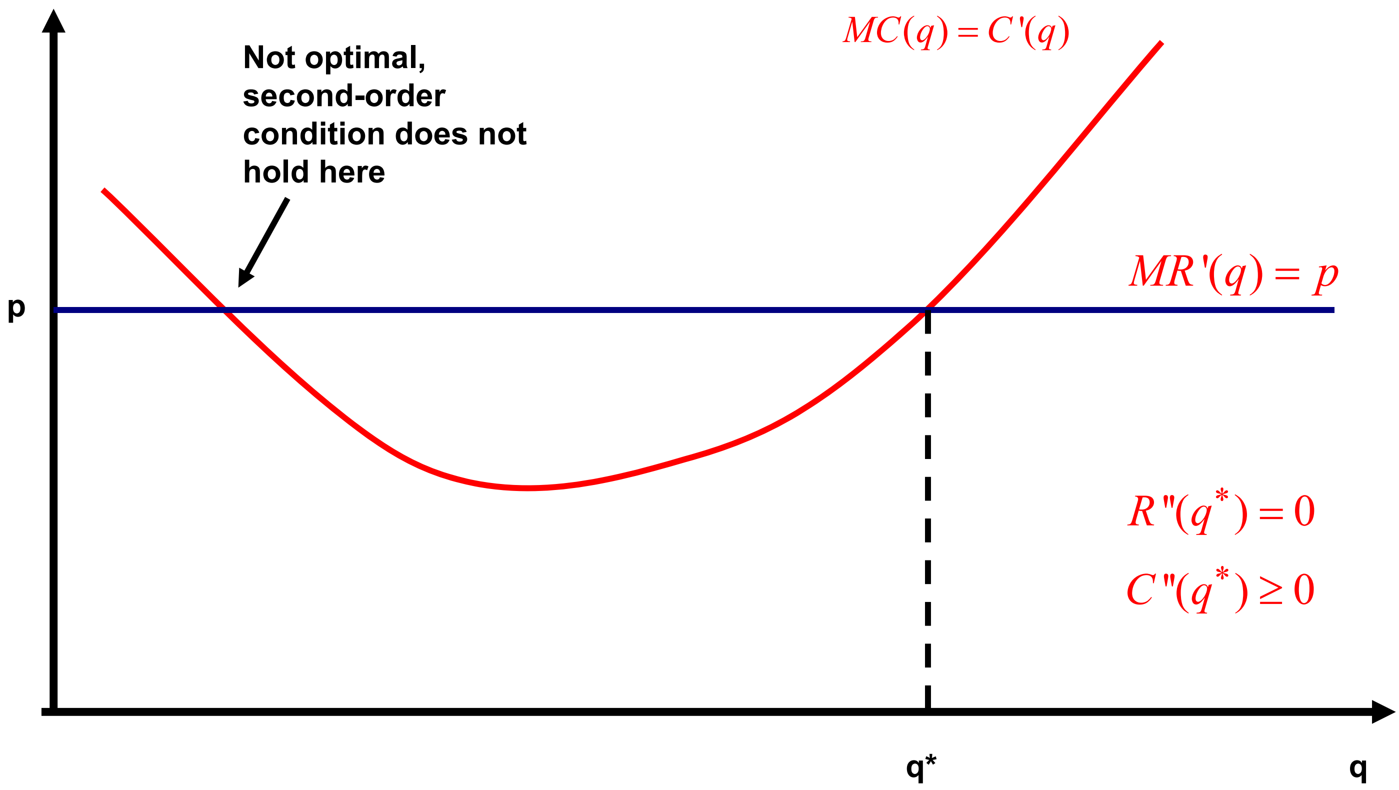
**Minimum Efficient Scale** (1)

$$\underline{q} = \min\{q \geq 0 : AC'(q) = 0\}$$

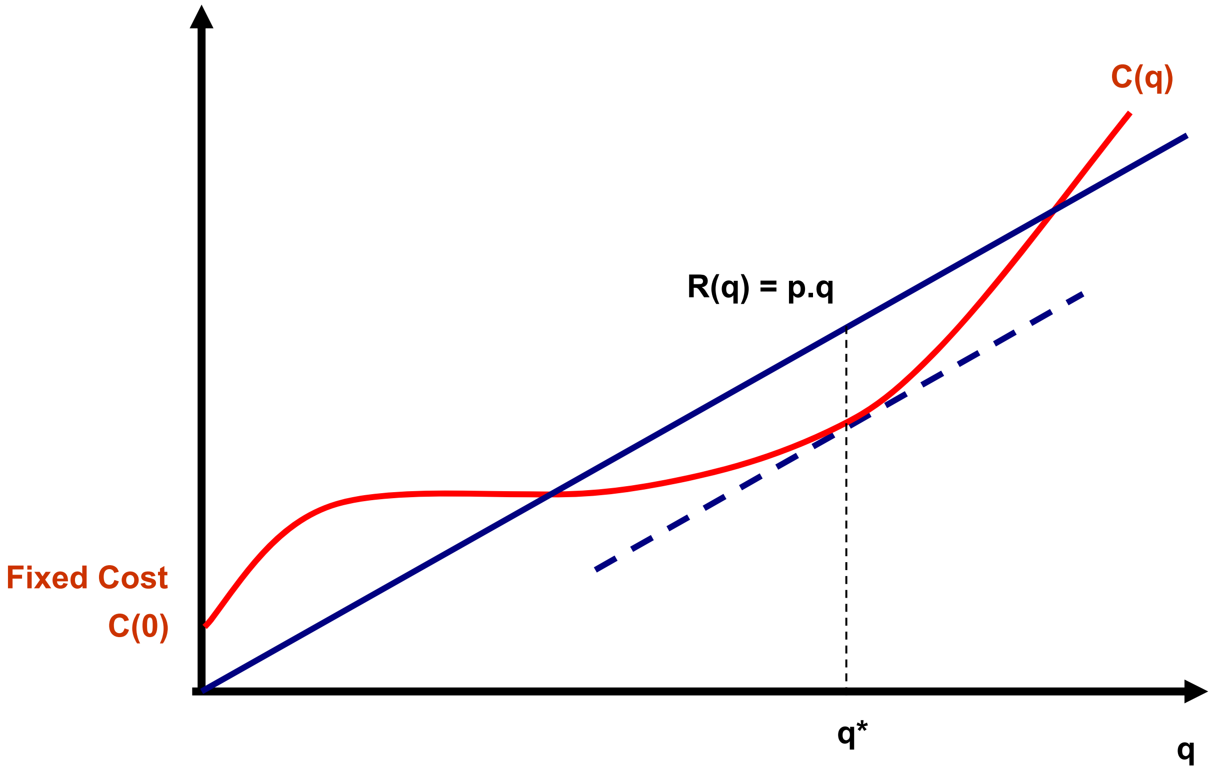
(1): More precisely: $\underline{q} = \min\left\{\arg\min_{q\geq 0}\{AC(q)\}\right\}$



# OPTIMAL CHOICE OF THE FIRM'S OUTPUT LEVEL



## OPTIMAL CHOICE OF THE FIRM'S OUTPUT LEVEL (Cont'd)

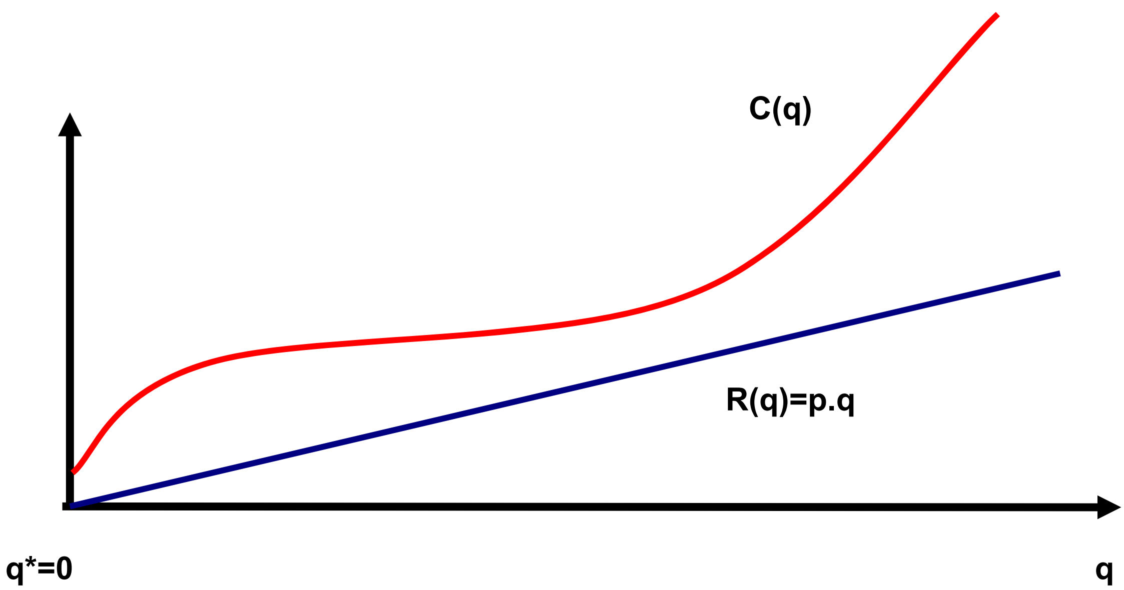



**Remark. The firm's variable cost is C(y) – C(0).**

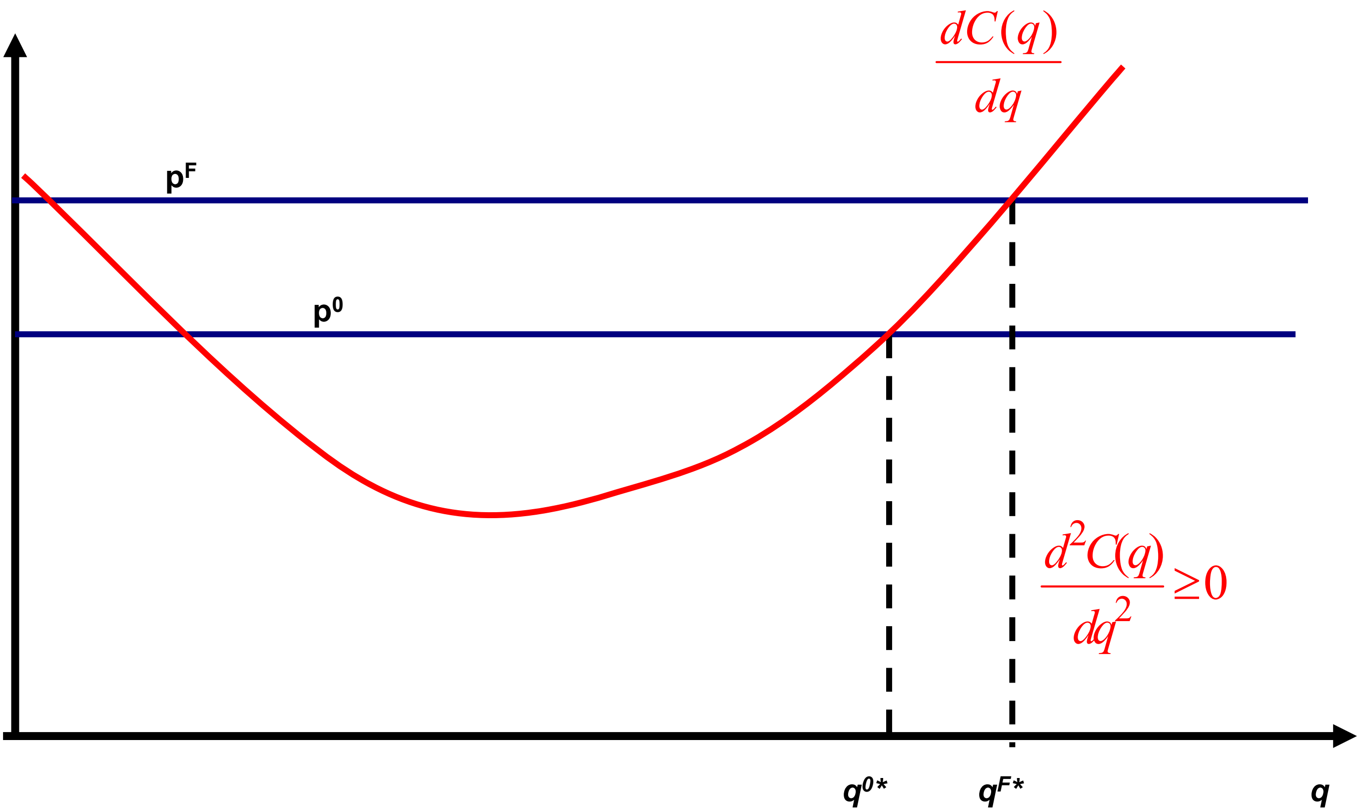


## OPTIMAL CHOICE OF THE FIRM'S OUTPUT LEVEL (Cont'd)

**First and second-order conditions are necessary for an interior optimum. Need to examine the boundaries of the action set as well (here: q* = 0)**



# COMPARATIVE STATICS OF COMPETITIVE FIRM'S OUTPUT






# AGENDA

**Introduction**

**Production Sets**

**Profit Maximization: Some Intuition**

**The Firm's Cost Function**

**Profit Maximization**

**Key Concepts to Remember**

# KEY CONCEPTS TO REMEMBER

- **Profit = Revenue – Cost**
- **(Net) Inputs vs. (Net) Outputs**
- **Profit Maximization Problem**
- **Production (Possibilities) Set**
- **Properties of Production Sets**
- **Production Function**
- **Increasing/Decreasing/Constant Returns to Scale**
- **Cost Function**
- **Cost Minimization Problem**
- **Average Cost, Marginal Cost**
- **Profit Maximization**
- **Marginal Revenue = Marginal Cost**

# MGT 621 – MICROECONOMICS

## Thomas A. Weber

## *5. Market Power*

## Autumn 2023

**École Polytechnique Fédérale de Lausanne**
**College of Management of Technology**



# AGENDA

**What is Market Power?**

**Monopoly**

**Monopsony**

**Price Discrimination**

**Key Concepts to Remember**

# MARKET POWER

**Definition. Market power is the ability of a firm to *increase its output prices above the competitive level*, and/or to *reduce its input prices below the competitive level*.**

- **Monopoly**
  - **Single seller of a product**
- **Oligopoly**
  - **Small number of sellers of a product**

**Sellers' Market**

- **Monopsony**
  - **Single buyer of a product**
- **Oligopsony**
  - **Small number of buyers of a product**

**Buyers' Market**



# ANALYSIS OF MARKET POWER
## Initial Focus on Single Firm

**We first examine the case where one single firm has market power, in a monopoly or a monopsony. Other market participants' actions are aggregated to a market demand (for monopoly) or a market supply (for monopsony).**

- **When more than one firm holds market power, it is necessary to model the interactions between those firms explicitly. For this, one needs the tools of *Game Theory***

**Since actions of all non-market-power-holding entities (the 'other' side of the market) are aggregated into a demand curve (or a supply curve), this is often referred to as partial equilibrium analysis.**

**In general equilibrium analysis, the optimizing behavior of all market participants is explicitly taken into account (they could be price takers or not).**

**We first focus on partial equilibrium analysis of monopoly and monopsony.**

# AGENDA

What is Market Power?

**Monopoly**

Monopsony

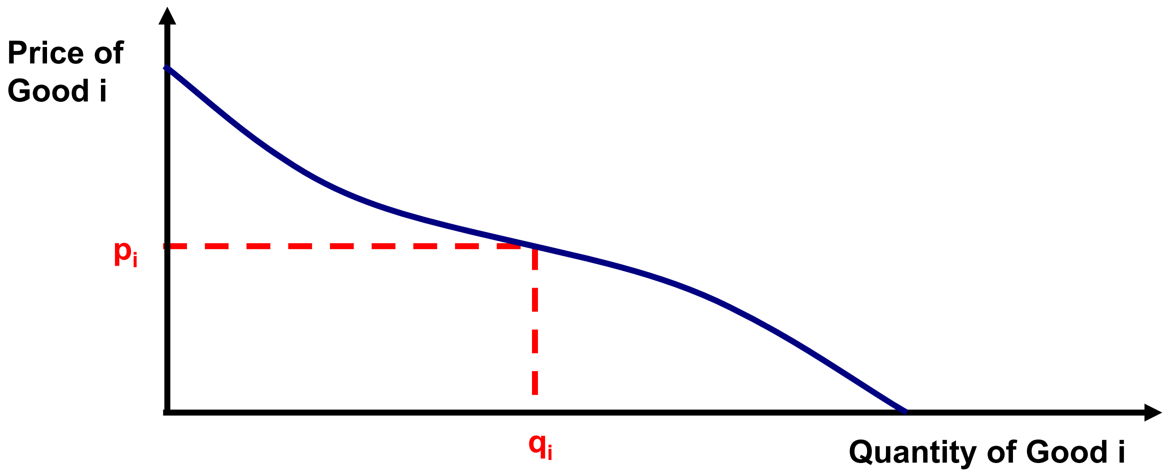
Price Discrimination

Key Concepts to Remember



# DEMAND CURVE

**The quantity of commodity i a monopolist can sell, its "demand" $D_i(p)$, is a decreasing function of the price $p_i$. Equivalently, the price at which the firm can sell the product, referred to as its "inverse demand" $p_i(q_i,q_{-i}; p_{-i})$, is a decreasing function of the quantity $q_i$.**




$$q_i = D_i(p_i, p_{-i})$$ ⟵ **Demand Curve**

$$p_i = p_i(q_i, q_{-i}; p_{-i})$$ ⟵ **Inverse Demand Curve**

# OPTIMAL CHOICE OF MONOPOLY OUTPUT

**Assume that a monopolist produces a quantity q of a single output and that the market price at that output is given by the *downward-sloping* inverse market demand p(q). The monopolist's cost function C(q) is *increasing and convex*.**

**Monopolist's profit:**

$$\Pi(q) = \underbrace{R(q)}_{\text{Revenue}} - \underbrace{C(q)}_{\text{Cost}} = p(q)q - C(q)$$

**First-order necessary optimality condition:**

$$\frac{d\Pi(q)}{dq} = \frac{dR(q)}{dq} - \frac{dC(q)}{dq} = 0 \quad \Leftrightarrow \quad \frac{dR(q)}{dq} = \frac{dC(q)}{dq}$$

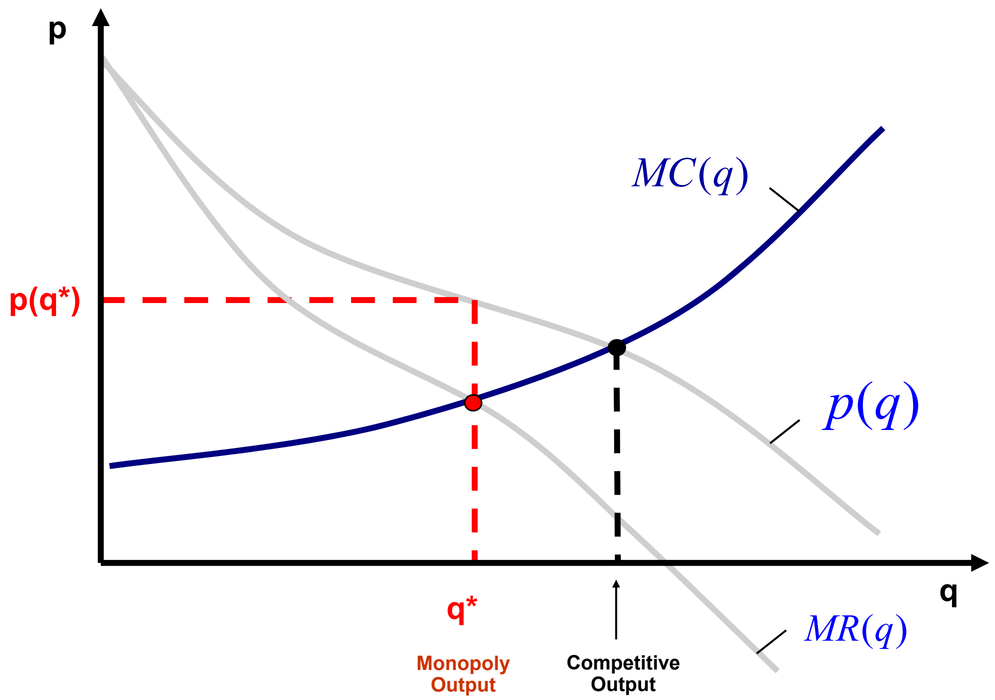
**Hence,**

$$\boxed{p(q) > p(q) + q\underbrace{\frac{dp(q)}{dq}}_{<0} = \frac{dC(q)}{dq}}$$

**In other words, the market price in a monopoly exceeds marginal cost!**



# OPTIMAL MONOPOLY OUTPUT (Cont'd)



# MONOPOLY PRICING
## Inverse Elasticity Rule

**Consider the monopolist's choice of a profit-maximizing price p, given its (downward-sloping) demand function D(p).**

**The (own-price) demand elasticity is** $\varepsilon(p) = -\frac{p}{D(p)}\frac{dD(p)}{dp}$

**Maximizing the monopolist's profit** $\Pi(p) = pD(p) - C(D(p))$

**yields the first-order necessary optimality condition**

$$D(p) + p\frac{dD(p)}{dp} = \frac{dC(D(p))}{dq}\frac{dD(p)}{dp} \quad \text{or} \quad 1 = \left(-\frac{D'(p)}{D(p)}\right)\left(p - MC(D(p))\right) = \varepsilon(p)\frac{p - MC(D(p))}{p}$$

**Hence, we obtain the "inverse elasticity rule" for monopoly pricing:**

$$1 \geq \underbrace{\frac{p - MC(D(p))}{p}}_{\text{Relative Markup}} = \underbrace{\frac{1}{\varepsilon(p)}}_{\text{Inverse Demand Elasticity (Lerner Index)}}$$

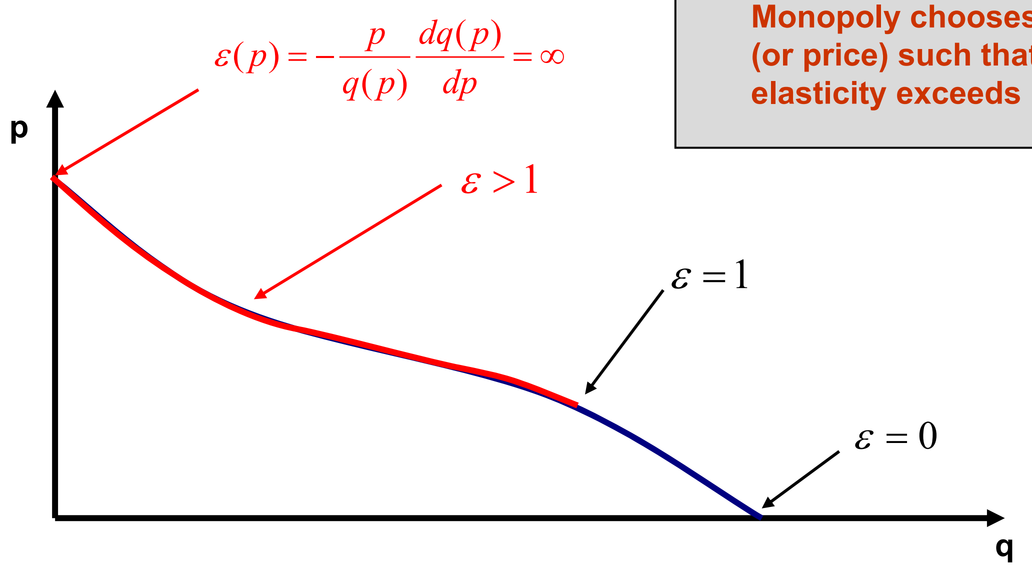


# RELATIVE MONOPOLY MARKUPS

| Demand Elasticity | Lerner Index: Markup as Percent of Price | Markup in Percent of Marginal Cost |
|---|---|---|
| 50 | 2% | 2% |
| 20 | 5% | 5% |
| 10 | 10% | 11% |
| 2 | 50% | 100% |
| 1.5 | 67% | 200% |
| 1.1 | 91% | 1,000% |
| 1.01 | 99% | 10,000% |
| 1 | 100% | Infinity |

# DEMAND ELASTICITY CHANGES ALONG DEMAND FUNCTION

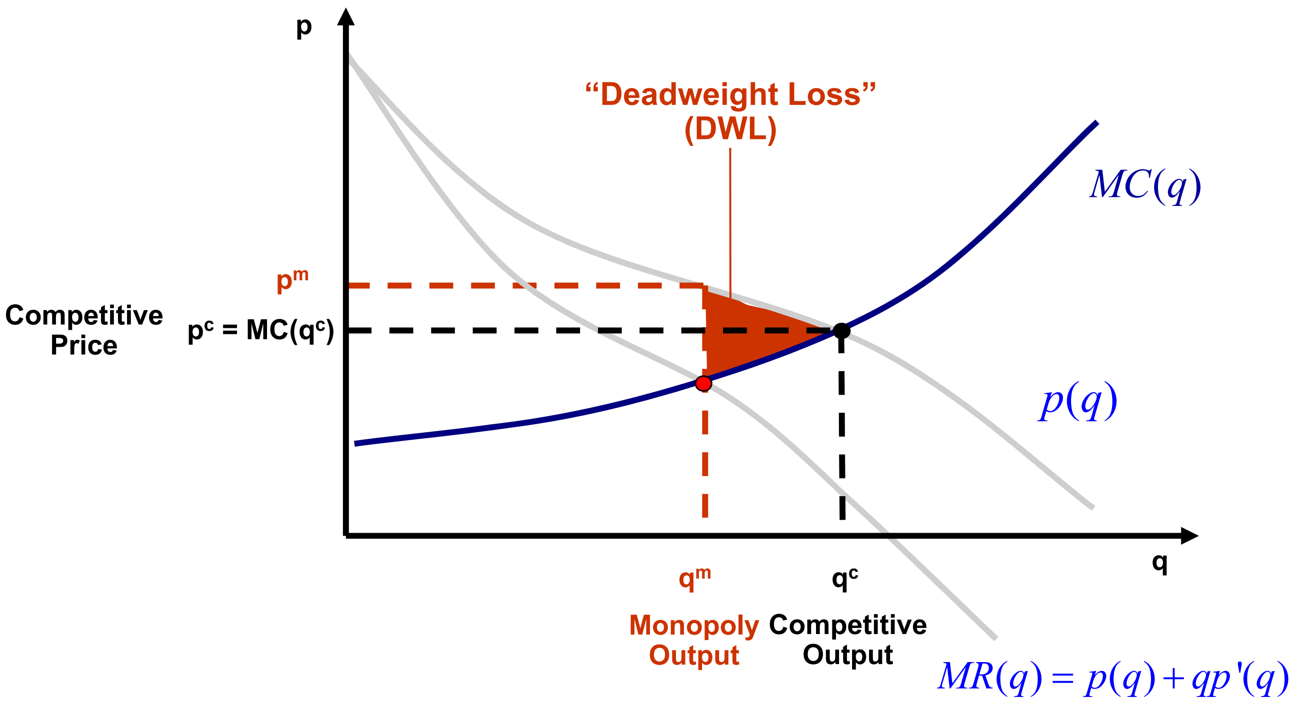
… typically from 0 to infinity




Monopoly chooses production (or price) such that demand elasticity exceeds 1



# INEFFICIENCY CREATED BY MONOPOLY




$$\text{DWL} = \int_{q^m}^{q^c} (p(q) - MC(q))dq = \int_{q^m}^{q^c} p(q)dq - \left(C(q^c) - C(q^m)\right)$$

# WHAT CAN A REGULATOR DO?
## Price Caps

**When trying to reduce the deadweight loss created by a monopolist, the typical difficulty a regulator faces, is that the marginal cost MC(q) as a function of output belongs to the monopolist's private information.**

**Hence, when imposing a price-cap $p^{reg}$ the regulator has no way of knowing if the regulated price is corresponds to the efficient market price**

**Two exceptions:**

- **When $p^{reg} > p^{m}$, then the observed market price is below the price cap**
- **When $p^{reg} < p^{c}$, then one may be able to observe excess demand**

**In general, in order to set an efficient market price (improving the performance of the market by reducing deadweight loss) a regulator needs to find ways to elicit the monopolist's private information about its cost structure.**



# AGENDA

**What is Market Power?**

**Monopoly**

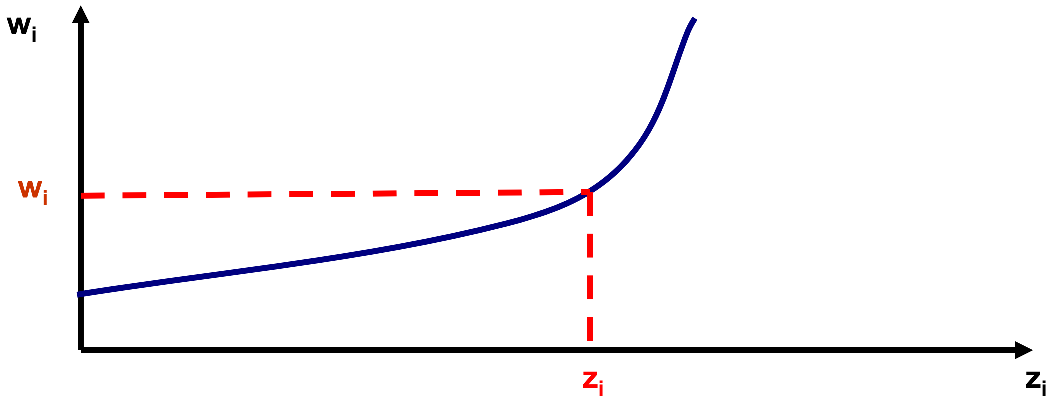
**Monopsony**

**Price Discrimination**

**Key Concepts to Remember**

# SUPPLY CURVE

**The quantity $z_i$ of commodity i a monopsonist can buy, its "supply" $S_i(w_i, w_{-i}; z_{-i})$, is an increasing function of the price $w_i$. Equivalently, the price at which the firm can buy the product, referred to as its "inverse supply" $w_i(z_i, z_{-i}; w_{-i})$, is a decreasing function of the quantity $z_i$.**




$$z_i = S_i(w_i, w_{-i}; z_{-i}) \longleftarrow \text{Supply Curve}$$

$$w_i = w_i(z_i, z_{-i}; w_{-i}) \longleftarrow \text{Inverse Supply Curve}$$



# OPTIMAL MONOPSONY INPUT

**Without loss of generality, consider input 1, and assume that the firm has one output q which is produced as a function of the input vector, i.e., q(z) is the firm's production function.**

**Profit:**

$$\Pi(z) = pq(z) - \sum_{i \neq 1} w_i z_i - w_1(z_1) z_1$$

**FOCs:**

$$\frac{\partial \Pi(z)}{\partial z_i} = p\frac{\partial q(z)}{\partial z_i} - w_i = 0 \quad \text{for } i \neq 1$$

$$\frac{\partial \Pi(z)}{\partial z_1} = p\frac{\partial q(z)}{\partial z_1} - w_1(z_1) - \frac{\partial w_1(z_1)}{\partial z_1} z_1 = 0$$

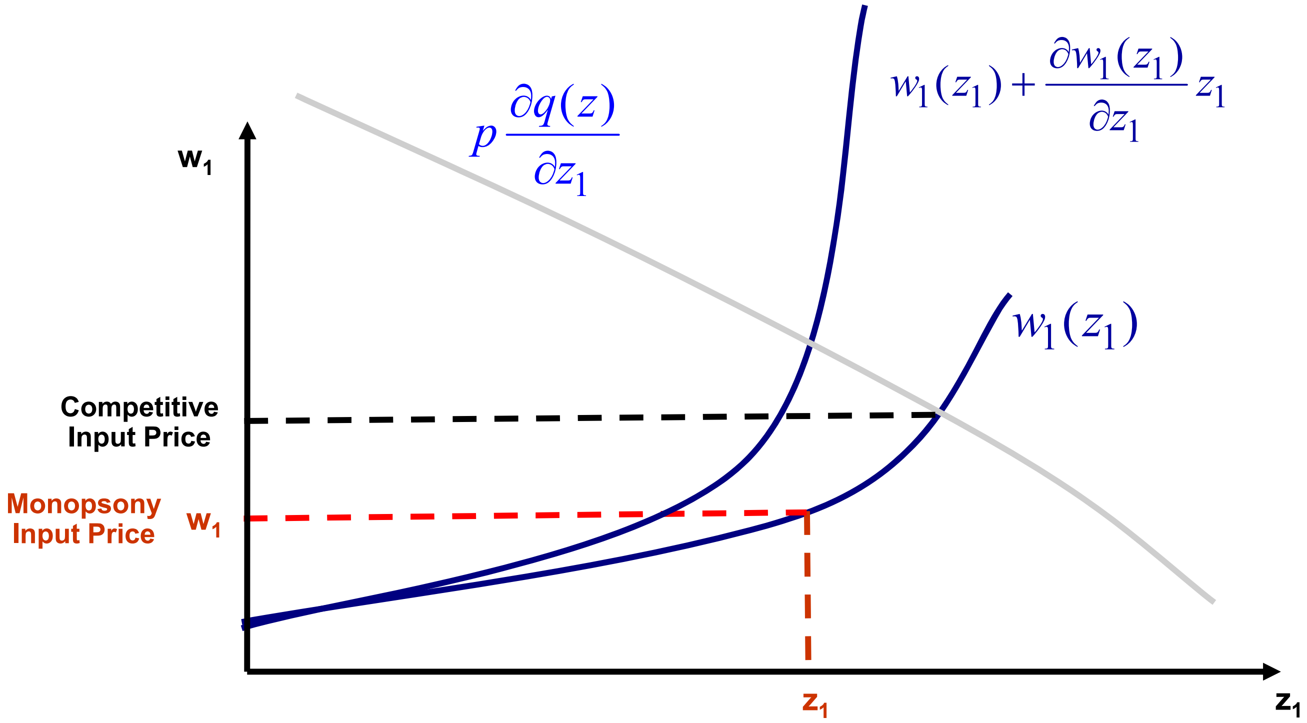
---

$$p\frac{\partial q(z)}{\partial z_i} = w_i \quad \text{for } i \neq 1$$

$$p\frac{\partial q(z)}{\partial z_1} = w_1(z_1) + \frac{\partial w_1(z_1)}{\partial z_1} z_1 > w_1(z_1)$$

# MONOPSONIST'S INPUT CHOICE

**Consider input 1.**






# AGENDA

**What is Market Power?**

**Monopoly**

**Monopsony**

**Price Discrimination**

**Key Concepts to Remember**

# WHAT IS PRICE DISCRIMINATION?

**Definition. Price discrimination exists if *different units of the same good are sold at different prices to one or more consumers*.**

**One commonly distinguishes three different degrees of price discrimination.**

- **First-Degree Price Discrimination: the seller charges a *price for each unit corresponding to the maximum willingness to pay* over all available consumers of that unit. This is also referred to as perfect price discrimination as it maximizes the seller's revenues.**
- **Second-Degree Price Discrimination: the seller charges *different amounts for different numbers of units* bought by the *same* consumer. This is also referred to as nonlinear pricing.**
- **Third-Degree Price Discrimination: the seller charges *different prices to different consumer groups* based on observable differences between the groups.**



# FIRST-DEGREE PRICE DISCRIMINATION

**If the maximum willingness to pay for each unit is available, then the seller can order these values so that the willingness to pay for additional units is nonincreasing. This yields a nonincreasing inverse demand curve p(q) as a function of the seller's output q.**

**The seller can choose the optimal output by maximizing**

$$\Pi(q) = \int_0^q p(\hat{q})d\hat{q} - C(q)$$

**with respect to q. The first-order necessary optimality condition is**

$$p(q) = MC(q)$$

**In other words, the *seller should increase output until the maximum willingness to pay for the next unit exactly equals her marginal cost of producing that unit*.**

**Note that with perfect price discrimination, the monopolist's deadweight loss vanishes, and so does the consumers' surplus.**

# SECOND-DEGREE PRICE DISCRIMINATION

- Second-degree price discrimination (or "nonlinear pricing," or "screening") is a mechanism-design problem. It is more difficult than first-degree or third-degree price discrimination, but it is also more realistic.
- It operates under the assumption that the seller knows that consumers have heterogeneous preferences but is unable to directly distinguish the different consumers. Information about a given consumer's preferences (his utility function) is assumed to be only privately available to that consumer.
- In order to incentivize a consumer to reveal his utility function (or his "type") the seller needs to offer several options for the consumer to choose from. Through his choice the consumer "reveals" his preference, and the seller may thereby be able to charge different consumers (or groups of consumers) different prices.
- The solution to the problem will naturally depend on the seller's model of the consumer heterogeneity.



# EXAMPLE: SELLING A REFRIGERATOR
## Screening Model

Instead of quantities (which can vary continuously) we take a very simple shot at this generally difficult problem and examine a special case where the seller has two refrigerators (of qualities $q_1$=1 and $q_2$ = 2) to sell to consumers who are heterogeneous but indistinguishable to the seller.

Question. How much should the seller charge for the two refrigerators?

What needs to be considered?

1.Buyer's private information:

- The seller does not know how much the buyer is willing to pay for the refrigerator
- She assumes that the buyer values a refrigerator of quality q at $u = \theta q$ , where $\theta$ is unknown to her
- She assumes that the buyer might be of two types $\theta \in \{\theta_L, \theta_H\}$ : with probability $\mu$ it is $\theta_H = 2$ and with probability 1-$\mu$ it is $\theta_L = 1$

2. The buyer's voluntary participation in the mechanism

- The seller cannot force the buyer to pay more than his WTP u
- The seller has to leave the choice of the refrigerator up to the buyer

# SELLING A REFRIGERATOR (Cont'd)

**Designing a mechanism amounts for the seller to choosing the best possible prices $p_1$ and $p_2$ for the two products (of qualities $q_1$ and $q_2$ respectively).**

**Seller's maximizes expected revenues and assuming that the high-type buys the product**

**Buyer's participation ("individual rationality"):**

- **Type** $\theta_L = 1$ **: participates if and only if** $p_1 \le \theta_L q_1 \Leftrightarrow p_1 \le 1$
- **Type** $\theta_H = 2$ **: participates if and only if** $p_2 \le \theta_H q_2 \Leftrightarrow p_2 \le 4$

**Buyer's choice ("incentive compatibility"):**

- **Type** $\theta_L = 1$ **: chooses $q_1$ over $q_2$ if and only if**
  $$\theta_L q_2 - p_2 \le \theta_L q_1 - p_1 \Leftrightarrow 2 - p_2 \le 1 - p_1 \Leftrightarrow 1 \le p_2 - p_1$$
- **Type** $\theta_H = 2$ **: chooses $q_2$ over $q_1$ if and only if**
  $$\theta_H q_1 - p_1 \le \theta_H q_2 - p_2 \Leftrightarrow 2 - p_1 \le 4 - p_2 \Leftrightarrow p_2 - p_1 \le 2$$



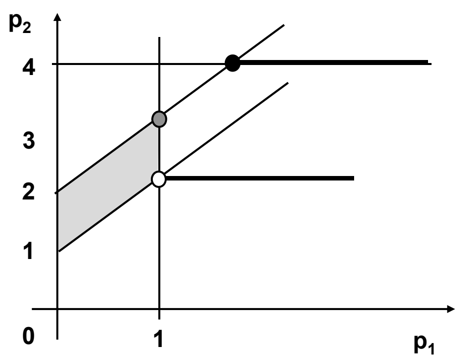
# SELLING A REFRIGERATOR (Cont'd)

**Hence, the seller solves the following revenue-maximization problem:**

$$\max_{p_1, p_2} \{\mu p_1 + (1-\mu) p_2\}$$

**subject to** $p_1 \le 1, p_2 \le 4$ **(individual rationality)**

**and** $1 \le p_2 - p_1 \le 2$ **(incentive compatibility)**




$p_1^* \ge 2, p_2^* = 4$ **(only $\theta_H$ participates; sell only product 2)**
**optimal for** $\mu \le 1/2$

$p_1^* = 1, p_2^* = 3$ **(full participation; sell both products)**
**optimal for** $\mu = 1/2$

**(*)** $p_1^* > 1, p_2^* = 2$ **(full participation; sell only product 2)**
**optimal for** $\mu \ge 1/2$

Remark. Solution (*) is typically "overlooked" in the more general case, as it does not satisfy the incentive-compatibility constraint for the low type but at the same time encourages full participation. We neglect it in the discussion that follows.

# CONCLUSIONS ABOUT THE SCREENING MODEL
## Second-Degree Price Discrimination

**Key Conclusions from the example: (generalizes to other nonlinear pricing models)**

1. **In the presence of asymmetric information, high consumer types typically obtain a positive surplus ("information rent")**

2. **Low-type consumers exert a positive externality on high-type consumers**

3. **As low-type consumers become less frequent, it becomes optimal for the seller to exclude them from the market ("shut-down solution")**

4. **When designing a good mechanism, the seller needs to take into account the consumers' individual rationality and incentive compatibility constraints**

5. **As long as the seller can commit to her mechanism she can, without any loss in generality, restrict her attention to "truthful" mechanisms in which all participating agents report their types truthfully ("revelation principle")**



# A MORE GENERAL EXAMPLE

**Question. At what qualities and what prices should a company offer a "vertically differentiated" product, such as an espresso maker?**

**For simplicity, we restrict attention to a firm which offers at most two products.**

# THERE ARE MANY OTHER EXAMPLES
## Memory Sticks





# CONSIDER A SIMPLE SCREENING MODEL

**Model Features**

- **Two Types ("high" $\theta_H$ and "low" $\theta_L$, with $\theta_H > \theta_L > 0$ )**
- **Utility increasing in instrument and in type, quasi-linear in wealth**
- **Outside option valued at zero**

- **Risk-neutral seller, maximizes expected profit**
- **Prior beliefs of principal (corresponding to the probability $\mu$ of a consumer being a high type) given**
- **Instrument (i.e., product quality) costly to provide, C(q) $\geq$ 0**

**What is missing? – SORTING CONDITION …**

$$\boxed{\hat{q} \geq q \quad \Rightarrow \quad u(\hat{q}, \theta_H) - u(q, \theta_H) \geq u(\hat{q}, \theta_L) - u(q, \theta_L)}$$

**u exhibits "increasing differences" (or is "supermodular")**

**The sorting condition enables the seller to separate high types from low types.**

# SELLER'S PROBLEM

**The seller chooses the qualities and prices of the products such as to maximize her expected profits, i.e., she solves the constrained optimization problem**

$$\max_{p_L, p_H, q_L, q_H \geq 0} \left\{ (1-\mu)\left(p_L - C(q_L)\right) + \mu\left(p_H - C(q_H)\right) \right\}$$

**subject to**

$$u(q_L, \theta_L) - p_L \geq 0 \qquad \textbf{(IR-L)}$$
$$u(q_H, \theta_H) - p_H \geq 0 \qquad \textbf{(IR-H)}$$

**Individual Rationality**

$$u(q_L, \theta_L) - p_L \geq u(q_H, \theta_L) - p_H \qquad \textbf{(IC-L)}$$
$$u(q_H, \theta_H) - p_H \geq u(q_L, \theta_H) - p_L \qquad \textbf{(IC-H)}$$

**Incentive Compatibility**



# THE SELLER'S PROBLEM CAN BE SIMPLIFIED

**Two constraints are binding.**

**1. <u>(IR-L) is binding at optimum</u>**

**Proof. Assume not. Then** $u(q_L, \theta_L) - p_L > 0$ **, so that**

$$u(q_H, \theta_H) - p_H \underset{\textbf{(IC-H)}}{\geq} u(q_L, \theta_H) - p_L \underset{\partial u / \partial \theta > 0}{\geq} u(q_L, \theta_L) - p_L > 0$$

**But this means that** $p_L$ **cannot be optimal, a contradiction. QED**

**2. <u>(IC-H) is binding at optimum</u>**

**Proof. Assume not. Then**

$$\textbf{(*)} \qquad u(q_H, \theta_H) - p_H \underset{\textbf{(IC-H)}}{>} u(q_L, \theta_H) - p_L \underset{\partial u / \partial \theta > 0}{\geq} u(q_L, \theta_L) - p_L = 0$$

**But this means that the seller could increase** $p_H$ **, a contradiction. QED**

# THE SELLER'S PROBLEM CAN BE SIMPLIFIED (Cont'd)

**Two constraints are redundant.**

**3. <u>(IC-L) can be neglected</u>**

**Proof. Since (IC-H) is binding, it is** $u(q_H,\theta_H)-p_H=u(q_L,\theta_H)-p_L$

**Hence,** $$p_H-p_L=u(q_H,\theta_H)-u(q_L,\theta_H)\underset{\text{(SC)}}{\geq}u(q_H,\theta_L)-u(q_L,\theta_L)$$

**Therefore** $u(q_L,\theta_L)-p_L\geq u(q_H,\theta_L)-p_H$ **QED**

**4. <u>(IR-H) can be neglected</u>**

**The proof follows directly from (*) in the proof of claim 2.**



# THE SIMPLIFIED PROBLEM

**The seller's nonlinear pricing problem is equivalent to**

$$\max_{p_L,p_H,q_L,q_H\geq 0}\left\{(1-\mu)\left(p_L-C(q_L)\right)+\mu\left(p_H-C(q_H)\right)\right\}$$

**subject to**

$$u(q_L,\theta_L)-p_L=0 \qquad \textbf{(IR-L)}$$

$$u(q_H,\theta_H)-p_H=u(q_L,\theta_H)-p_L \qquad \textbf{(IC-H)}$$

**The constraints (IR-L) and (IC-H) can be directly substituted into the objective function.**

# THE SIMPLIFIED PROBLEM

**The seller's nonlinear pricing problem is equivalent to**

$$\max_{q_L, q_H \geq 0} \left\{(1-\mu)\big(u(q_L,\theta_L) - C(q_L)\big) + \mu\Big(\big(u(q_H,\theta_H) - u(q_L,\theta_H) + u(q_L,\theta_L)\big) - C(q_H)\Big)\right\}$$

**Hence, the seller's optimal quality levels obtain as follows (for $\mu > 0$):**

$$q_L^* \in \arg\max_{q_L \geq 0} \left\{\big(u(q_L,\theta_L) - C(q_L)\big) - \frac{\mu}{1-\mu}\big(u(q_L,\theta_H) - u(q_L,\theta_L)\big)\right\}$$ **(Distorted Quality Level)**

$$q_H^* \in \arg\max_{q_H \geq 0} \left\{u(q_H,\theta_H) - C(q_H)\right\}$$ **(Efficient Quality Level)**

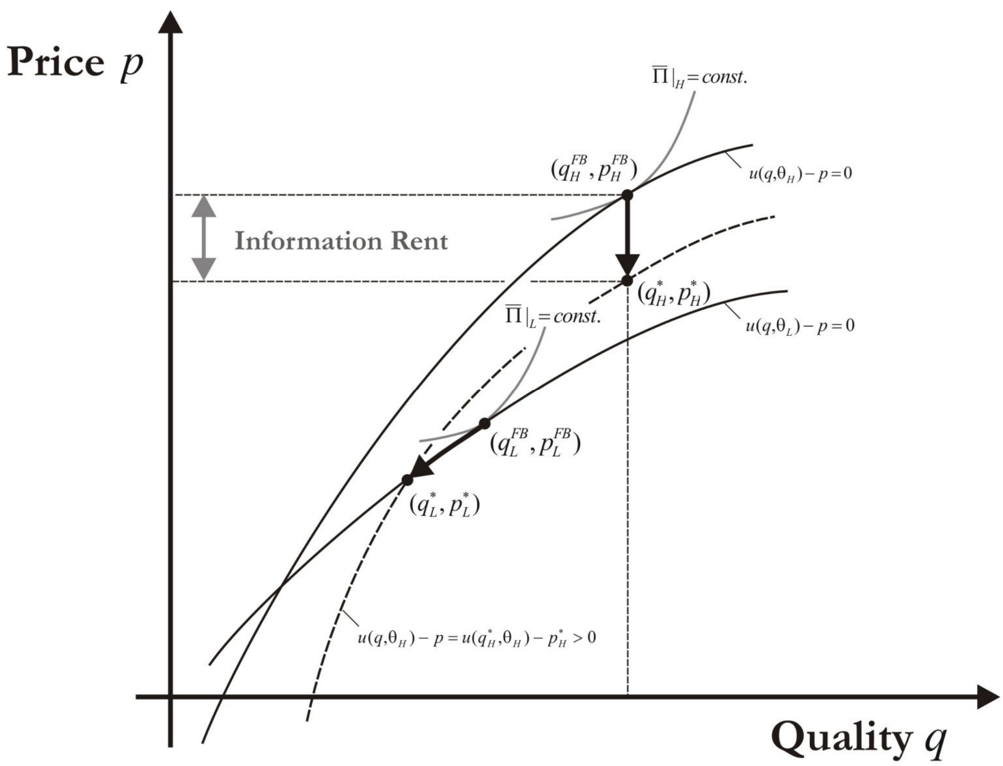
**From (IR-L) and (IC-H) we then get**

$$p_L^* = u(q_L^*,\theta_L)$$ **(Efficient Price Level)**

$$p_H^* = u(q_H^*,\theta_H) - \underbrace{\big(u(q_L^*,\theta_H) - u(q_L^*,\theta_L)\big)}_{\text{Information Rent } (\geq 0)}$$ **(Distorted Price Level)**

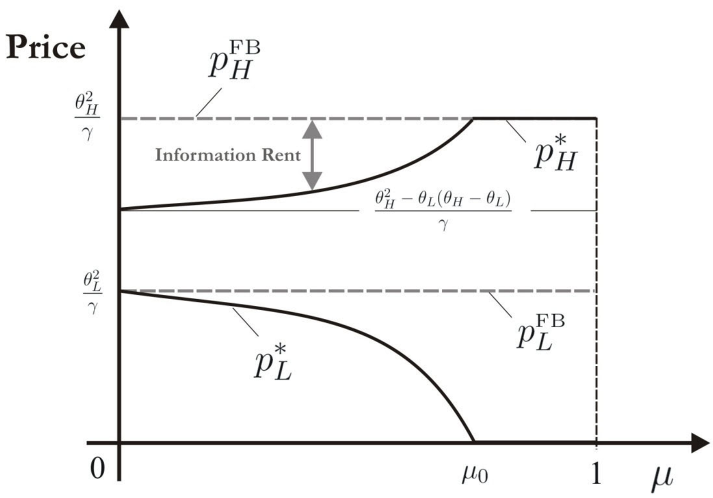


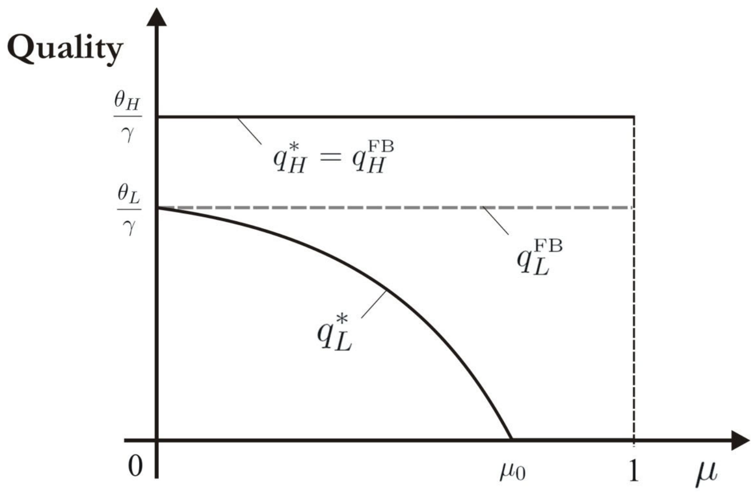
# FIRST-BEST AND SECOND-BEST SOLUTIONS IN THE SCREENING MODEL



# PRICE AND QUALITY IN THE TWO-TYPE SCREENING MODEL

**Example:** $u(q,\theta) = \theta q \qquad C(q) = \gamma q^2 / 2,\ \gamma > 0$






# THIRD-DEGREE PRICE DISCRIMINATION

**For simplicity, let us assume that there are two different consumer groups, 1 and 2, that the seller can distinguish and which can legally be charged different prices for the same product. Let the inverse demand curve of consumer group $i \in \{1,2\}$ be given by $p_i(q_i)$ , where $q_i$ is the amount consumed by that group.**

**Given a standard (increasing, convex) cost function C(q), the monopolist then solves the profit-maximization problem**

$$\max_{q_1, q_2 \geq 0} \{ p_1(q_1)q_1 + p_2(q_2)q_2 - C(q_1 + q_2) \}$$

**which for $q_1, q_2 > 0$ leads to the first-order necessary optimality conditions**

$$p_1(q_1) + q_1 p_1{}'(q_1) = C'(q_1 + q_2)$$
$$p_2(q_2) + q_2 p_2{}'(q_2) = C'(q_1 + q_2)$$

**Hence, at an optimum, the marginal revenues from the two consumer groups are equal to each other and equal to the marginal cost at the combined output.**

## THIRD-DEGREE PRICE DISCRIMINATION (Cont'd)

**More generally, the two consumer groups may not be fully separable. Each group's demand may be influenced by the amount sold to the other group. Then the inverse demand curve of consumer group $i \in \{1,2\}$ is given by $p_i(q_1,q_2)$, where $q_i$ is the amount consumed by that group.**

**Given an increasing, jointly convex cost function $C(q_1,q_2)$, the monopolist then solves the profit-maximization problem**

$$\max_{q_1,q_2\geq 0}\left\{p_1(q_1,q_2)q_1 + p_2(q_1,q_2)q_2 - C(q_1,q_2)\right\}$$

**which for $q_1,q_2>0$ leads to the first-order necessary optimality conditions**

$$p_1(q_1,q_2)+q_1\frac{\partial p_1(q_1,q_2)}{\partial q_1}+q_2\frac{\partial p_2(q_1,q_2)}{\partial q_1}=\frac{\partial C(q_1,q_2)}{\partial q_1}$$

$$p_2(q_1,q_2)+q_1\frac{\partial p_1(q_1,q_2)}{\partial q_2}+q_2\frac{\partial p_2(q_1,q_2)}{\partial q_2}=\frac{\partial C(q_1,q_2)}{\partial q_2}$$

**At an optimum, the marginal revenue from each of the two consumer groups is equal to the marginal cost of increasing the output for that group (sometimes equal to the marginal cost of increasing output for the other group, e.g., when the cost depends only on $q_1$ + $q_2$).**



## AGENDA

**What is Market Power?**

**Monopoly**

**Monopsony**

**Price Discrimination**

**Key Concepts to Remember**

# KEY CONCEPTS TO REMEMBER

- Market Power
- Monopoly/Monopsony
- (Own-)Price Elasticity
- Inverse Elasticity Pricing Rule
- Lerner Index
- Deadweight Loss
- Price Caps
- Price Discrimination (first/second/third degree)

# MGT 621 – MICROECONOMICS


## Thomas A. Weber


## *6. Oligopoly*

## Autumn 2023


## École Polytechnique Fédérale de Lausanne
## College of Management of Technology




# OLIGOPOLY THEORY
## Introduction

**So far in this course we have not emphasized strategic interactions between firms.**

- **We have seen that externalities can lead to significant distortions of the market outcome, even if all firms are price takers**
- **When a monopolist has market power, it can use second-degree price discrimination to segment a heterogeneous consumer base. For that analysis we did consider strategic interactions, but obtained a pure optimization problem, since the monopolist is able to move first by committing to a pricing scheme, anticipating the consumers' actions**

**When multiple firms select their actions simultaneously, and those actions directly influence each others' payoffs (i.e., there are externalities), then we need game theory to produce reasonable predictions about the outcome of the interaction.**

> **Game theory is a fundamental tool in the analysis of strategic interactions between multiple firms with market power.**

# AGENDA

**What is Game Theory?**

**Building Blocks and Key Assumptions**

**Market Structure & Strategy Analysis**

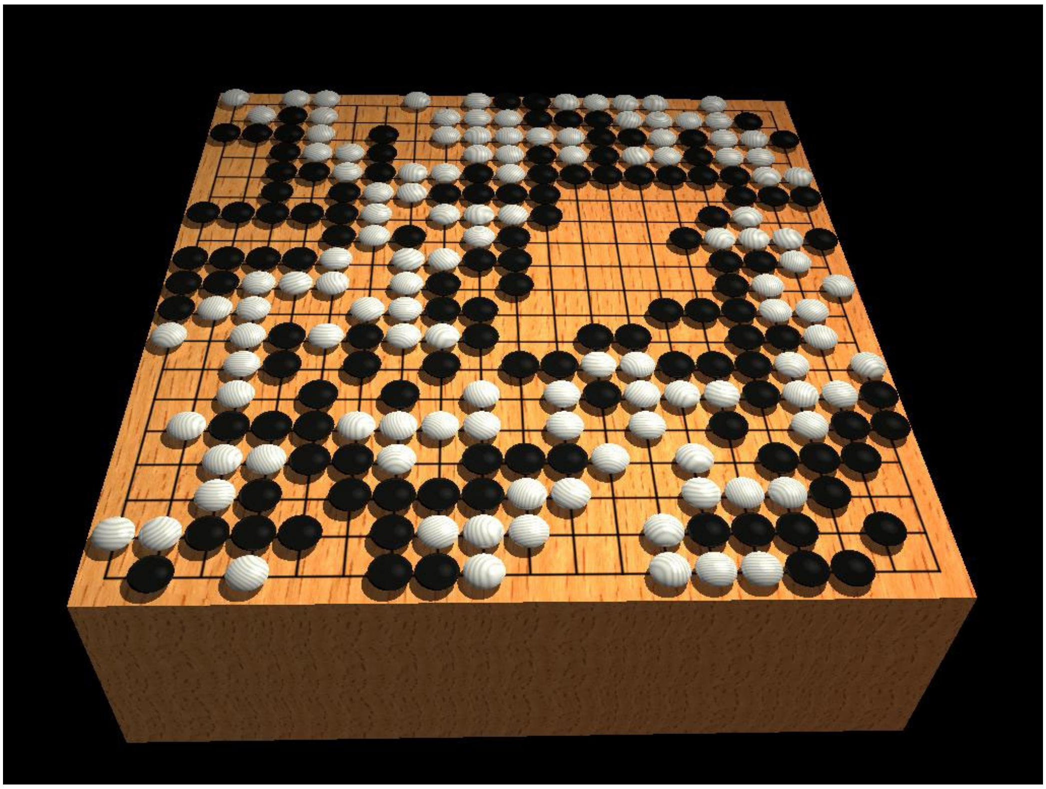
- **Cournot Quantity Competition**
- **Bertrand Price Competition**

**Key Concepts to Remember**



# GAME THEORY

# GAME THEORY

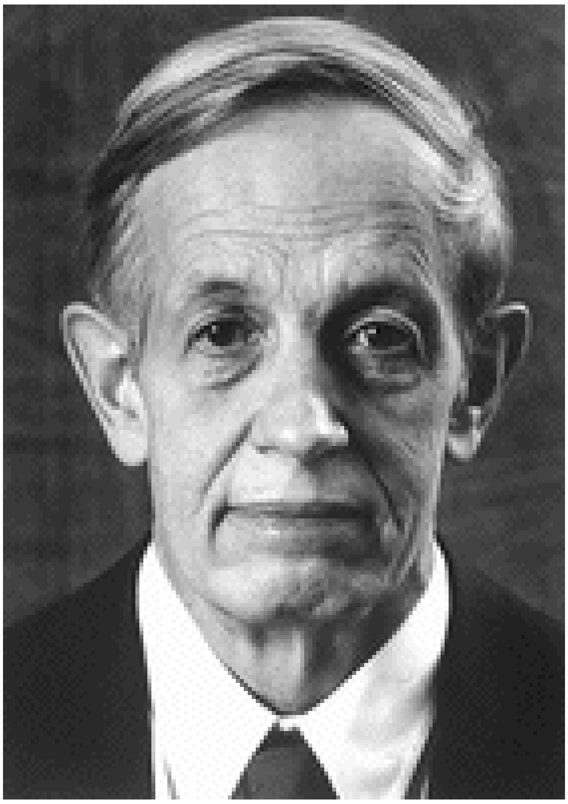




# JOHN VON NEUMANN
# (1903 – 1957)

**Oskar Morgenstern**
**(1902 – 1976)**

# JOHN FORBES NASH
# (1928 – 2015)





# GAME THEORY

**Game Theory is the *analysis of strategic interactions among agents*.**

**A *strategic interaction* is a situation in which each agent, when selecting his or her most preferred action, takes into account the likely decisions of the other agents.**

**Example: War**

> ***"In war the will is directed at an animate object that reacts."***
>
> **- Carl von Clausewitz, *On War***

**The objective of game theory is to provide predictions about the behavior of agents (players) in strategic interactions. The more precise these predictions are, the higher their "predictive power."**

(1) Cf. von Clausewitz, C. (1976) *On War*, Princeton University Press, Princeton, NJ. Clausewitz lived from 1780 to 1831; for more details about his life and work, see http://www.clausewitz.com/. The first systematic academic treatment of game theory is von Neumann, J., Morgenstern (1944) *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ.

# AGENDA

**What is Game Theory?**

**Building Blocks and Key Assumptions**

**Market Structure & Strategy Analysis**

- **Cournot Quantity Competition**
- **Bertrand Price Competition**

**Key Concepts to Remember**



# NORMAL-FORM GAME

**Building Blocks**

- **Players**, $i \in N = \{1,\ldots,n\}$
- **Action Sets (Strategy Spaces)**, $A_i$, **with elements** $a_i \in A_i$
- **Individual Payoffs**[(1)], $u_i(a)$, **where** $a = (a_i, a_{-i}) \in A = A_1 \times \cdots \times A_n$ **and** $a_{-i} \in A_{-i} = A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n$
- **(Mixed) Strategies**,[(2)] $\sigma_i \in \Delta(A_i)$ **and** $\sigma_{-i} \in \Delta(A_{-i})$

**Definition: A *Normal-Form Game* $\Gamma_N$ is a collection of players, action sets, and payoffs,**

$$\Gamma_N = \left\{N, \{\Delta(A_i)\}, \{u_i(\cdot)\}\right\}$$

(1) The payoff functions $u_i(\cdot)$ are generally taken to be von Neumann-Morgenstern utility functions. You can think of them as bounded measurable functions (functionals) mapping actions (probability distributions over actions) into real values. The payoff of a mixed strategy profile is simply the expected value over the random action profile, $u_i(\sigma) = u_i(\sigma_i, \sigma_{-i}) = \sum_{a \in A} [\sigma_1(a_1) \cdot \ldots \cdot \sigma_n(a_n)] u_i(a)$

(2) Instead of pure strategies (elements of $A_i$) we allow "mixed" strategies (elements of $\Delta(A_i)$). If $A_i$ is finite containing $m_i$ elements, then $\Delta(A_i)$ corresponds to the $(m_i - 1)$-dimensional simplex over $A_i$.

## PRISONER'S DILEMMA
### Example

**Two suspects, 1 and 2, are being interrogated separately about a crime**

- **If both confess, each is sentenced to five years in prison**
- **If both deny their involvement, each is sentenced to one year in prison**
- **If just one confesses, he is released but the other one is sentenced to ten years in prison**

**Assume that each player's payoffs are proportional to the length of time of his prison sentence.**

**Formulate this game in normal form.**



## PRISONER'S DILEMMA (Cont'd)
### Example

**Normal-Form Representation**

- **Players,** $i \in N = \{1,2\}$
- **Action Sets,** $A_i = \{Deny, Confess\}$
- **Individual Payoffs,** $u(a_1, a_2)$ **, defined by "payoff matrix"**
- **(Mixed) Strategies,** $\sigma_i = (\sigma_i(Deny), \sigma_i(Confess)) \geq 0$**, with** $\sigma_i(Deny) + \sigma_i(Confess) = 1$

**Payoff Matrix**(1)

| Player 1 \ Player 2 | Confess | Deny |
|---|---|---|
| Confess | (-5,-5) | (0,-10) |
| Deny | (-10,0) | (-1,-1) |

(1) The payoff matrix is *equivalent* to a normal-form representation for (static) two-player games with finite action sets.

# PRISONER'S DILEMMA (Cont'd)
## Example

**Find Prediction about Outcome of this Game**

| | | Player 2 | |
|---|---|---|---|
| | | Confess | Deny |
| Player 1 | Confess | (-5,-5) | (0,-10) |
| | Deny | (-10,0) | (-1,-1) |

- Consider player 1's "best response" when fixing player 2's strategy
- Consider player 2's "best response" when fixing player 1's strategy

**Hence, each player has a dominant strategy: no matter what the other player does, it is optimal (i.e., payoff-maximizing) for player i to select $a_i = Confess$ .**

**Note also that the outcome is inefficient (i.e., does not maximize social surplus).**



# FUNDAMENTAL ASSUMPTIONS

**Question:** What assumptions are necessary to arrive at predictions about outcomes of normal-form games?

**Assumption 1: All players are rational, i.e., they maximize (expected) payoffs.**

**Assumption 2: The players' payoff functions and action sets are common knowledge, i.e.,(1)**

- Each player knows the rules of the game
- Each player knows that each player knows the rules
- Each player knows that each player knows that each player knows the rules
- Each player knows that each player knows that each player knows that each player knows the rules
- Each player knows that each player knows that each player knows that each player knows that each player knows the rules
- ...

**Assumptions 1 and 2 imply a unique prediction in the Prisoner's Dilemma game; we will maintain these assumptions throughout this course**

(1) For a formal definition of common knowledge, see Osborne, M.J., Rubinstein, A. (1994) *A Course in Game Theory*, MIT Press, Cambridge, MA, pp. 73—75.

# WHAT HAPPENS IF PLAYERS ARE NOT RATIONAL?
## Relaxing Assumption 1

**Relaxing the rationality assumption leads to boundedly rational agents, which is compatible with empirical observations. Some features of real-world agents which violate the rationality assumption are:**

- **Overconfidence**
- **Sensitivity to framing of the problem**
- **Satisficing behavior**
- **Intransitive preferences over outcomes (e.g., Allais Paradox, Ellsberg Paradox)**
- **Limited information-processing capabilities**
- **Availability heuristic**
- **Status-quo bias (e.g., endowment effect, regret avoidance, cognitive dissonance)**
- **…**

**There is a fast growing literature on "behavioral game theory" (1)**

(1) See e.g., Camerer, C.F. (2003) *Behavioral Game Theory: Experiments in Strategic Interaction*, Princeton University Press, Princeton, NJ. For more on behavioral decision making, see Kahnemann, D., Tversky, A. (2000) *Choices, Values, and Frames*, Cambridge University Press, Cambridge, UK.



# UNDERSTANDING RATIONALITY

**Consider the following normal-form game (for which we just provide the payoff matrix):**

**Player 2**

| | | L | R |
|---|---|---|---|
| **Player 1** | U | (4,4) | (-1000,3.9) |
| | D | (3.9,3.9) | (4,3.8) |

**Player 2 has a strictly dominant strategy; his dominated strategy can thus be eliminated. This leads to a unique prediction of the outcome (U,L) in this game.**

**Note though that player 1 has to be absolutely sure of the rationality of player 2!**

# PURE-STRATEGY NASH EQUILIBRIUM

**Definition: For any normal-form game** $\Gamma_N = \{N, \{\Delta(A_i)\}, \{u_i(\cdot)\}\}$ **a** **pure-strategy Nash equilibrium** **is a strategy profile** $a^* = (a_i^*, a_{-i}^*)$**, such that for every** $i \in N$ **:**

$$\boxed{a_i^* \in \arg\max_{a_i \in A_i} u_i(a_i, a_{-i}^*)}$$

**In other words,**

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \qquad \forall a_i \in A_i, i \in N$$

**or equivalently**

$$a_i^* \in B_i(a_{-i}^*) \qquad \forall i \in N$$

**Examples: The Prisoner's Dilemma game has a unique pure-strategy Nash equilibrium (NE), in the Matching Penny game such an equilibrium does not exist**



# MIXED-STRATEGY NASH EQUILIBRIUM

**To increase the predictive power in games such as Matching Pennies, we extend the definition of Nash Equilibrium to include mixed strategy profiles of the form** $\sigma \in \Delta(A_1) \times \cdots \times \Delta(A_n)$**.**

**Definition: For any normal-form game** $\Gamma_N = \{N, \{\Delta(A_i)\}, \{u_i(\cdot)\}\}$ **a** **mixed-strategy Nash equilibrium** **is a strategy profile** $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$**, such that for every** $i \in N$ **:**

$$\boxed{\sigma_i^* \in \arg\max_{\sigma_i \in \Delta(A_i)} u_i(\sigma_i, \sigma_{-i}^*)}$$

**where**

$$u_i(\sigma) = u_i(\sigma_i, \sigma_{-i}) = \sum_{a \in A} [\sigma_1(a_1) \cdot \ldots \cdot \sigma_n(a_n)] u_i(a)$$

# MATCHING PENNIES

**In the Prisoner's Dilemma game the assumptions of common knowledge and rationality were enough to generate a unique prediction about the outcome, the reason being that each player found it strictly dominant to confess.**

**As we see below, rationality and common knowledge, are generally not enough to generate a prediction for the outcome of a normal-form game.**

**Example: Matching Pennies**

**In a game of Matching Pennies, Ann and Bert, show each other a penny with either heads (H) or tails (T) up. If they choose the same side of the penny, Ann gets both pennies, otherwise Bert gets them.**

**(Note that this is a zero-sum game, as are most games people play for leisure.)**



# MATCHING PENNIES (Cont'd)

**Normal-Form Representation**

$N = \{Ann, Bert\}$

$A_i = \{H, T\}, i \in N$

$u_i(\cdot)$ **defined by the following payoff matrix**

| | | Bert: H | Bert: T |
|---|---|---|---|
| Ann | H | (1,-1) | (-1,1) |
| | T | (-1,1) | (1,-1) |

**Question: What is the outcome of this game?**

# MATCHING PENNIES (Cont'd)

**Consider each player's best-response correspondence**

$$B_i(a_{-i}) = \arg\max_{a_i \in A_i} u(a_i, a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(\hat{a}_i, a_{-i}), \forall \hat{a}_i \in A_i\}$$

$$B_{Ann}(H) = H \qquad B_{Bert}(H) = T$$

$$B_{Ann}(T) = T \qquad B_{Bert}(T) = H$$

| | | **Bert** | |
|---|---|---|---|
| | | **H** | **T** |
| **Ann** | **H** | **(1,-1)** | **(-1,1)** |
| | **T** | **(-1,1)** | **(1,-1)** |

**Result: The players' best-response correspondences do not "intersect."**



# MATCHING PENNIES (Cont'd)

**Let us try to find a mixed-strategy Nash equilibrium in the Matching Pennies game.**
**For simplicity set Ann = Player 1 and Bert = Player 2, so that** $N = \{1,2\}$

**The player's mixed-strategy spaces are**

$$\Delta(A_i) = \{(\sigma_i(H), \sigma_i(T)) : \sigma_i(H), \sigma_i(T) \geq 0 \text{ and } \sigma_i(H) + \sigma_i(T) = 1\}$$

**Without loss of generality, let** $\sigma_1(H) = p$ **and** $\sigma_2(H) = q$ **.**

**Then**

$$\begin{aligned}
u_1(\sigma) &= p\big(qu_1(H,H) + (1-q)u_1(H,T)\big) + (1-p)\big(qu_1(T,H) + (1-q)u_1(T,T)\big) \\
&= p\big(q - (1-q)\big) + (1-p)\big(-q + (1-q)\big) \\
&= p(2q-1) + (1-p)(1-2q) \\
&= (1-2p)(2q-1) \\
&= -u_2(\sigma)
\end{aligned}$$

# MATCHING PENNIES (Cont'd)

**This is a linear optimization problem for each player. Note that player 1 has only control over $p$ and player 2 has only control over $q$.**

**Player 1 can make player 2 indifferent about any of his strategies by choosing $p = .5$ i.e., $\hat{\sigma}_1 = (p, 1-p) = (0.5, 0.5)$ and thus**

$$\Delta(A_2) = \arg\max_{\sigma_2 \in \Delta(A_2)} u_2(\sigma_2, \hat{\sigma}_1)$$

**If player 1 chooses a different strategy, player 2 is not indifferent and strictly prefers to play either $q = 0$ (for $p > .5$) or $q = 1$ (for $p < .5$).**

**On the other hand, if player 2 chooses anything other than $q = .5$, player 1 is not indifferent about her actions and will strictly prefer to play a pure strategy.**

**As a result, $\sigma^* = (\sigma_1^*, \sigma_2^*)$ with $\sigma_i^* = (.5, .5)$ is the unique mixed-strategy Nash equilibrium of the Matching Pennies game.**



# ROLE OF INDIFFERENCE

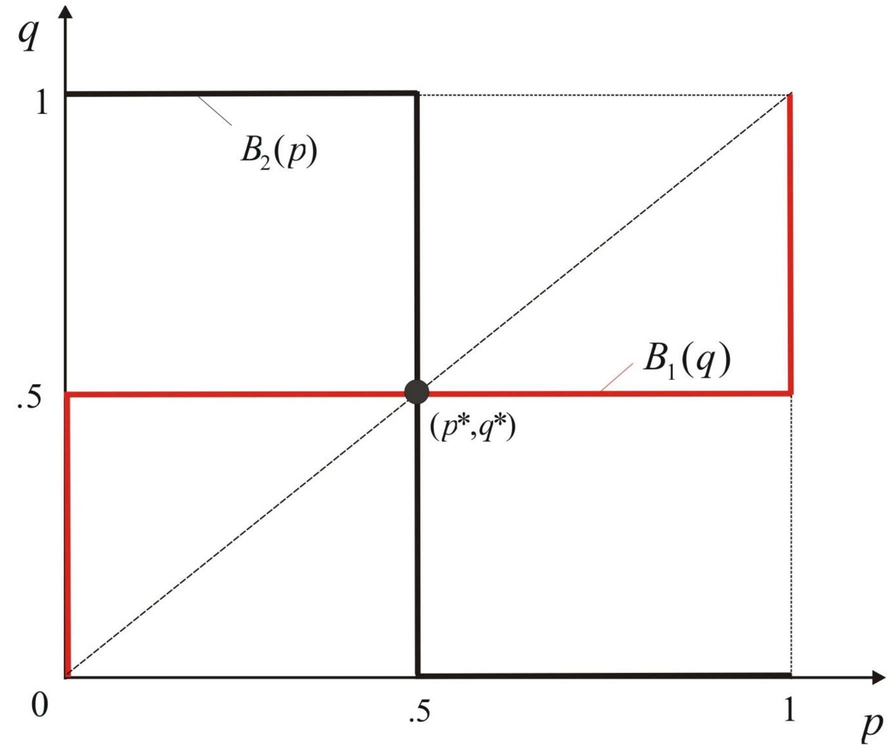
**We emphasize the role that the players' indifference played in determining the NE in the Matching Pennies game. The following assumption is maintained for the rest of the course.**

**Assumption: provided indifference between two or more actions in a player's (mixed-strategy) best-response correspondence, this player will select an action that is part of a (mixed-strategy) Nash equilibrium.[(1)]**

(1) In other words, if a player is indifferent between a strategy that is not a part of a particular Nash equilibrium and a strategy that is part of a particular Nash equilibrium, we assume that this player plays the strategy that is part of the equilibrium (i.e., he makes the equilibrium happen).

# MATCHING PENNIES (Cont'd)

**It is possible to graph the players' best-response correspondences. The unique intersection is at** $(p^*, q^*) = (.5, .5)$ **.**






# AGENDA

**What is Game Theory?**

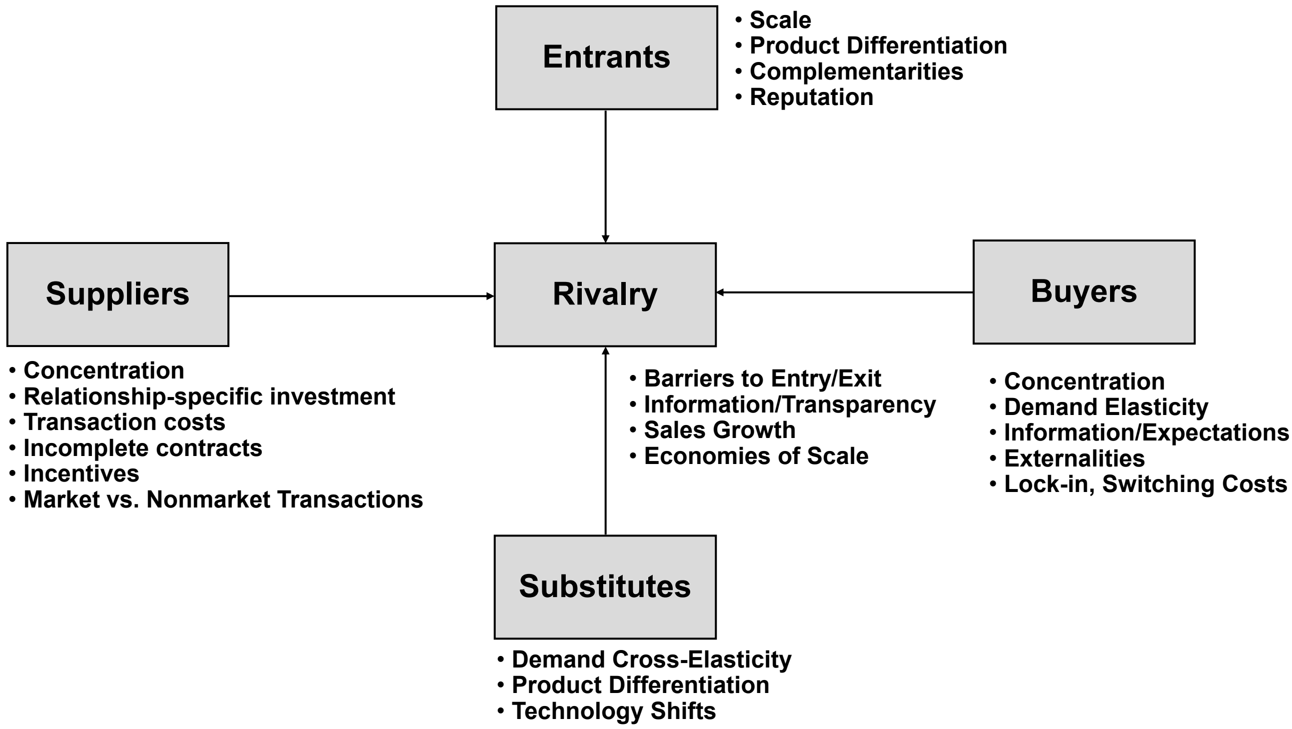
**Building Blocks and Key Assumptions**

**Market Structure & Strategy Analysis**

- **Cournot Quantity Competition**
- **Bertrand Price Competition**

**Key Concepts to Remember**

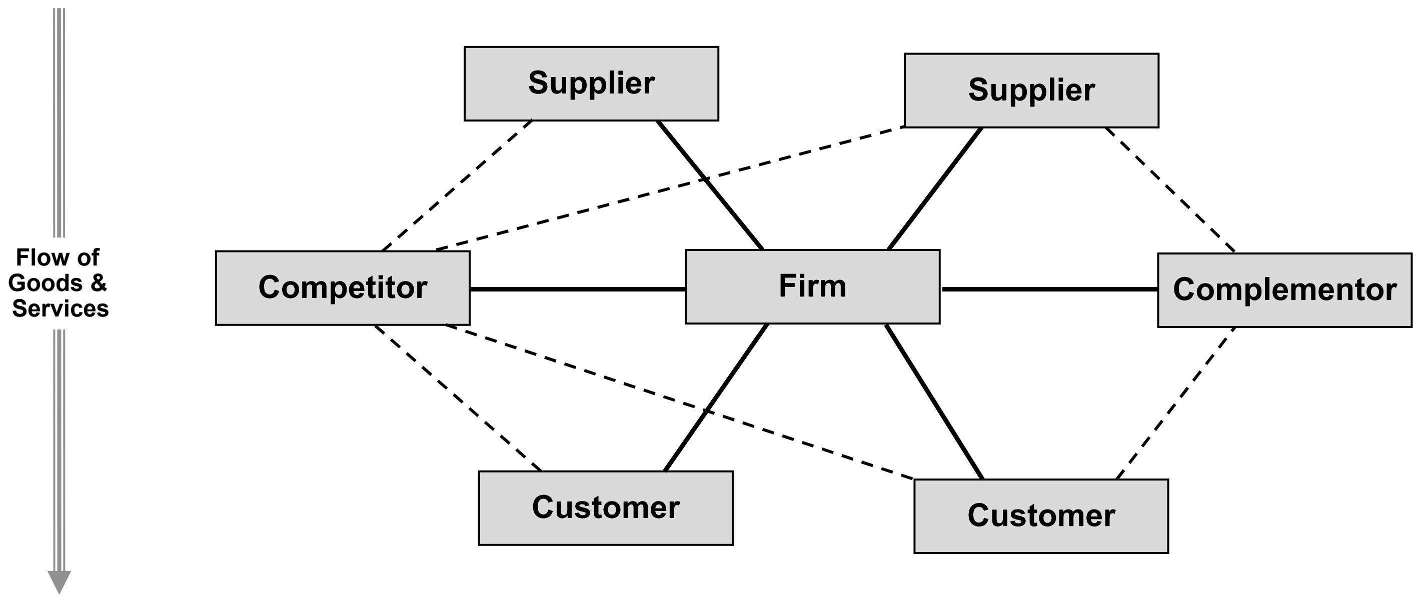
# PORTER'S FIVE FORCES
## … and what influences them




Note: For the original presentation of the Five-Forces Model, see Porter, M.E. (1980) *Competitive Strategy*, Free Press, New York, NY.



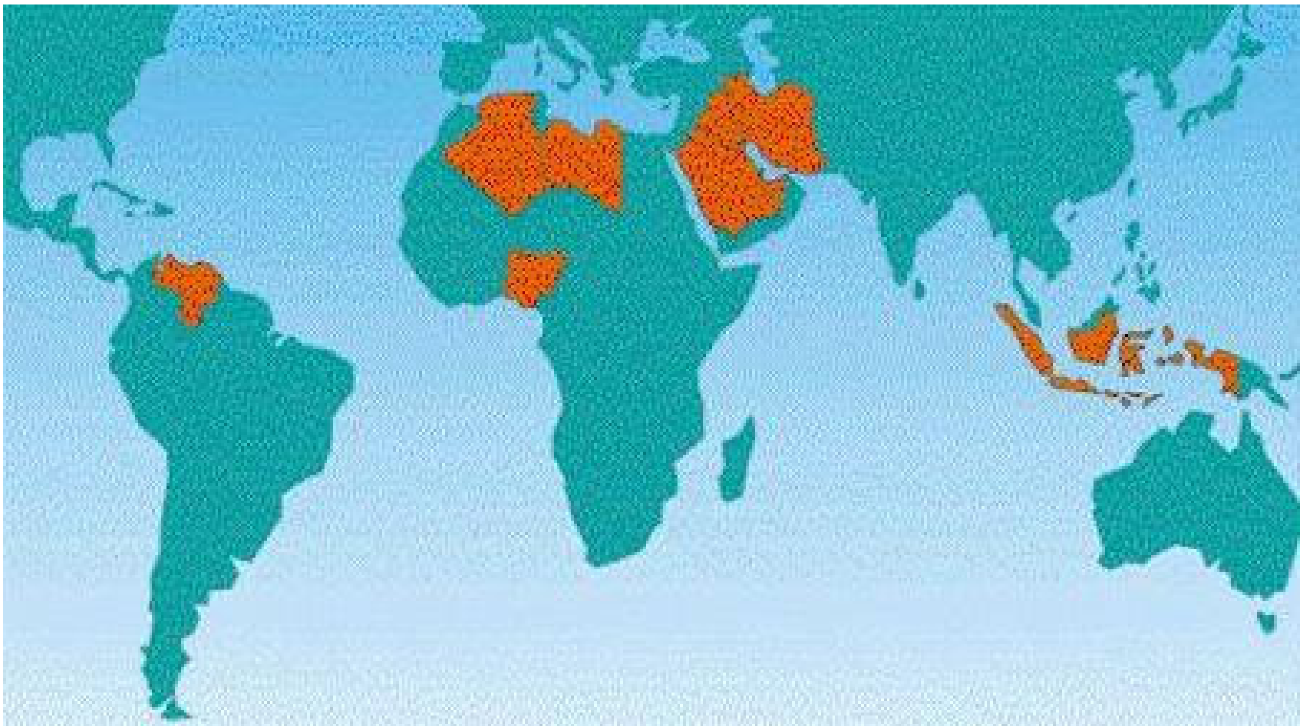
# BRANDENBURGER AND NALEBUFF'S VALUE NET
## The Firm and Its Network of Transaction Relationships




**Note that the firm and its competitors/complementors can have relationships in different markets at the same time ("multimarket contact")**

Note: For the original presentation of the value net, see Brandenburger, A.M., Nalebuff, B.J. (1995) "The Right Game: Use Game Theory to Shape Strategy," *Harvard Business Review*, Vol. 73, No. 4, pp. 57—71. The presentation here is close to the one in McAfee, R.P. (2002) *Competitive Solutions: The Strategist's Toolkit*, Princeton University Press, Princeton, NJ, p. 25.

## INDUSTRY ANALYSIS
### Example





## CHOOSING QUANTITIES: COURNOT DUOPOLY

**Consider two firms, 1 and 2, choosing their production outputs $q_1$ and $q_2$ simultaneously. Each firm has a unit production cost of c (with 0 < c < 1).**

- **The market (inverse) demand is given by** $p(q_1, q_2) = 1 - q_1 - q_2$

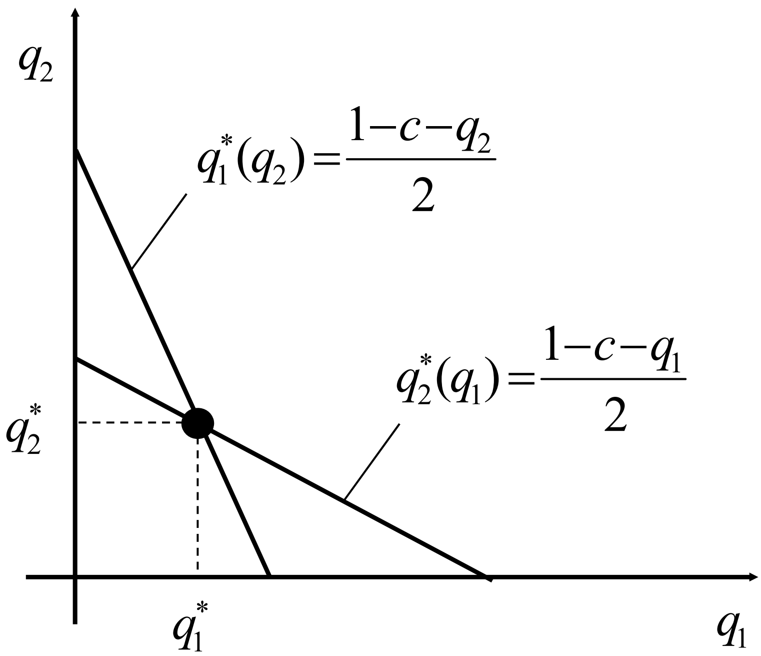
**Question. Determine a Nash equilibrium of this game.**

**Solution.**

**Firm i's profit is** $\Pi_i(q_1, q_2) = \left(p(q_1, q_2) - c\right)q_i = \left(1 - c - q_1 - q_2\right)q_i$

- **Its optimality condition is** $\dfrac{\partial \Pi_i(q_1, q_2)}{\partial q_i} = 1 - c - 2q_i - q_j \overset{!}{=} 0$
- **Its best-response to $q_j$ is therefore** $q_i^*(q_j) = \dfrac{1 - c - q_j}{2}$
- **Symmetry implies that at the Nash equilibrium** $q_i^* = \dfrac{1 - c - q_i^*}{2}$

# COURNOT DUOPOLY (Cont'd)




**Unique Nash Equilibrium:** $\boxed{q_1^* = q_2^* = \frac{1-c}{3}}$



# COURNOT OLIGOPOLY
## Generalization of Previous Example

**Consider the symmetric linear model and perfect substitutability, where**

$$p_i(q) = p(Q) = a - bQ$$

**with** $Q = q_1 + ... + q_n$, $a, b > 0$ **, and**

$$C_i(q_i) = cq_i$$

**where** $c \in (0, a)$ **.**

- **We find that at the unique NE each firm produces** $q_i^* = \frac{a-c}{(n+1)b}$
- **The total supply is in a Cournot NE is thus** $Q^* = nq_i^* = \frac{a-c}{(1+1/n)b}$
- **The Cournot NE price is** $p^* = a - bQ^* = \frac{a/n+c}{1+1/n}$
- **The industry Cournot profits are** $\Pi_T^* = (p^*-c)Q^* = \frac{n}{b}\left(\frac{a-c}{n+1}\right)^2$

# COURNOT OLIGOPOLY (Cont'd)

## Comparison with Perfect Competition

**The market power of each firm can be measured using the Lerner index $L_i$,[(1)] which corresponds to the inverse of the own demand elasticity $\varepsilon_i$ in equilibrium**

$$\varepsilon_i(n) = -\frac{\partial \log q_i}{\partial \log p_i}\bigg|^* = -\frac{p_i}{q_i}\frac{dq_i}{dp_i}\bigg|^* = -\frac{p^*}{q_i^*}\left(-\frac{1}{b}\right) = \frac{a+nc}{a-c}$$

**Let $p^c$ and $Q^c$ denote the price and total output in a symmetric equilibrium under perfect competition. Under perfect competition we have that necessarily $p^c = c$ and $Q^c$ therefore solves $c = a - bQ^c$**

**We have that** $$\lim_{n\to\infty} Q^*(n) = \lim_{n\to\infty} \frac{a-c}{(1+1/n)b} = \frac{a-c}{b} = Q^c$$

**Note also:** $$\lim_{n\to\infty} L_i(n) = \lim_{n\to\infty} \frac{a-c}{a+nc} = 0$$

(1) Lerner, A.P. (1934) "The Concept of Monopoly and the Measurement of Monopoly Power," *Review of Economic Studies*, Vol. 1, No. 3, pp. 157—175.



# STACKELBERG QUANTITY LEADERSHIP GAME

**Consider firms in industries producing goods that are perfect or at least close substitutes**

- **Duopoly**
- **Oligopoly with one dominant firm**
- **Dominant firm and 'competitive fringe'**

**Examples**

- **OPEC or Saudi Arabia**
- **Certain airlines or particular hubs**

# STACKELBERG GAME: COURNOT WITH LEADER

**Suppose there are two symmetric firms. Firm 1 is the leader and gets to choose its quantity at t = 0. Firm 2 is the follower and chooses its quantity at t = 1.**

**Any SPNE (the "Stackelberg Equilibrium") can be found using backward induction, i.e., we start at t = 1. Firm 2 solves**

$$q_2^*(q_1) = \arg\max_{\hat{q}_2 \geq 0} \{(a - c - b(q_1 + \hat{q}_2))\hat{q}_2\}$$

**so that the best-response for firm 2 given the leader's output choice $q_1$ becomes**

$$q_2^*(q_1) = \frac{a-c}{2b} - \frac{q_1}{2}$$



# STACKELBERG GAME (Cont'd)

**Let us now examine the leader's optimal policy at t = 0. Firm 1's residual demand is given by**

$$\hat{p}(q_1) = a - b(q_1 + q_2^*(q_1)) = \frac{a + c - bq_1}{2}$$

**The elasticity of firm 1's residual demand curve is**

$$\hat{\varepsilon}_1 = -\frac{\hat{p}_1(q_1)}{q_1}\frac{dq_1}{d\hat{p}_1} = -\frac{a+c-bq_1}{2q_1}\frac{1}{\frac{d\hat{p}_1(q_1)}{dq_1}} = \frac{a+c}{bq_1} - 1 > \frac{a}{bq_1} - 1$$

Demand elasticity in the absence of followers

**Firm 1 maximizes its profits with respect to residual demand,**

$$q_1^* = \arg\max_{\hat{q}_1 \geq 0}\{(a - c - b\hat{q}_1)\hat{q}_1 / n\} = \arg\max_{\hat{q}_1 \geq 0}\{(a - c - b\hat{q}_1)\hat{q}_1\} = \frac{a-c}{2b} = q^m$$

Monopoly quantity!

**Hence, the follower produces**

$$q_2^* = \frac{a-c}{2b} - \frac{q^m}{2} = \frac{a-c}{4b}$$

## STACKELBERG GAME (Cont'd)

**Total Stackelberg equilibrium output of all firms is therefore**

$$Q^* = q_1^* + q_2^*(q_1^*) = \left(1 - \frac{1}{4}\right)\frac{a-c}{b}$$

**and equilibrium market price is**

$$p^* = a - bQ^* = c + \frac{a-c}{4}$$

**The leader's equilibrium profit is**

$$\Pi_1^* = \frac{(a-c)^2}{8b}$$

**while the follower obtains in equilibrium**

$$\Pi_2^* = \frac{(a-c)^2}{16b}$$



## BERTRAND DUOPOLY

**Consider two firms selling a homogeneous product at a unit cost** $c_i \in [0,1], i \in N = \{1,2\}$

- **The firms simultaneously set their prices** $p_i \in [0,\infty) = A_i$
- **Let the total number of consumers be normalized to one. All consumers buy from the cheaper firm and randomize evenly between the two firms if their prices are equal**
- **The value of the consumers' (common) outside option is zero; their (net) value from the product if they buy from firm i is** $Y$. **(Assume that** $Y > c_i$ **)**

***Question*: Determine the NE of this simultaneous-move game.**

***Answer*: 1. Determine the firms' payoff functions,** $u_i(\cdot)$ **:**

$$u_i(p_i, p_{-i}) = \begin{cases} p_i - c_i, & p_i < p_{-i}, p_i \le Y, \\ (p_i - c_i)/2, & p_i = p_{-i}, p_i \le Y, \\ 0, & otherwise. \end{cases}$$

# BERTRAND DUOPOLY (Cont'd)

**2. Determine the firms' best-response correspondences**

- **Assume that** $c_1 < c_2$
- **Find the set of strategies that *survive iterated deletion of strategies which are never a best response*:**
  - **For player i, the strategies** $p_i < c_i$ **and** $p_i > Y$ **are dominated by** $p_i = Y$
  - **All strategies** $p_i \in [c_i, Y]$ ***could* be rationalizable → not very useful**
- **Find best-response correspondences**
  - **Start with** $p_2 \in [c_2, Y]$**: then** $p_1 \in [p_2, Y]$ **is strictly dominated by any** $p_1 \in (c_1, p_2)$
  - **Player 1's payoffs are strictly increasing in** $p_1 \in (c_1, p_2)$ **. Thus, there is no best-response for player 1, since the payoff from any particular strategy in** $(c_1, p_2)$ **can be strictly improved upon**
    **However, if increments are finite, of arbitrarily small size** $\varepsilon > 0$**, then**[(1)]

$$B_i(p_{-i}) = \begin{cases} Y, & p_{-i} > \max\{c_i, Y+\varepsilon\}, \\ \max\{c_i, p_{-i} - \varepsilon\}, & p_{-i} \in (c_i, Y+\varepsilon], \\ \{c_i + k\varepsilon\}_{k=0}^{\infty}, & p_{-i} = c_i, \\ \{p_{-i} + k\varepsilon\}_{k=1}^{\infty}, & p_{-i} \leq c_i. \end{cases}$$

(1) We approximate the Bertrand game at this point by a series of discrete games $\{\Gamma_N(\varepsilon_n)\}_{n=1}^{\infty}$ with $\varepsilon_n \to 0+$, as $n \to \infty$.



# BERTRAND DUOPOLY (Cont'd)

**3. Find the intersection of the best-response correspondences**

**Continuum of Nash equilibria:**

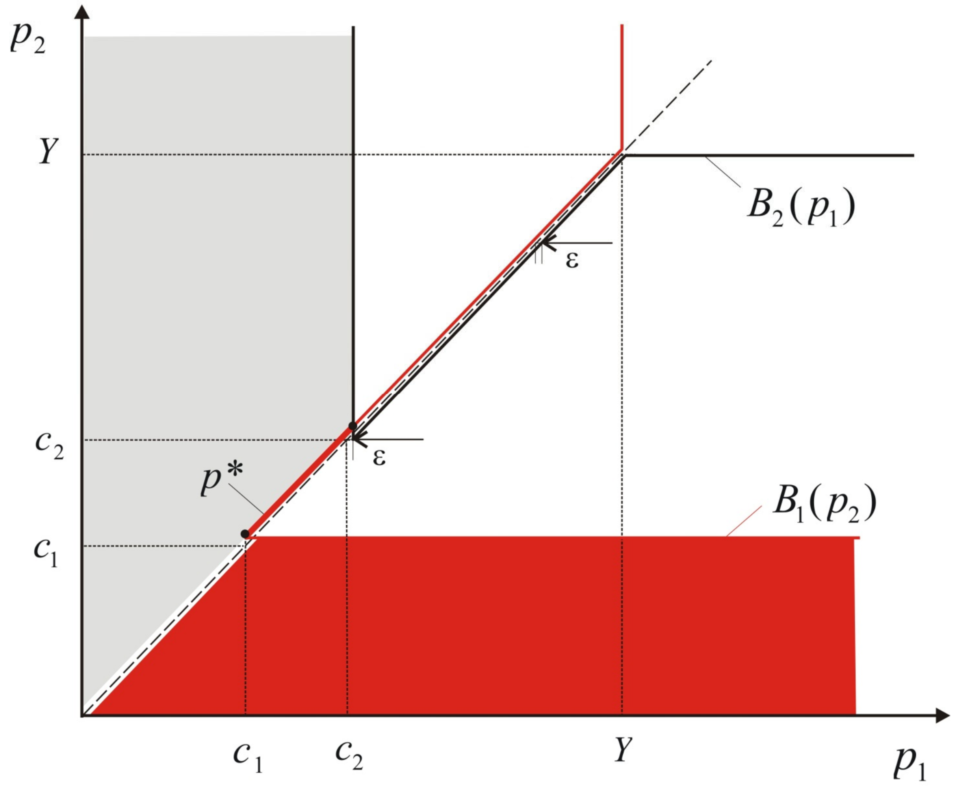
$p^* = (p_1^*, p_2^*)$

**with**

$p_1^* \in [c_1, c_2]$

$p_2^* = p_1^* + \varepsilon$

**Note that all NE involve at least one player playing a weakly dominated strategy**



## BERTRAND DUOPOLY (Cont'd)

**Additional Notes**

- **In equilibrium with $c_1 < c_2$, firm 2 plays a weakly dominated strategy**
- **A tiebreaking rule that assigns all profits to firm 1 in case of equal prices guarantees a set of NE $p^*$ for $\varepsilon \to 0+$:**

> **"In a Bertrand equilibrium, firms charge a price between the first- the second-most efficient firm's costs."(1)**

**It is possible that all firms play a weakly dominated strategy in equilibrium.**



(1) This finding generalizes to n firms with possible equalities between the different marginal costs.



## DIFFERENTIATION SOFTENS PRICE COMPETITION
### Generalization: Imperfect Substitutes

**Given:** **Demands for products of firm 1 and firm 2: $q_1(p_1,p_2)$ and $q_2(p_1,p_2)$ [& the firms' cost functions: $C_1(q_1)$ and $C_2(q_2)$]**

$$\max_{p_2} \{p_2 q_2(p_1, p_2) - C_2(q_2(p_1, p_2))\}$$

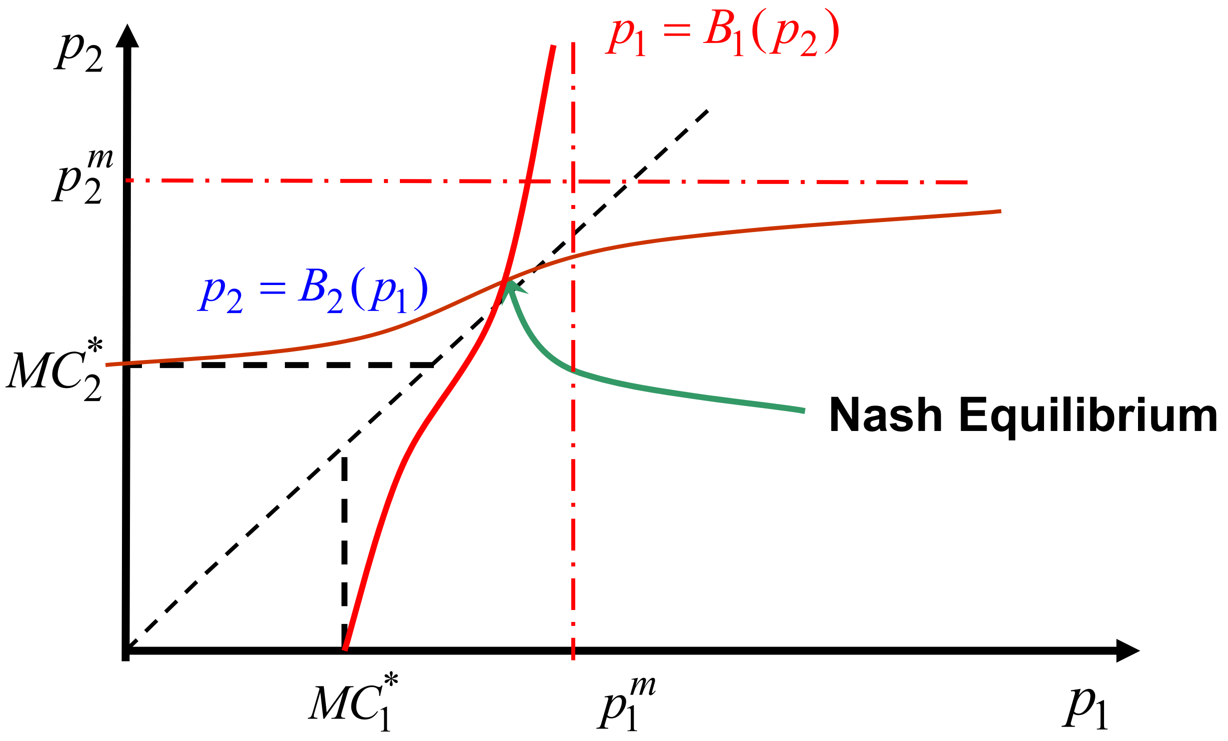
$$FOC: \quad MR_2(q_2) = MC_2(q_2) \quad \Rightarrow \quad p_2 = B_2(p_1)$$

$$\max_{p_1} \{p_1 q_1(p_1, p_2)) - C_1(q_1(p_1, p_2))\}$$

$$FOC: \quad MR_1(q_1) = MC_1(q_1) \quad \Rightarrow \quad p_1 = B_1(p_2)$$

$$\left[p_1 - \frac{dC_1(q_1)}{dq_1}\right] = -\frac{q_1(p_1, p_2)}{\frac{\partial q_1(p_1, p_2)}{\partial p_1}}$$

## BERTRAND WITH IMPERFECT SUBSTITUTES (Cont'd)






## AGENDA

**What is Game Theory?**

**Building Blocks and Key Assumptions**

**Market Structure & Strategy Analysis**

- **Cournot Quantity Competition**
- **Bertrand Price Competition**

**Key Concepts to Remember**

# KEY CONCEPTS TO REMEMBER

- **Predictive Power**
- **Payoff Matrix**
- **Pure/Mixed Strategy**
- **Dominant Strategy**
- **Best-Response**
- **Nash Equilibrium**
- **Cournot and Bertrand Game**
- **Stackelberg Sequential-Move Games**

# MGT 621 – MICROECONOMICS

Thomas A. Weber

*7. Externalities & Regulation*

Autumn 2023

École Polytechnique Fédérale de Lausanne
College of Management of Technology



## AGENDA

**What are externalities?**

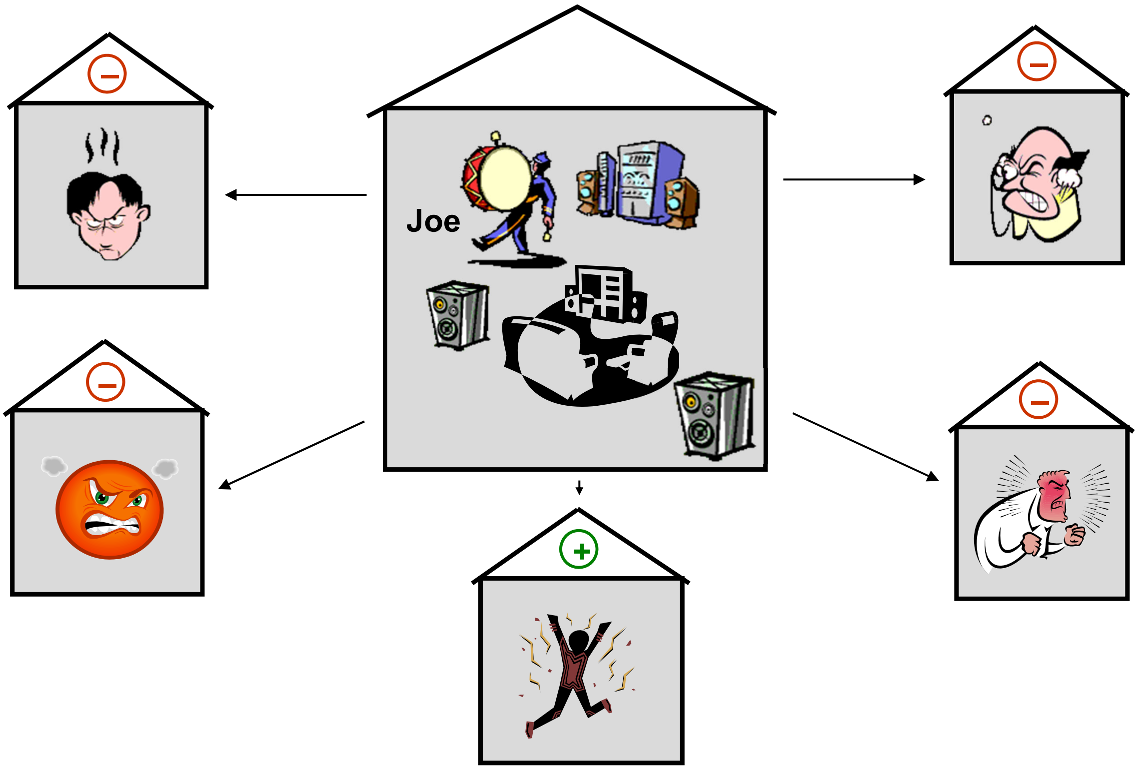
**Example: Production Externalities**

**Some Regulatory Options**

**Uncertainty Matters: Prices vs. Quantities**

**Key Concepts to Remember**

# WHAT ARE EXTERNALITIES?
## Example






# WHAT ARE EXTERNALITIES? (Cont'd)

**Definition. An externality exists whenever the well-being (utility) of a consumer or the production possibility set of a firm are *directly* affected by the action of another agent in the economy.**

**Externalities can be "positive" or "negative."**

# WHAT ARE EXTERNALITIES? (Cont'd)
## Some Examples

- Drivers' cars release pollutants that deteriorate the air quality
- Cigarette smoke increases the probability of lung cancer for smokers and others
- Chemical plant releases wastes in river; fishing industry becomes less productive
- Fish caught by one fishing boat cannot be caught be another fishing boat
- High-tech patents lead to public disclosure of inventions that can be used by other firms (be they complementors or competitors)



# AGENDA

What are externalities?

**Example: Production Externalities**

Some Regulatory Options

Uncertainty Matters: Prices vs. Quantities

Key Concepts to Remember

# EXAMPLE: PRODUCTION EXTERNALITIES

**One firm's decisions may have a direct impact on another firm's payoff.**

- **Our discussion of game theory shows (see, e.g., the Prisoners' Dilemma) that, in general, if each firm individually maximizes its profits, then the sum of both firms' profits may not be maximal.**

**Consider the following situation for two firms, 1 and 2:**

- **Firm 1's production produces wastewater, resulting in an externality for a downstream fishing company**
- **Firm 2, impacted by firm 1's waste production could reduce harmful effects, say by treatment of water, but at a cost**

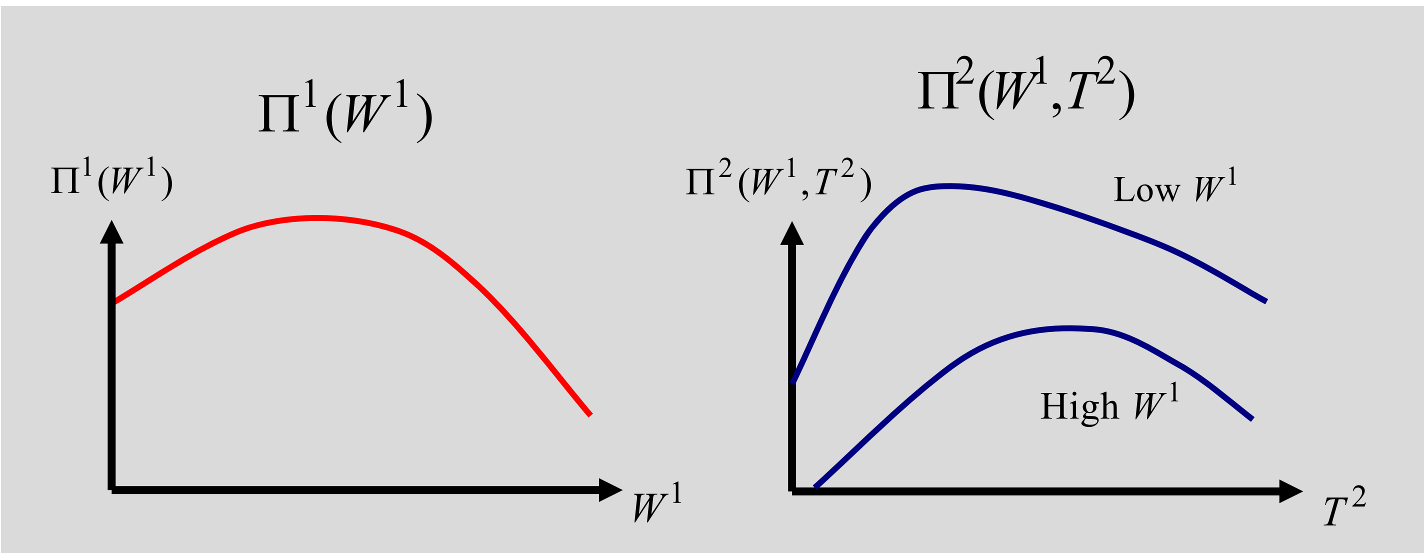


# PRODUCTION EXTERNALITIES (Cont'd)

**Firm 1 produces a nonnegative amount of waste, $W^1$**

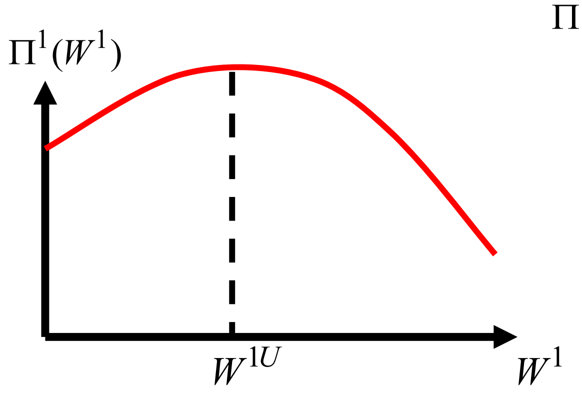
**Firm 2, negatively impacted by firm 1's waste production, could reduce the harmful effects through treatment $T^2$, which comes at a cost.**

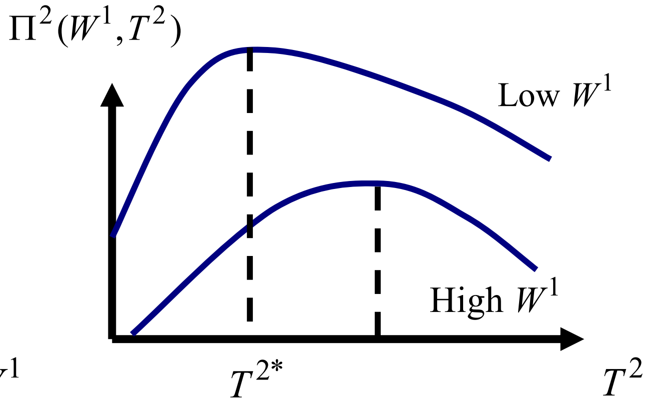
**Payoff Functions:**



# PRODUCTION EXTERNALITIES (Cont'd)

$$\frac{d\Pi^1(W^1)}{dW^1} = 0 \qquad\qquad \frac{\partial\Pi^2(W^1,T^2)}{\partial T^2} = 0$$




**Optimal treatment $T^{2*}$**
**Depends on $W^1$**



# PRODUCTION EXTERNALITIES (Cont'd)

**Efficient outcome maximizes total profits,**

$$\max_{W^1,T^2\geq 0} \left\{\Pi^1(W^1) + \Pi^2(W^1,T^2)\right\}$$

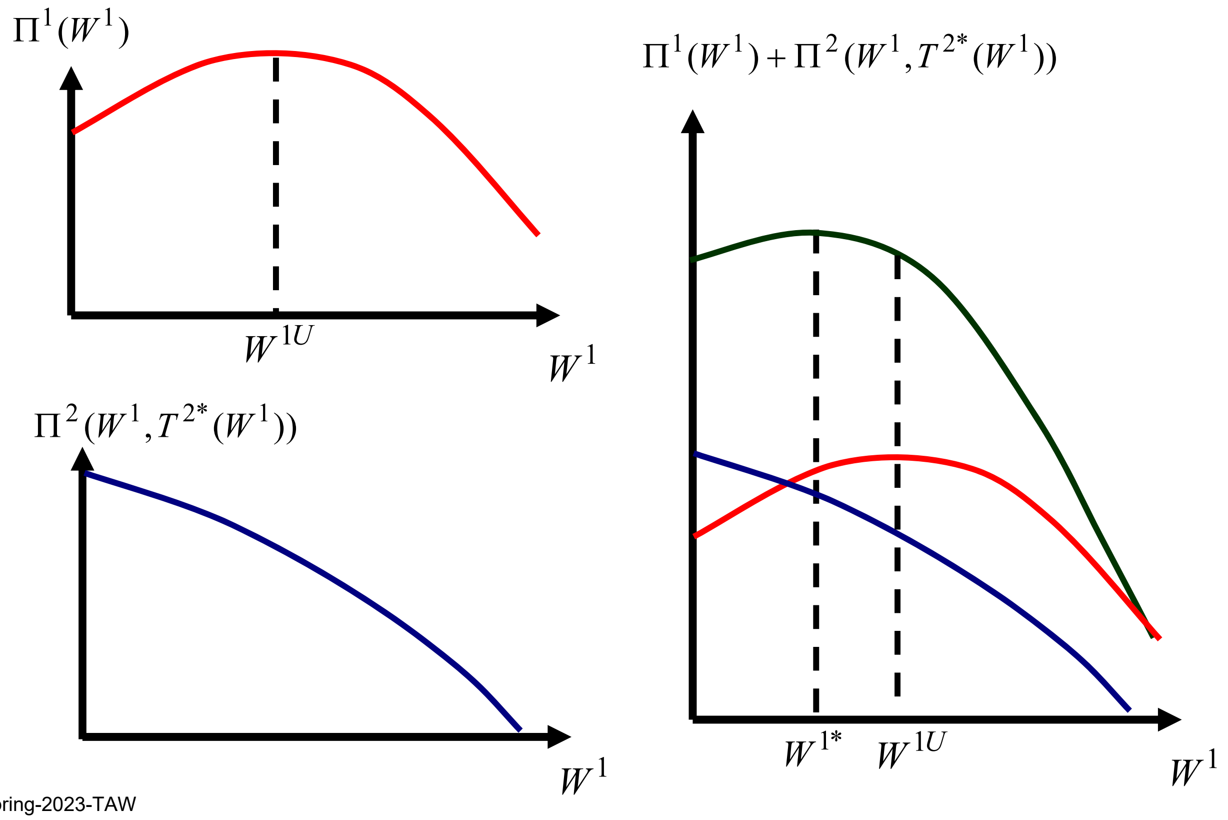
**Firm 1: optimality condition for socially optimal $W^1$ differs from the individual optimality condition,**

$$\frac{d\Pi^1(W^1)}{dW^1} + \frac{\partial\Pi^2(W^1,T^2)}{\partial W^1} = 0$$

**Firm 2: optimality condition for socially optimal $T^2$ is same as individually optimal, for any given $W^1$**

$$\frac{\partial\Pi^2(W^1,T^2)}{\partial T^2} = 0$$

# PRODUCTION EXTERNALITIES (Cont'd)






# PRODUCTION EXTERNALITIES (Cont'd)

## Define Externalities in Terms of Harm

**Define harm to firm 2 as a function of W¹ (corresponds to the externality that firm 1 exerts on firm 2)**

$$h(W^1) = \Pi^2(0, T^{2*}(0)) - \Pi^2(W^1, T^{2*}(W^1))$$

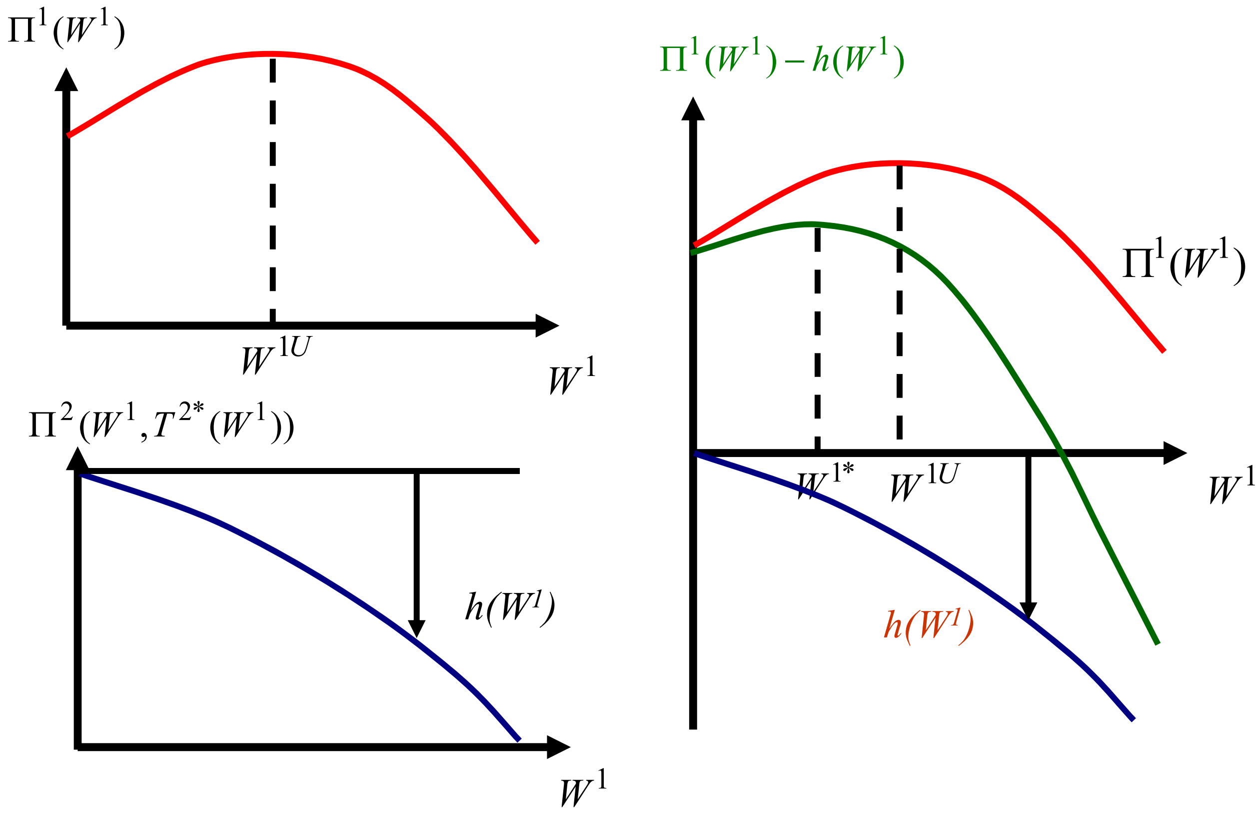
**Choose W¹ to maximize profit of firm 1, minus harm:**

$$\max_{W^1 \geq 0} \left\{ \Pi^1(W^1) - h(W^1) \right\}$$

**Choose T² to maximize profit of firm 2:**

$$\max_{T^2 \geq 0} \Pi^2(W^{1*}, T^2)$$

# PRODUCTION EXTERNALITIES (Cont'd)






# PRODUCTION EXTERNALITIES (Cont'd)

**Gives FOC for $T^2$ identical to optimal**

$$\frac{\partial \Pi^2(W^1, T^2)}{\partial T^2} = 0$$

**Gives FOC for $W^1$ identical to optimal**

$$\frac{d\Pi^1(W^1)}{dW^1} - \frac{\partial h(W^1)}{\partial W^1} = 0$$

$$\frac{d\Pi^1(W^1)}{dW^1} - \left\{ -\frac{\partial \Pi^2(W^1, T^{2*})}{\partial W^1} \right\} - \underbrace{\left\{ -\frac{\partial \Pi^2(W^1, T^{2*})}{\partial T^{2*}} \right\} \left\{ \frac{\partial T^{2*}}{\partial W^1} \right\}}_{0} = 0$$

$$\frac{d\Pi^1(W^1)}{dW^1} + \frac{\partial \Pi^2(W^1, T^{2*})}{\partial W^1} = 0$$

# AGENDA

What are externalities?

Example: Production Externalities

**Some Regulatory Options**

Uncertainty Matters: Prices vs. Quantities

Key Concepts to Remember



# REMEDIES FOR MARKET FAILURE

Idea: How can we correct market to move back toward competitive norm?

- "Polluter Pays" Principle
- Litigation to recover harms: "damages"
- Tax per unit on externality: "pollution tax"
- Marketable emissions rights
  - Create market for rights to produce the externality
- Regulation of emissions or other waste
  - Restriction against hazardous waste
  - Limits on emissions rate
- Assign property rights and allow negotiation (Coase Theorem)

# "POLLUTER PAYS" PRINCIPLE

**Polluter may be required to pay a fee to the government, offsetting the entire damage. In that case, the polluter (firm 1) solves the problem**

$$\max_{W^1 \geq 0} \left\{ \Pi^1(W^1) - h(W^1) \right\}$$

**If firm 1 maximizes its profit minus harm, it will choose optimal waste ($W^1$). No money paid to harmed firm (firm 2), who solves the following problem:**

$$\max_{T^2 \geq 0} \Pi^2(W^{1*}, T^2)$$

**If firm 2 is not compensated for damages, it will choose optimal treatment.**



# LITIGATION RECOVERS DAMAGES

**Polluter may be required to pay fee to harmed firm equal to the damage. Then the polluter solves problem:**

$$\max_{W^1 \geq 0} \left\{ \Pi^1(W^1) - h(W^1) \right\}$$

**If firm 1 maximizes its profit minus harm, it will choose optimal waste ($W^1$) for whatever is the level of $T^2$.**

**Firm 1 has an incentive to select a socially efficient waste level.**

# LITIGATION RECOVERS DAMAGES

**Damages paid to harmed firm, with damages determined for <u>optimal level of $T^2$</u>. Then firm 2 chooses $T^2$ to solve problem:**

$$\max_{T^2 \geq 0} \left\{ \Pi^2(W^1, T^2) + h(W^1) \right\}$$

**Monetary damages – harm – do not depend on actual choice of $T^2$, so $h(W^1)$ is a constant, from perspective of firm 2.**

**If firm 2 is compensated a <u>fixed amount</u> for damages, it will maximize its before-harm-payment profit and will choose optimal treatment.**

$$h(W^1) = \Pi^2(0, T^{2*}(0)) - \Pi^2(W^1, T^{2*}(W^1))$$



# LITIGATION RECOVERS DAMAGES (Cont'd)

**Damages paid to harmed firm, with damages determined for <u>actual level of $T^2$</u>. Then firm 2 solves the problem:**

$$\max_{T^2 \geq 0} \left\{ \Pi^2(W^1, T^2) + h(W^1) \right\}$$

**Monetary damages – harm – do depend on actual choice of $T^2$, so $h(W^1)$ is not a constant, from perspective of firm 2, in this case.**

$$h(W^1) = \Pi^2(0, T^2(0)) - \Pi^2(W^1, T^2)$$

$$\max \ \underbrace{\left\{ \Pi^2(W^1, T^2) - \Pi^2(W^1, T^2) \right\}}_{0} + \Pi^2(0, T^2(0))$$

**Profit after damage payment is independent of $T^2$; firm 2 has no incentive to choose optimal treatment.**

# POLLUTION TAX

**The polluter may be required to pay a fee to government equal to a fixed amount t <u>per unit</u> of pollution. Per-unit amount is set equal to the marginal harm. Then polluter solves problem:**

$$\max_{W^1 \geq 0} \left\{ \Pi^1(W^1) - t\,W^1 \right\}$$

**Firm 1, maximizing its own after-tax profit, leads to efficient level of waste ($W^1 = W^{1*}$) :**

$$t = \left. \frac{\partial h(W^1)}{\partial W^1} \right|_{W^1 = W^{1*}}$$

$$\frac{d\Pi^1(W^1)}{dW^1} = \frac{\partial h(W^{1*})}{\partial W^1}$$



# MARKETABLE EMISSIONS RIGHTS

**Government determines optimal total waste, and issues emissions rights. The number of rights equal to optimal waste. Rights can be bought and sold. Market clearing price will equal marginal cost of waste reduction. Thus, if the optimal total waste is produced, then the price will be equal to marginal harm,**

$$p_{pr} = \frac{\partial h(W^1)}{\partial W^1}$$

**Firm 1's optimization problem:**

$$\max_{W^1 \geq 0} \left\{ \Pi^1(W^1) - p_{pr}\; W^1 \right\}$$

**Profit maximization condition leads Firm 1, maximizing its own after-tax profit, to an efficient level of waste:**

$$\frac{d\Pi^1(W^1)}{dW^1} = p_{pr} = \frac{\partial h(W^1)}{\partial W^1}$$

# DIRECT REGULATIONS OF EMISSIONS

**Government sets maximum allowable waste regulation, after determining the solution of the problem:**

$$\max_{W^1 \geq 0} \left\{ \Pi^1(W^1) - h(W^1) \right\}$$

**Firm required to meet regulation or else face a penalty higher than the cost of meeting the regulation. Incentive to just meet the regulation.**



# PROPERTY RIGHTS AND RENEGOTIATION

**Generally applicable only with a very small number of firms**

**General approach**

- **Assign property rights to either party**
  - **Right to pollute or Right to no pollution**
- **Allow negotiation**
- **Efficient outcome either way**
- **Distribution of profits differs**

**Coase Theorem**

# PROPERTY RIGHTS AND RENEGOTIATION

**Give property rights to either (→ Case 1 and 2)**

**Case 1. Assume give right to polluter for some high pollution level. Then impacted firm will pay to firm 1 some amount of money, B, to reduce pollution.**

$$\Pi^1(W^{1*}) + \Pi^2(W^{1*}, T^{2*}) > \Pi^1(W^{1U}) + \Pi^2(W^{1U}, T^{2U})$$

$$\Pi^2(W^{1*}, T^{2*}) - \Pi^2(W^{1U}, T^{2U}) > \Pi^1(W^{1U}) - \Pi^1(W^{1*}) > 0$$

**Then B can be chosen such that**

$$\Pi^2(W^{1*}, T^{2*}) - \Pi^2(W^{1U}, T^{2U}) > B > \Pi^1(W^{1U}) - \Pi^1(W^{1*}) > 0$$

**whence**

$$\Pi^2(W^{1*}, T^{2*}) - \Pi^2(W^{1U}, T^{2U}) - B > 0$$ **Firm 2 increases profit**

$$\Pi^1(W^{1*}) - \Pi^1(W^{1U}) + B > 0$$ **Firm 1 increases profit**



# PROPERTY RIGHTS AND RENEGOTIATION

**Case 2. Assume give to affected firm right to face no pollution. Then polluting firm will pay firm 2 some amount of money, F, to allow pollution. (Assume no abatement, i.e., $T^2 = 0$, when there is no pollution)**

$$\Pi^1(W^{1*}) + \Pi^2(W^{1*}, T^{2*}) > \Pi^1(0) + \Pi^2(0,0)$$

$$0 > \Pi^2(W^{1*}, T^{2*}) - \Pi^2(0,0) > \Pi^1(0) - \Pi^1(W^{1*})$$

**Then F can be chosen such that**

$$0 > \Pi^2(W^{1*}, T^{2*}) - \Pi^2(0,0) > -F > \Pi^1(0) - \Pi^1(W^{1*})$$

**whence**

$$\Pi^2(W^{1*}, T^{2*}) - \Pi^2(0,0) + F > 0$$ **Firm 2 increases profit**

$$\Pi^1(W^{1*}) - \Pi^1(0) - F > 0$$ **Firm 1 increases profit**

# COASE THEOREM

Coase Theorem. If all parties can negotiate with each other costlessly and with perfect information, then bargaining will lead to an efficient outcome.

The outcome will be efficient, no matter how the initial property rights are determined.

**A Caveat**

**If property rights are not firmly established, so that agents spend resources trying to reallocate property rights, then the final outcome, including the costs of reallocating property rights, will not be efficient.**



# AGENDA

**What are externalities?**

**Example: Production Externalities**

**Some Regulatory Options**

**Uncertainty Matters: Prices vs. Quantities**

**Key Concepts to Remember**

# PRACTICAL PROBLEM: REDUCING GLOBAL CARBON OUTPUT

**To bound global warming to less than 2 degrees Celsius by 2050, worldwide carbon emissions need to be reduced by 50% in that timeframe (IPCC 2008).**

**Hence, need to provide incentives for …**

- **carbon abatement by implementing an efficient carbon pricing policy**
- **technological innovation by encouraging the necessary investments**

**Question: Design a simple (i.e., implementable) regulatory scheme which *jointly* accounts for innovation and abatement.**

Remark: For details on the material in the following slides, see Weber, T.A., Neuhoff, K. (2010) "Carbon Markets and Technological Innovation," *Journal of Environmental Economics and Management*, Vol. 60, No. 2, pp. 115—132; an earlier version is also available at http://ssrn.com/abstract=1333244



# THE MODEL
## Primitives

- **Unit mass of firms, indexed by $\theta \in \Theta$, distributed on type space $\Theta$ such that**

$$\mu = \int_{\Theta} \theta dF(\theta) < \infty \quad \textbf{and} \quad \sigma_\theta^2 = \int_{\Theta} (\theta - \mu)^2 dF(\theta) < \infty$$

- **Each firm $\theta$ has business-as-usual (BAU) level of emissions $e_0(\theta) > 0$**
- **BAU emissions levels of all firms are subject to a common macroeconomic shocks, modeled by the additive zero-mean noise $\tilde{\varepsilon}$ such that**

$$\sigma_\varepsilon^2 = \int_{\Theta} \varepsilon^2 dG(\varepsilon) < \infty$$

- **Expected total emissions:**

$$e_0 = E[\int_{\Theta} (e_0(\theta) + \tilde{\varepsilon}) dF(\theta)] < \infty$$

# THE MODEL (Cont'd)
## Timing

**The actions take place in three periods, indexed by** $t \in \{0,1,2\}$

- **<u>Regulation Stage</u> (t=0)**
  - **Regulator commits to a regulatory policy in the form of a cap-and-trade scheme with price controls, denoted by** $R=(E,L,U)$

    $E$ **: Emissions cap (e.g., set by number of issued emissions permits)**

    $L$ **: Price floor in market for emissions permits**

    $U$ **: Price cap in market for emissions permits**

- **<u>Innovation Stage</u> (t=1)**
  - **Each firm** $\theta$ **decides about its innovation activity** $y \geq 0$ **at the cost of** $K(y)=cy^2/2$ **where** $c \geq 0$ **determines the slope of the marginal cost**
  - **An innovation activity of** $y$ **results in the realization** $\rho$ **of a random cost improvement** $\tilde{\rho}(y)>0$**, where** $y=E[\max\{\tilde{\rho}(y),1\}\,|\,y]-1$

    $\rho \leq 1$ **: Innovation unsuccessful – current practice is (weakly) better**

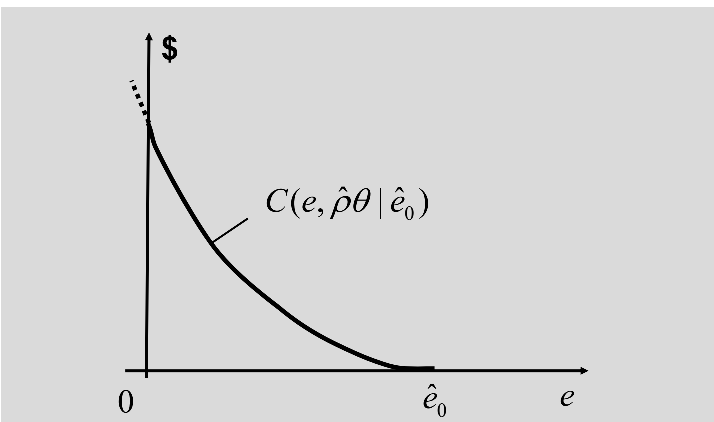
    $\rho > 1$ **: Innovation successful – firm exercises option of using it**



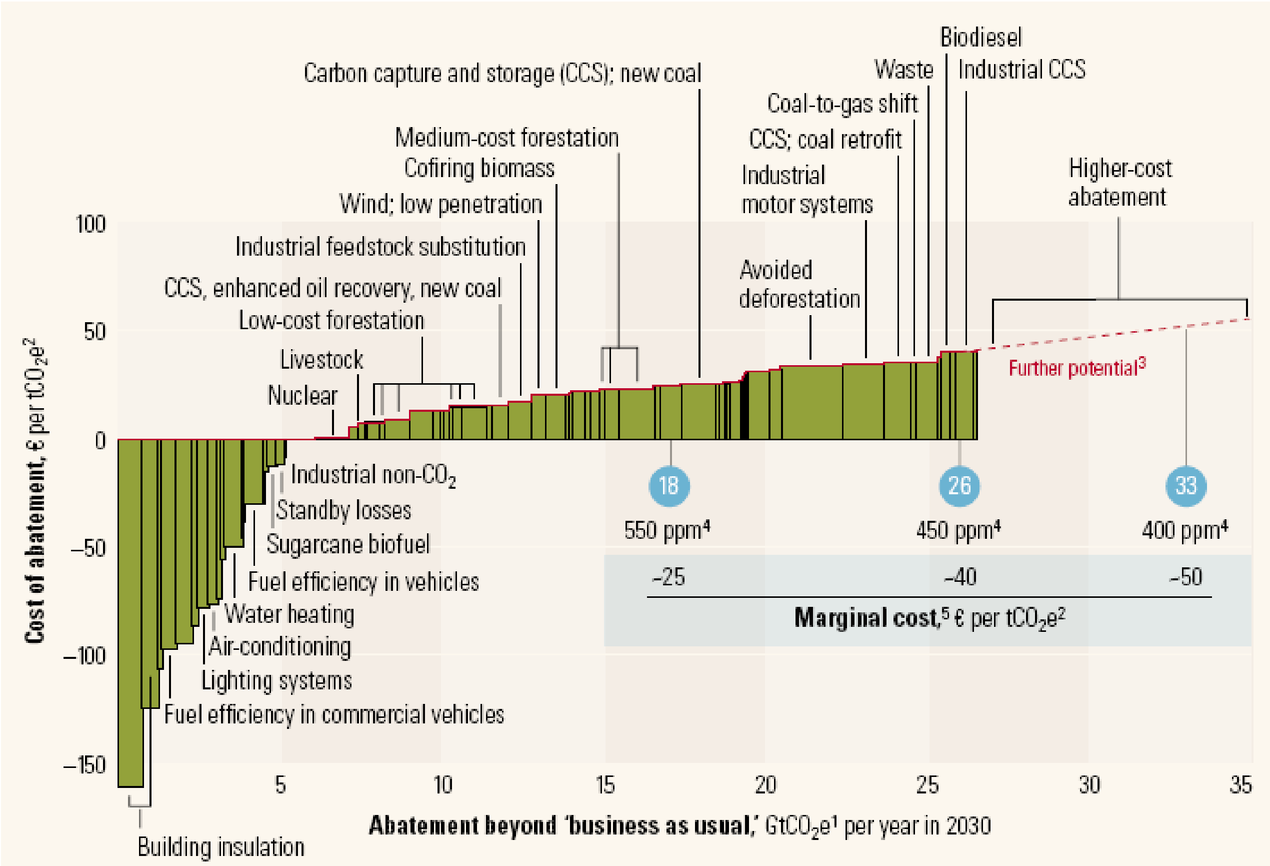
# THE MODEL (Cont'd)
## Timing

- **<u>Implementation Stage</u> (t=2)**
  - **The macroeconomic uncertainty** $\varepsilon$ **realizes**
  - **Each firm** $\theta$**, based on the outcome** $\hat{\rho}=\max\{\rho,1\}$ **of its innovation activity in the last stage and the current price** $p$ **for carbon emissions, decides about its emission level** $e(p,\hat{\theta})=e(p,\hat{\rho}\theta)\geq 0$
  - **Firm** $\theta$**'s total cost of abating its emissions to a level** $e \leq \hat{e}_0 = e_0(\theta)+\varepsilon$ **is**

$$TC(e,p,\hat{\rho}\theta\,|\,\hat{e}_0)=C(e,\hat{\rho}\theta\,|\,\hat{e}_0)+pe=\frac{(\hat{e}_0-e)^2}{2\hat{\rho}\theta}+pe$$



# MARGINAL ABATEMENT COST ALMOST LINEAR

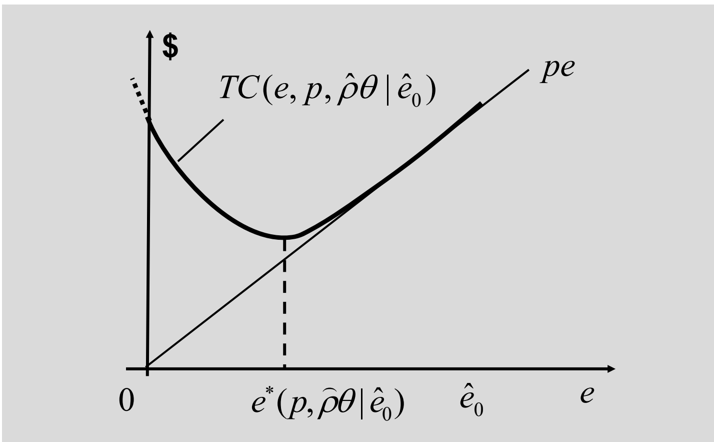



Source: Enkvist et al. (2007)



# MODEL SOLUTION
## Implementation (t = 2)

**Each firm $\theta$ chooses emissions so as to minimize its total emissions cost.**




$$e^*(p, \hat{\rho}\theta \mid \hat{e}_0) = \hat{e}_0 - \hat{\rho}\theta p$$

$$TC^*(p, \hat{\rho}\theta \mid \hat{e}_0) = \hat{e}_0 p - \frac{\hat{\rho}\theta p^2}{2}$$

# MODEL SOLUTION (Cont'd)
## Innovation (t = 1)

**Each firm $\theta$ chooses a level of innovation $y$ to maximize its expected net payoff,**

$$\pi(p,y,\theta) = \frac{\theta y p^2}{2} - K(y) = \frac{\theta y p^2}{2} - \frac{cy^2}{2}$$

**resulting in the optimal innovation of**

$$y^*(p,\theta) = \frac{\theta p^2}{2c}$$

**and the positive expected payoff**

$$\pi^*(p,\theta) = \frac{\theta^2 p^4}{8c}$$



# MODEL SOLUTION (Cont'd)
## Regulation (t = 0)

**The set of feasible cap-and-trade schemes is**

$$\Re = \{(E,L,U) \in \mathbf{R}_+^3 : L \le U\}$$

**The total carbon emissions output in the economy conditional on the market price for carbon and the macroeconomic condition is**

$$Q(p,\varepsilon) = \int_\Theta e^*(p,(1+y^*(p,\theta))\theta \,|\, e_0(\theta)+\varepsilon)dF(\theta) = e_0 + \varepsilon - \mu p(1 + \underbrace{\frac{\mu^2+\sigma_\theta^2}{2\mu c}}_{\beta} p^2)$$

**Environmental damages (measured in \$) are assumed to be quadratic in total emissions,**

$$D(Q) = \frac{dQ^2}{2}$$

**describes innovation**

**(for $\beta = 0$: no innovation)**

# MODEL SOLUTION (Cont'd)
## Regulation (t = 0)

**Given a feasible cap-and-trade scheme $R=(E,L,U)$ , the market-clearing condition set by the regulator,**

$$H(p,\varepsilon,R)=(U-p)(p-L)(E-Q(p,\varepsilon))=0$$

**determines the price $p\in[L,U]$ for carbon**

**Hence, <u>expected environmental damages</u> are**

$$\overline{D}(R)=E[D(Q(\tilde{p},\tilde{\varepsilon}))\,|\,H(\tilde{p},\tilde{\varepsilon},R)=0]$$

**Similarly, <u>expected abatement cost</u> are**

$$\overline{C}(R)=E[C(e^{*}(\tilde{p},(1+y^{*}(\tilde{p},\tilde{\theta}))\tilde{\theta})\,|\,e_0+\tilde{\varepsilon})\,|\,H(\tilde{p},\tilde{\varepsilon},R)=0]$$



# MODEL SOLUTION (Cont'd)
## Regulation (t = 0)

**In addition, the regulator may want to consider the firms' cost of innovation**

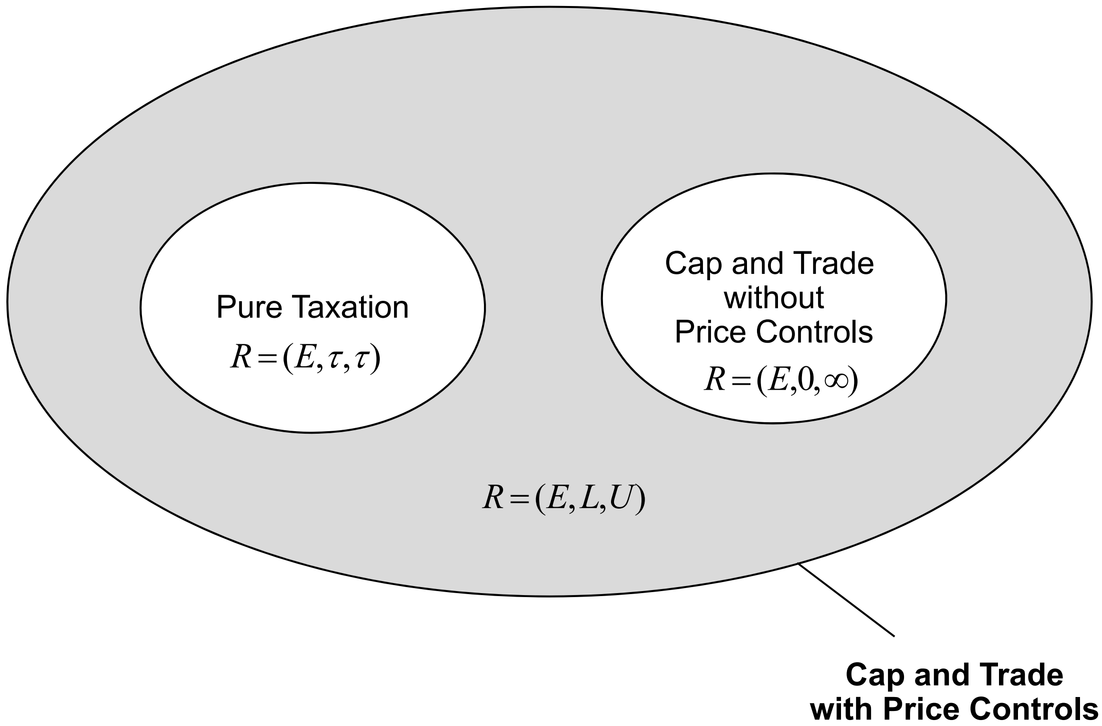
$$\overline{K}(R)=E[\lambda K(y^{*}(\tilde{p},\tilde{\theta}))\,|\,H(\tilde{p},\tilde{\varepsilon},R)=0]$$

**The regulator's objective function is**

$$\boxed{\overline{W}(R)=-\left(\overline{C}(R)+\overline{D}(R)+\overline{K}(R)\right)=-\overline{SC}(R)}$$

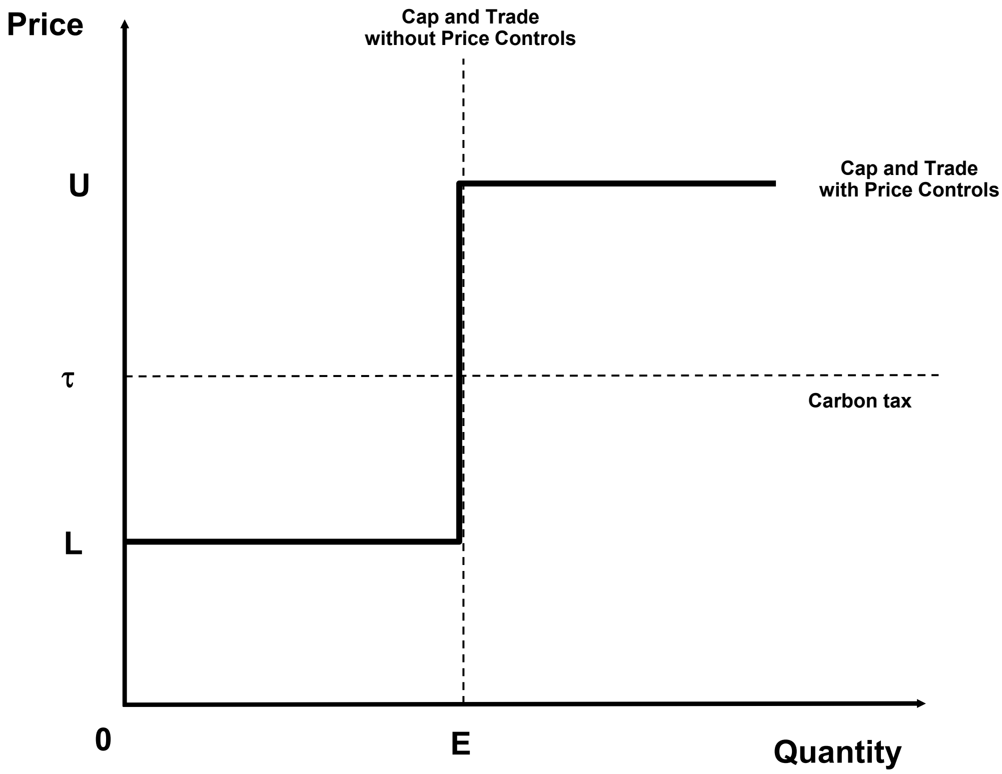
**<u>Problem</u>: Find optimal $R^{*}\in\Re$**

# RELATION BETWEEN COMMON REGULATORY SCHEMES
## Cap and Trade with Price Control = True Generalization






# COMMON REGULATORY SCHEMES (Cont'd)



# PURE TAXATION

**Under pure taxation, there is no price uncertainty, but there is uncertainty about the environmental damage.**

**Let** $\overline{W}(\tau)$ **be the planner's objective function;**

$$\overline{W}(\tau) = -\frac{\tau^2}{2} - \frac{d}{2}(\sigma_\varepsilon^2 + (e_0 - \tau)^2)$$

**and the optimal tax becomes**

$$\boxed{\tau^* = \frac{de_0}{1+d}}$$



# Optimal carbon tax – numerical example

**Assume:**

**Without innovation** $\tau$ **=40\$/tCO$_2$**

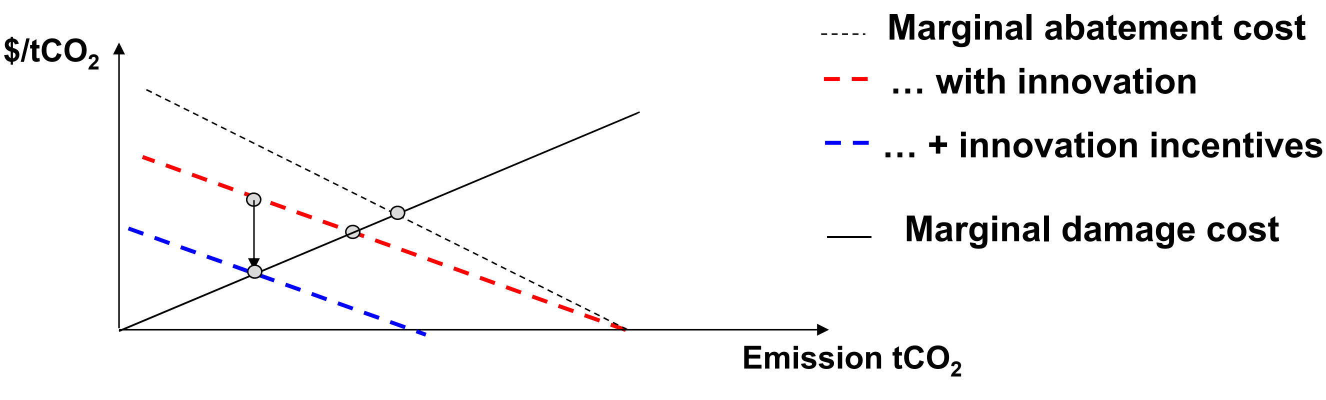
**$E_0$ = 13.5 GT (OECD)**

**10% reduction from existing technologies:** $\mu$ **= 33 $10^6$ tCO$_2^2$/\$$^2$**

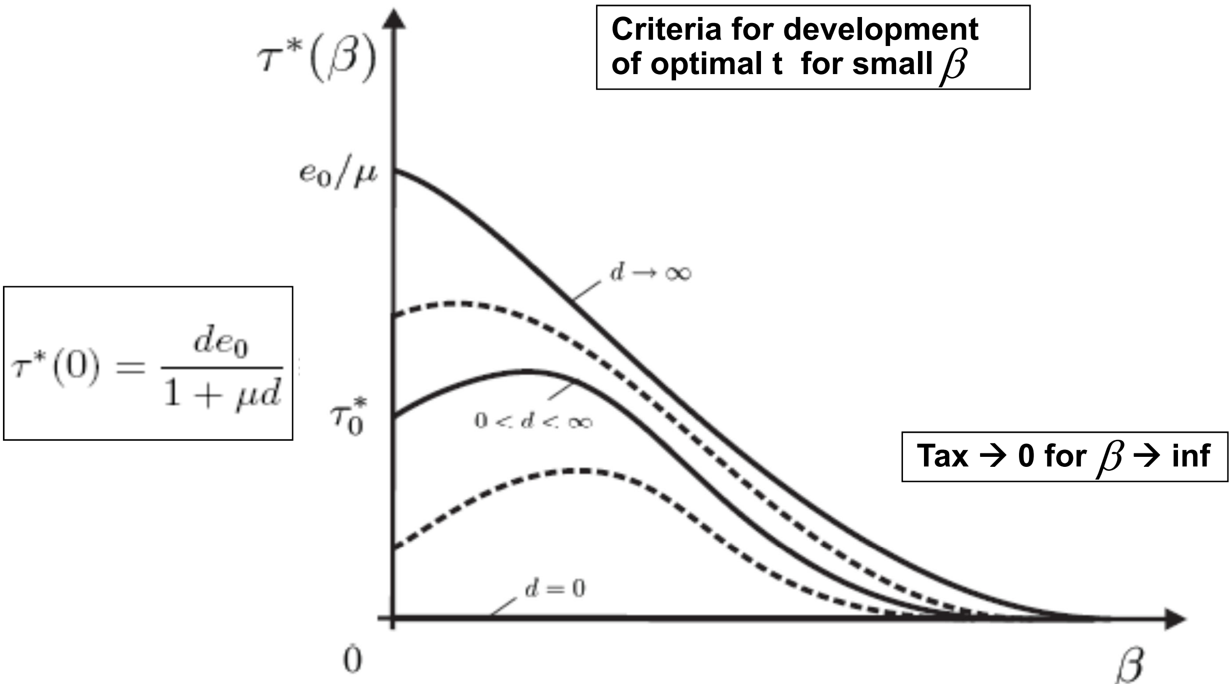
**Innovation delivers additional 33% reductions: c = 100\$ * $10^9$**

**Result**

$(\lambda = 0)$ $\tau$ **= 46 \$/t CO$_2$**



# Optimal Carbon Tax




Innovation effectiveness

$$\beta = \frac{\mu^2 + \sigma_\theta^2}{2\mu c}$$



# BASIC CAP AND TRADE (WITHOUT PRICE CONTROLS)

**Under basic cap and trade, there is no output uncertainty, but there is price uncertainty.**

**Let** $\overline{W}(E)$ **be the planner's objective function;**

$$\overline{W}(E) = -\frac{(e_0 - E)^2 + \sigma_\varepsilon^2}{2} - \frac{dE^2}{2}$$

**and the optimal emissions cap becomes**

$$\boxed{E^* = \frac{e_0}{1+d}}$$

**Note that expected price under this cap is the same as the optimal tax, i.e.,**

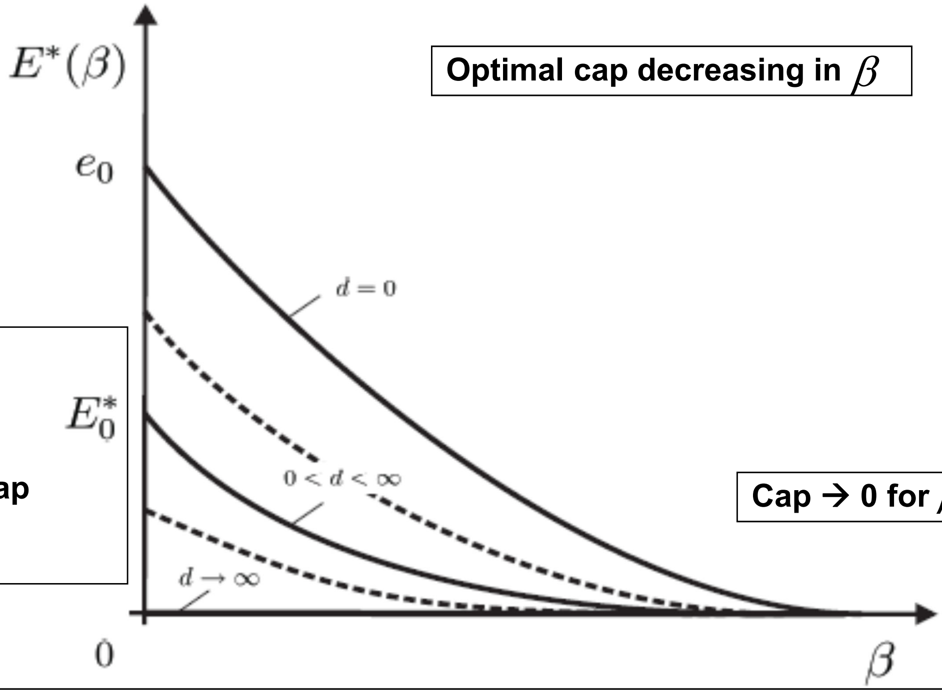
$$\overline{p}^* = E[p(\widetilde{\varepsilon}, E^*) \mid E^*] = \tau^*$$

**BUT, this does not mean that the two are equivalent!**

# OPTIMAL CAP

Optimal cap decreasing in $\beta$

Analytic solution optimal cap




Cap → 0 for $\beta$→inf

Increase in $\beta$ increases relative attractiveness of quantity based regulation



# PRICES (TAXES) VS. QUANTITIES (EMISSIONS CAP)
## Weitzman (1974)

**Depending on the relative magnitude of the marginal abatement cost (1) and the environmental damages (d), it may be better to either impose a pure tax or a basic cap-and-trade scheme:**

$$\overline{W}^*_{Tax} - \overline{W}^*_{Basic\ C\&T} = (1-d)\frac{\sigma_\varepsilon^2}{2}$$

**Tax is strictly better if and only if** $d < 1$

| | | |
|---|---|---|
| **Small damage cost d** | → | **Pure Tax** |
| **Large damage cost d** | → | **Basic Cap and Trade** |

# CAP AND TRADE WITH PRICE CONTROLS
## A Simple Example

**Example. Assume that the macroeconomic shock $\tilde{\varepsilon}$ is uniformly distributed on $[-\delta, \delta]$, where $\delta > 0$. Then,**

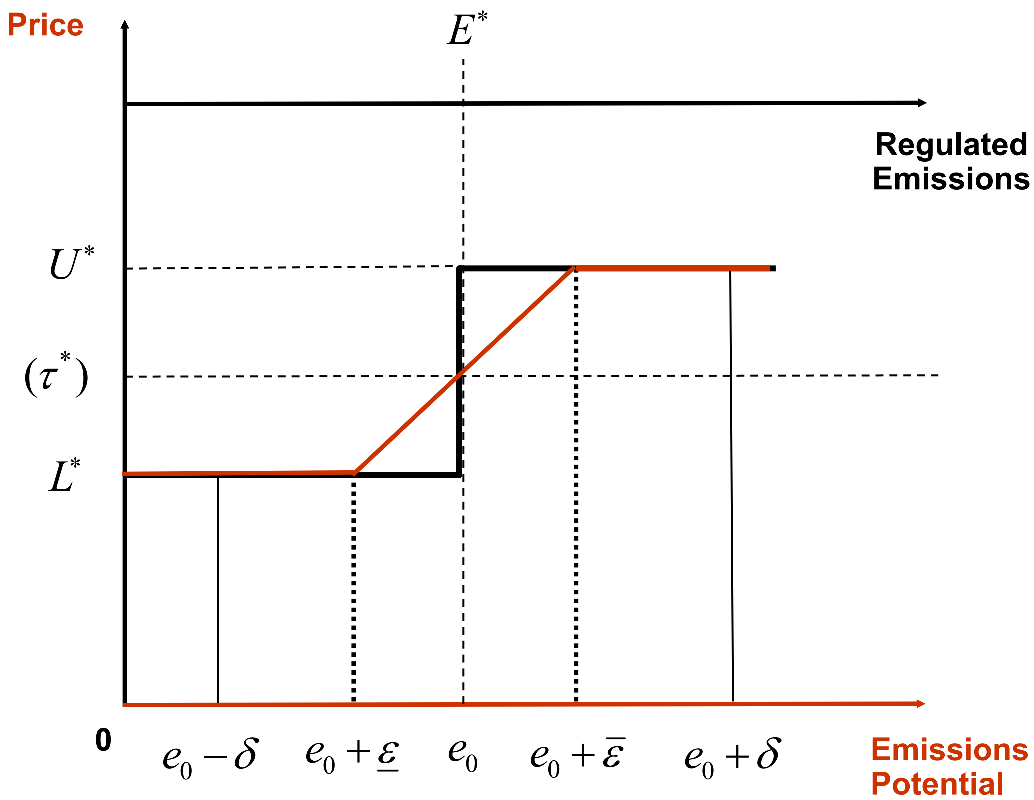
$$E^* = \frac{e_0}{1+d}$$

$$L^* = d\left(\frac{e_0}{1+d} - \frac{\delta}{2+d}\right)$$

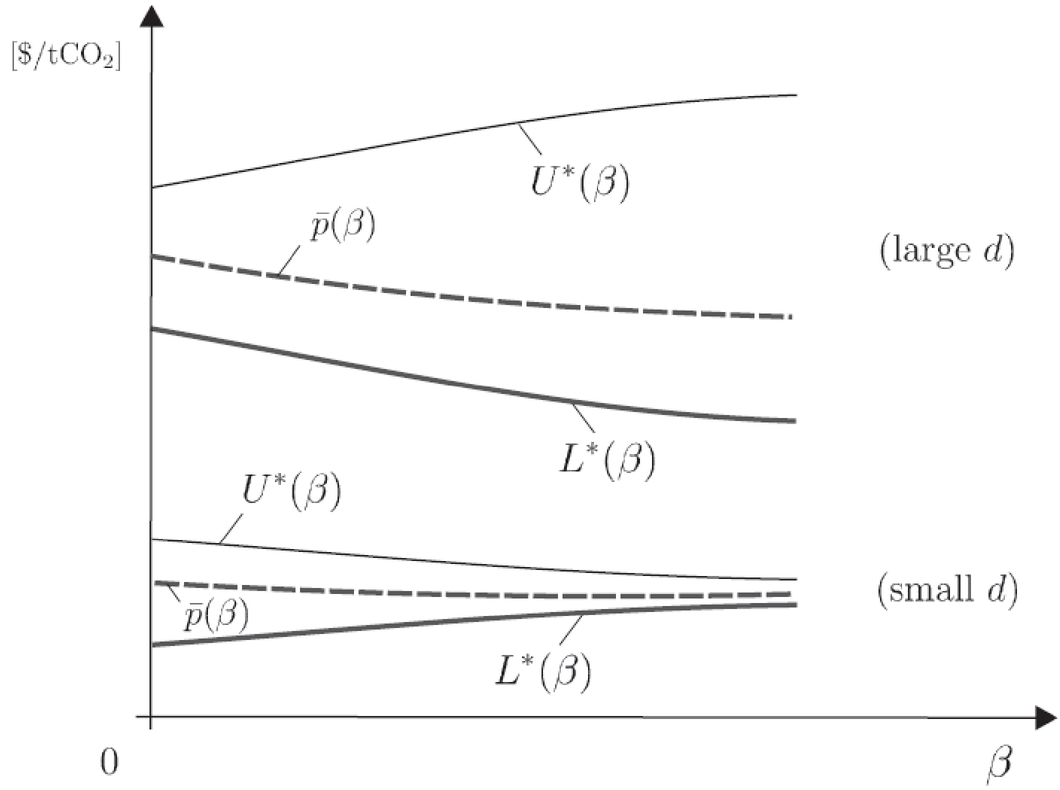
$$U^* = d\left(\frac{e_0}{1+d} + \frac{\delta}{2+d}\right)$$



# CAP AND TRADE WITH PRICE CONTROLS
## A Simple Example (Cont'd)

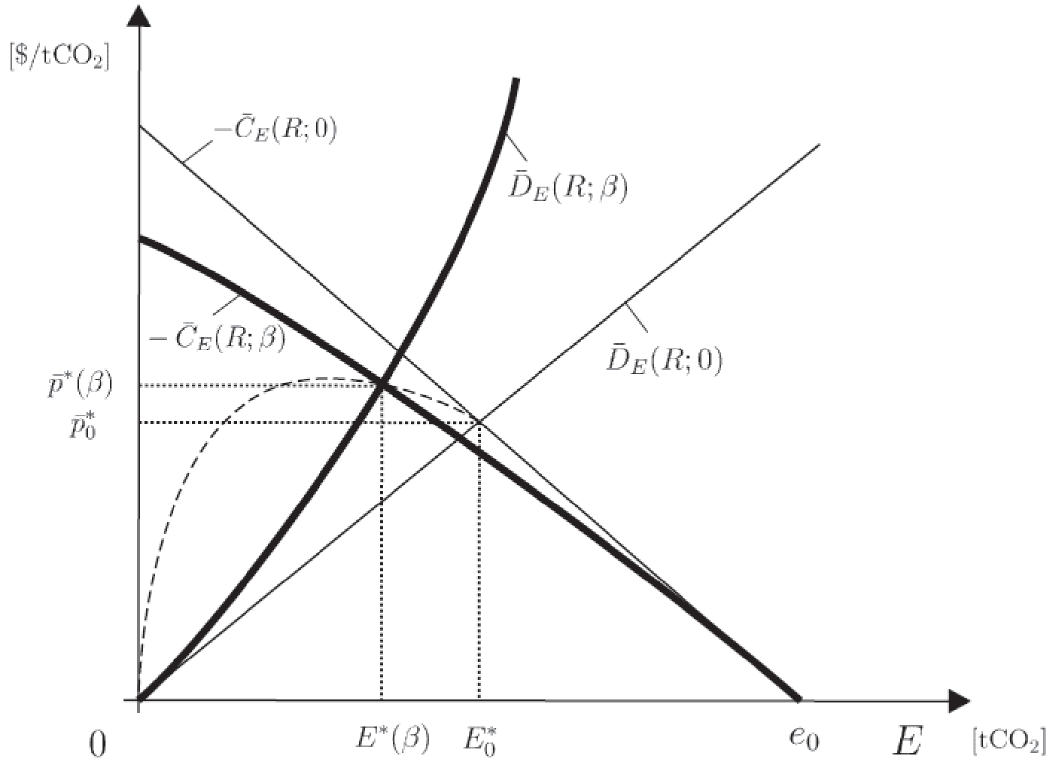


# OPTIMAL HYBRID SCHEME

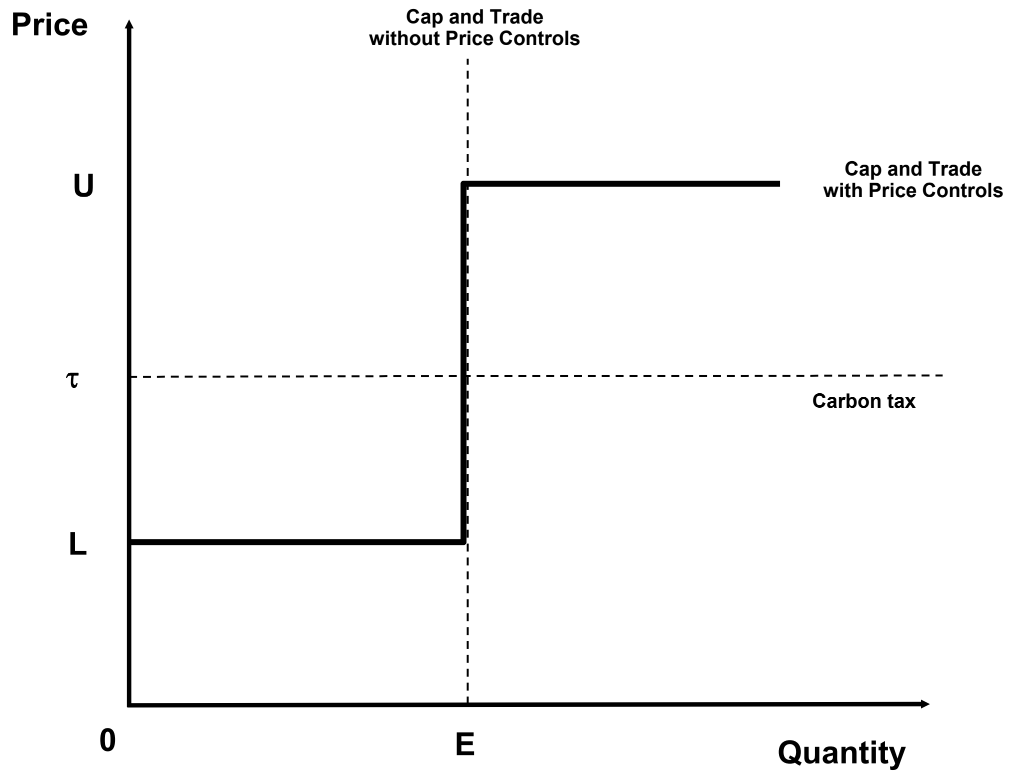





# EXPECTED MARKET PRICE VARIES WITH INNOVATION EFFECTIVENESS

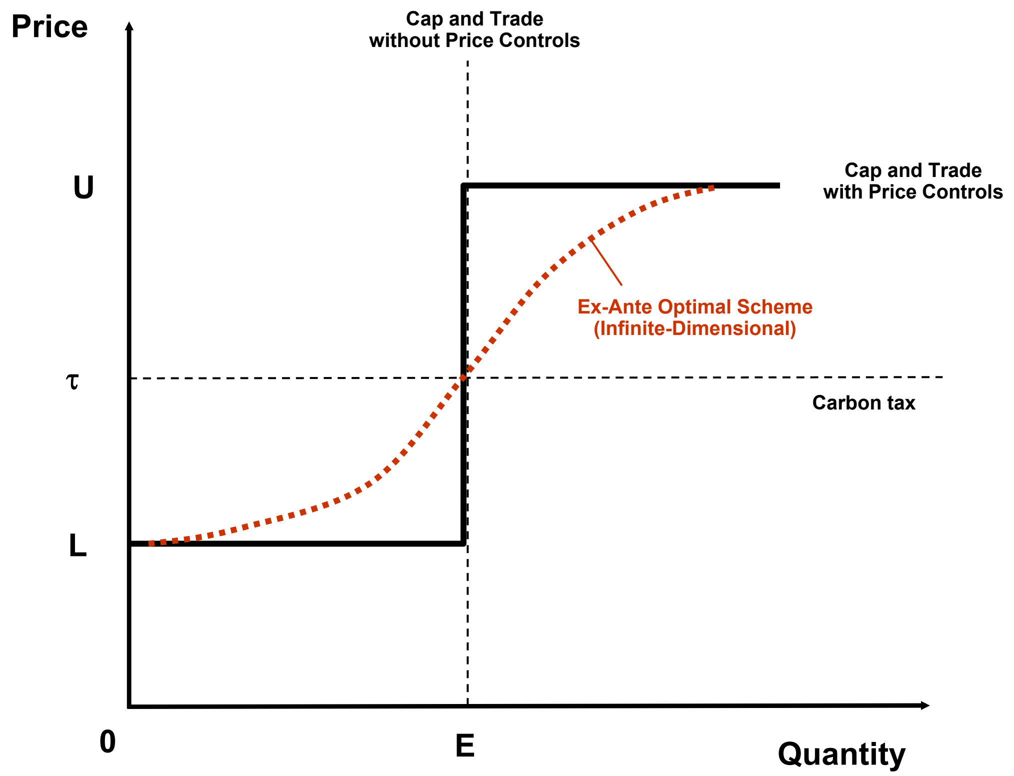


# OPTIMAL REGULATION
## Is Cap and Trade with Price Controls Really the Best One Can Do?

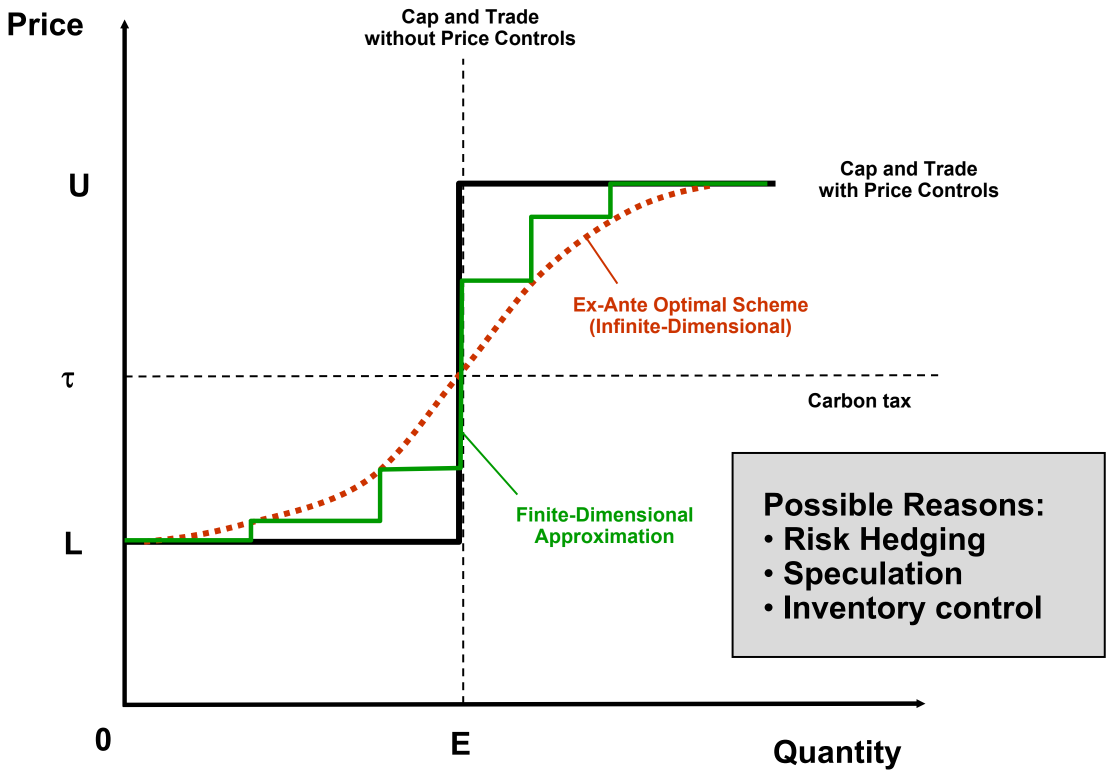





# OPTIMAL REGULATION
## Ex-Ante (Infinite-Dimensional) Solution Might Be Quite Different



# MULTI-CAP AND TRADE
## Implementation (e.g.) via Derivative Securities such as Options






# ANOTHER IMPORTANT QUESTION: REGULATORY COMMITMENT

**When the macroeconomic uncertainty has realized the regulator may want to deviate from his announced regulatory policy R and deviate to R'**

- **Is it good for a regulator to commit *ex ante* to a scheme R?**
- **What are *credible* commitment devices?**
- **What *degree* of commitment is optimal?**

**Additional policy instruments available at the implementation stage**

- **Incentives**
- **Supplementary regulation**
- **Emissions banking**
- **Mode of permit allocation**

## DYNAMIC POLICY ISSUES
## A consistent policy mix is credible

**Carbon pricing**

- **Short-term – address risk from extreme carbon prices (e.g., via price controls)**
- **Medium-term – flexible price response to deliver target (e.g., national targets, minimize leakage)**
- **Long-term – global mechanism with joint carbon price, where equity can be implemented via “green fund” (e.g., on per-capita-emission basis)**

**Complementary policies**

- **Common trajectory to ensure action across governments**
- **Fairness in the design of regional and global mechanism**

**Technology policy**

- **Innovation incentives (e.g., provided by law, technology competitions)**
- **Aggressive standard setting**
- **Certification (e.g., green labelling)**



## AGENDA

**What are externalities?**

**Example: Production Externalities**

**Some Regulatory Options**

**Uncertainty Matters: Prices vs. Quantities**

**Key Concepts to Remember**

# KEY CONCEPTS TO REMEMBER

- Positive/Negative Externalities
- Production Externalities
- Market Failure
- Regulatory options to deal with market failure due to externalities
- Coase Theorem
- Prices vs. Quantities, and how to regulate both!

# MGT 621 – MICROECONOMICS

Thomas A. Weber

*8. General Equilibrium, Part I*

Autumn 2023

École Polytechnique Fédérale de Lausanne
College of Management of Technology



# AGENDA

**General Equilibrium: The Standard Model**

**Pure Exchange**

**Production Economies**

**Key Concepts to Remember**

# THE STANDARD MODEL

**Basic Assumptions**:

- $N$ **goods**, $i \in \{1,...,N\}$
- $C$ **consumers**, $c \in \{1,...,C\}$
  - **Each consumer** $c$ **has a *rational* (i.e., complete and transitive) *preference ordering* over** $\Re_+^N$ **representable by a *continuous utility function*** $u_c : \Re_+^N \to \Re$. **Consumers are price takers**
- $F$ **firms**, $f \in \{1,...,F\}$
  - **Each firm** $f$ **has a production set** $Y_f \subset \Re^N$
    - ***Nonempty* and *closed***
    - ***No free lunch*** **(** $Y_f \cap \Re_+^N \subset \{0\}$ **; can't produce something from nothing)**
    - ***Possibility of inaction*** **(** $0 \in Y_f$ **)**
    - ***Free disposal*** **(** $Y_f - \Re_+^N \subset Y_f$ **)**
    - ***Irreversibility*** **(** $y \in Y_f \setminus \{0\} \Rightarrow -y \notin Y_f$**)**
- **Initial endowments: each consumer** $c$ **starts with an endowment vector** $\omega^c \in \Re_+^N$ **and a fractional share distribution** $\theta^c = \left(\theta_1^c,...,\theta_F^c\right)$ **, where** $\theta_f^c \in [0,1]$ **for each firm** $f$ **with**

$$\sum_{c=1}^{C} \theta_f^c = 1$$ **(Private Ownership Economy)**



# WALRASIAN EQUILIBRIUM

**Definition: A Walrasian equilibrium (WE) is a specification of a price vector** $p \in \Re_+^N$**, a demand vector** $x^c \in \Re_+^N$ **for each consumer** $c$ **, and a supply vector** $y^f \in Y_f$ **for each firm** $f$**, such that**

- **profit maximization, i.e.,** $y^f \in \arg\max_{y \in Y_f} p \cdot y$
- **utility maximization, i.e.,** $x^c \in \arg \max_{x \in B(p, I^c)} u_c(x)$

  **where *consumer c's* budget set is given by** $B_c = B(p, I^c) = \{x \in \Re_+^N : p \cdot x \le I^c\}$

  **with total income** $$I^c = p \cdot \omega^c + \sum_{f=1}^{F} \theta_f^c (p \cdot y^f)$$
- **demand = supply,** $$\sum_{c=1}^{C} x^c = \sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} y^f$$ **(i.e., allocation is feasible)**

**hold.**

# AGENDA

**General Equilibrium: The Standard Model**

**Pure Exchange**

**Production Economies**

**Key Concepts to Remember**



# PURE EXCHANGE IS A SPECIAL CASE OF THE STANDARD MODEL

**Consider an "exchange economy" *without firms* and *without production***

- **Two goods, 1 and 2**
- **M consumers of each of two types, 1 and 2**
- **Each consumer of type $c \in \{1,2\}$ begins with an allocation $\omega^c \in \Re_+^2$ (his endowment) and solves a utility maximization problem given a price vector $p \in \Re_+^2$, which defines his "offer curve",**

$$x^c(p) \in \arg \max_{x \in B(p, p \cdot \omega^c)} u_c(x)$$

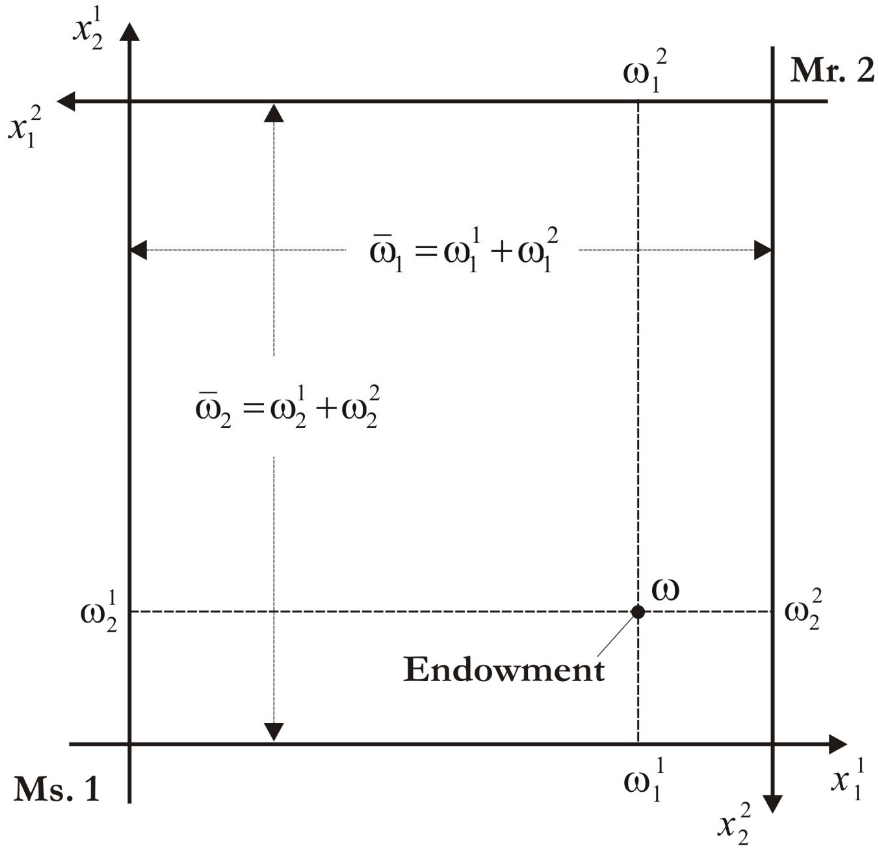
- **Any feasible allocation satisfies** $x^1 + x^2 = \omega^1 + \omega^2$

**This two-consumer two-good exchange economy can be represented graphically using a "Edgeworth box" (sometimes also referred to as "Edgeworth-Bowley Diagram").**

# EDGEWORTH BOX

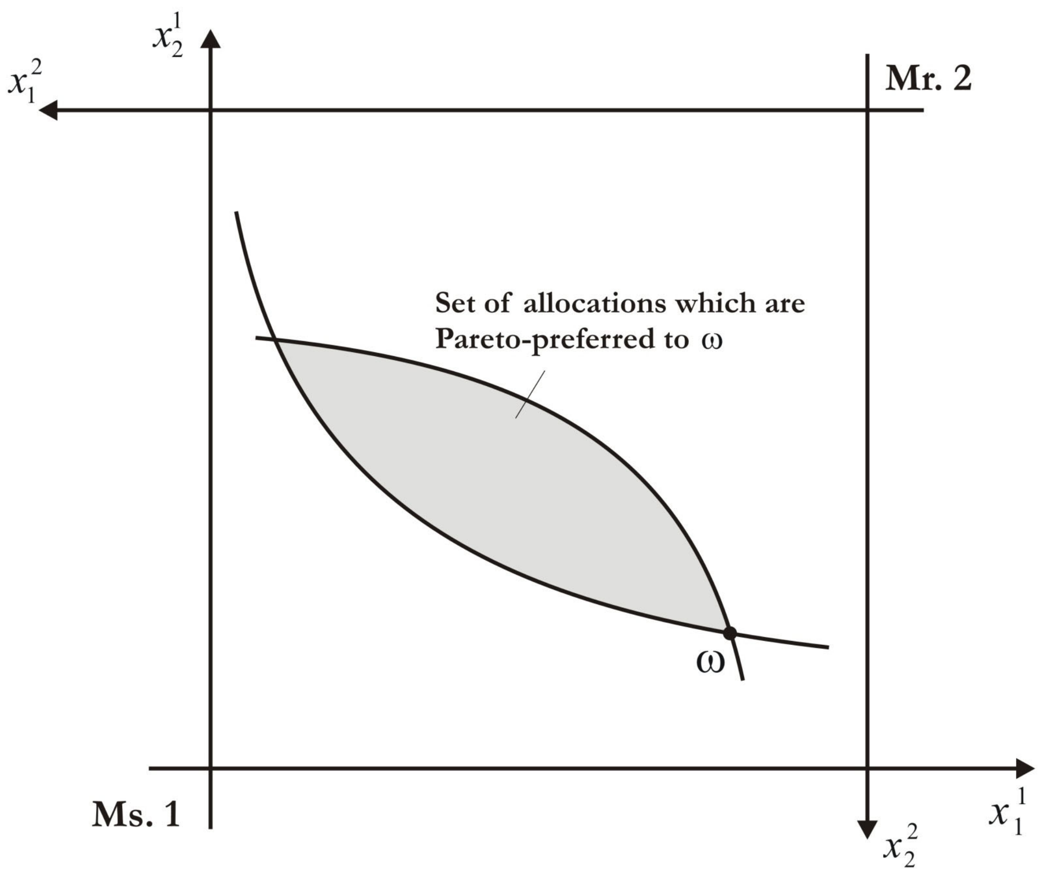
Consider one consumer of each type, Ms. 1 and Mr. 2

Any allocation of the ***totally available quantities*** $\overline{\omega}_i = \omega_i^1 + \omega_i^2$ of good i can be represented as a point in the Edgeworth box

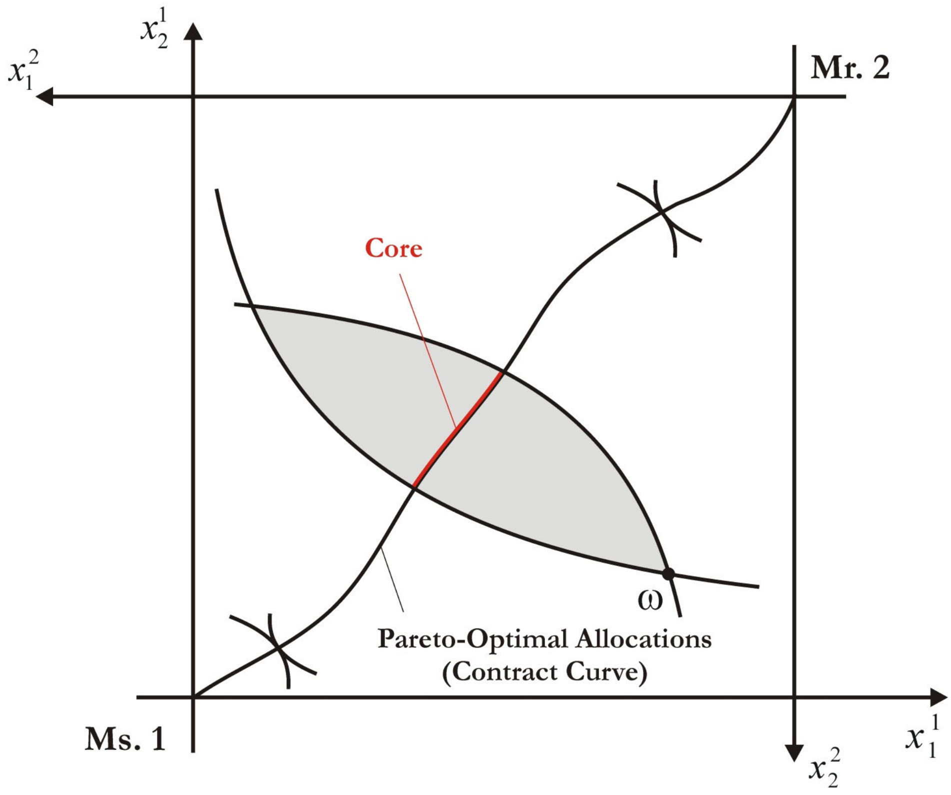





# THERE MAY BE GAINS FROM TRADE

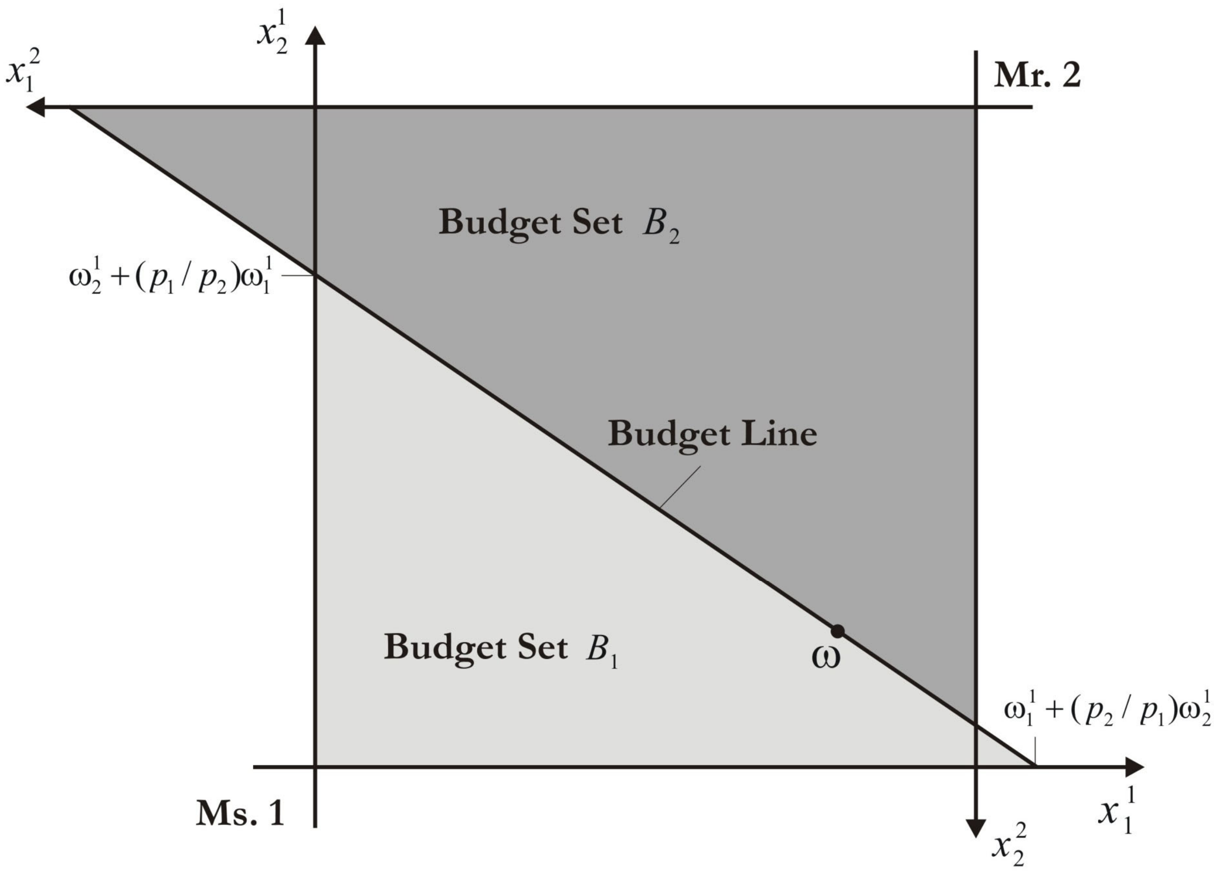


## PARETO-OPTIMAL ALLOCATIONS AND CONTRACT CURVE

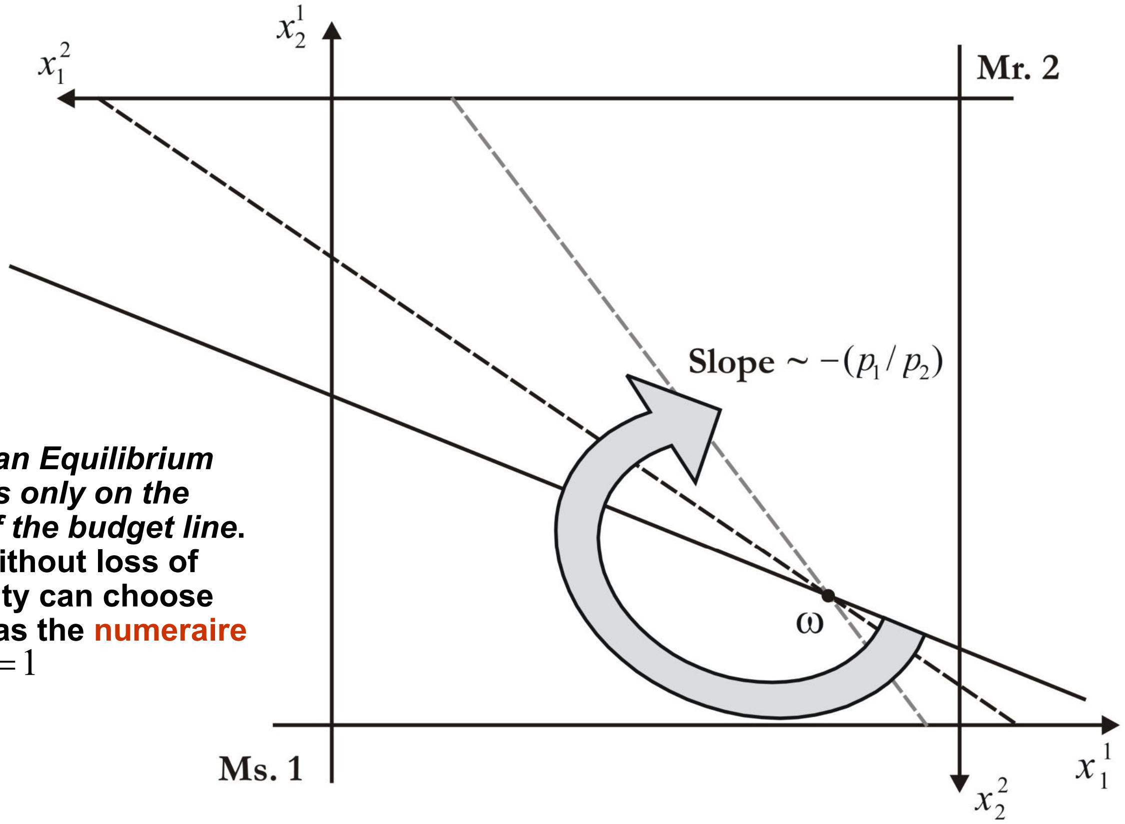





## BUDGET LINE AND BUDGET SETS



# THE SLOPE OF THE BUDGET LINE IS $(- p_1/p_2)$




***Walrasian Equilibrium depends only on the slope of the budget line*. Thus, without loss of generality can choose good 1 as the numeraire with $p_1 = 1$**

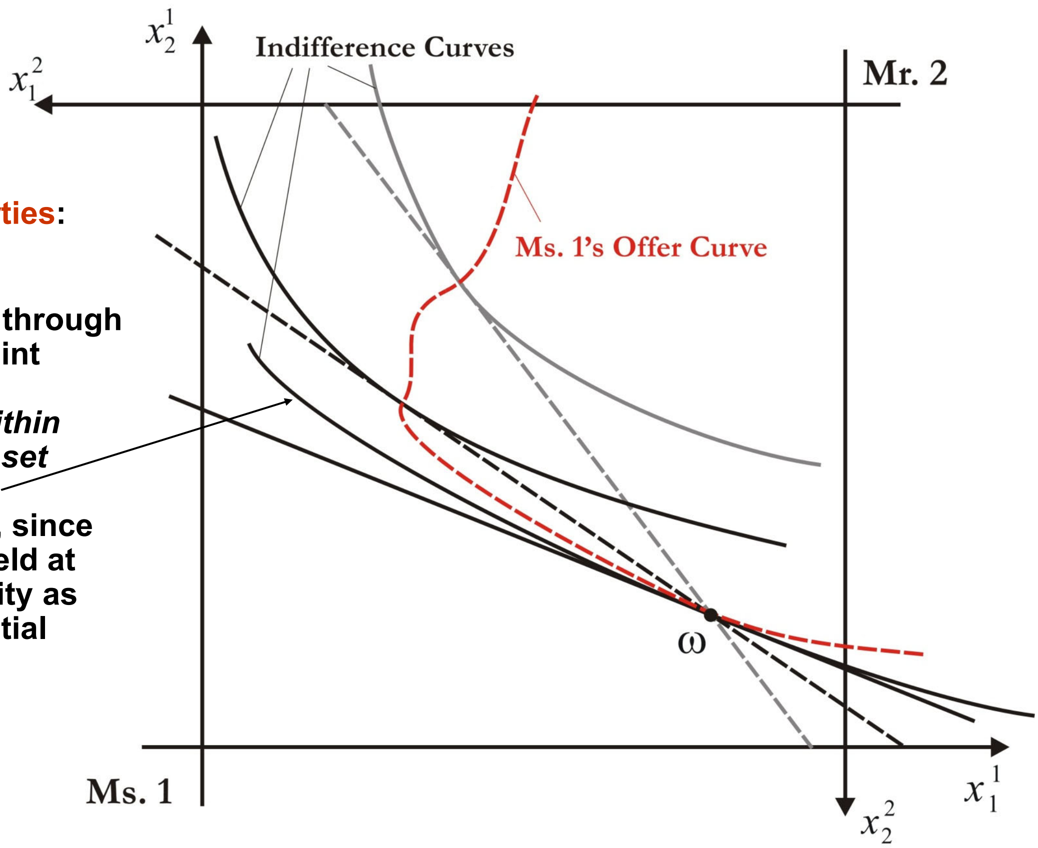


# THE OFFER CURVE DESCRIBES THE OPTIMAL CONSUMPTION CHOICE AS A FUNCTION OF MARKET PRICES

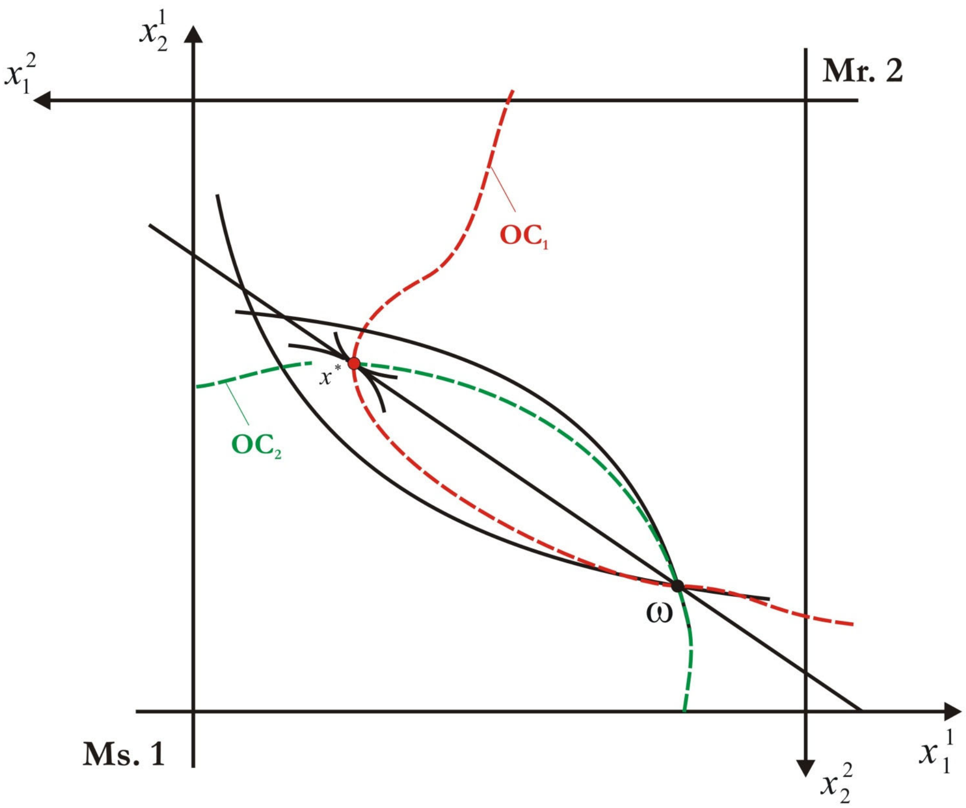
**Offer Curve Properties:**

**OC is *tangent to indifference curve* through the endowment point**

**It also *has to lie within the upper contour set* delineated by that indifference curve, since any offer has to yield at least the same utility as the consumer's initial endowment**



# IN A WALRASIAN EQUILIBRIUM THE OFFER CURVES INTERSECT






# EXAMPLE: EXCHANGE WITH COBB-DOUGLAS UTILITIES

## Compute the Walrasian Equilibrium

**Consider two consumers, 1 and 2, with Cobb-Douglas utility functions**

$$u_c(x_1^c, x_2^c) = (x_1^c)^{\alpha}(x_2^c)^{1-\alpha}$$

**where** $\alpha \in (0,1)$ **is a constant and** $c \in \{1,2\}$ **. The endowment vectors are** $\omega^1 = (1,2)$ **and** $\omega^2 = (2,1)$ **respectively**

**Given a price vector** $p = (p_1, p_2)$**, consumer 1's utility maximization problem can be stated (after taking the logarithm) in the form**

$$x^1(p) = \arg \max_{(x_1,x_2)\in B(p,p_1+2p_2)} \{\alpha \log x_1 + (1-\alpha)\log x_2\} = \underbrace{\left(\frac{\alpha(p_1+2p_2)}{p_1}, \frac{(1-\alpha)(p_1+2p_2)}{p_2}\right)}_{\textit{Consumer 1's offer curve}}$$

**Similarly, we find *consumer 2's offer curve*,**

$$x^2(p) = \left(\frac{\alpha(2p_1+p_2)}{p_1}, \frac{(1-\alpha)(2p_1+p_2)}{p_2}\right)$$

# EXCHANGE WITH COBB-DOUGLAS UTILITIES (cont'd)

**Using the Demand = Supply condition for the Walrasian equilibrium we can clear the market for good 1,**

$$x_1^1(p) + x_1^2(p) = \omega_1^1 + \omega_1^2$$

**or equivalently,**

$$\frac{\alpha(p_1 + 2p_2)}{p_1} + \frac{\alpha(2p_1 + p_2)}{p_1} = 3$$

**which yields** $\dfrac{p_1}{p_2} = \dfrac{\alpha}{1-\alpha}$ **and** $\hat{x}_1^1 = \hat{x}_2^1 = 2-\alpha, \quad \hat{x}_1^2 = \hat{x}_2^2 = 1+\alpha$

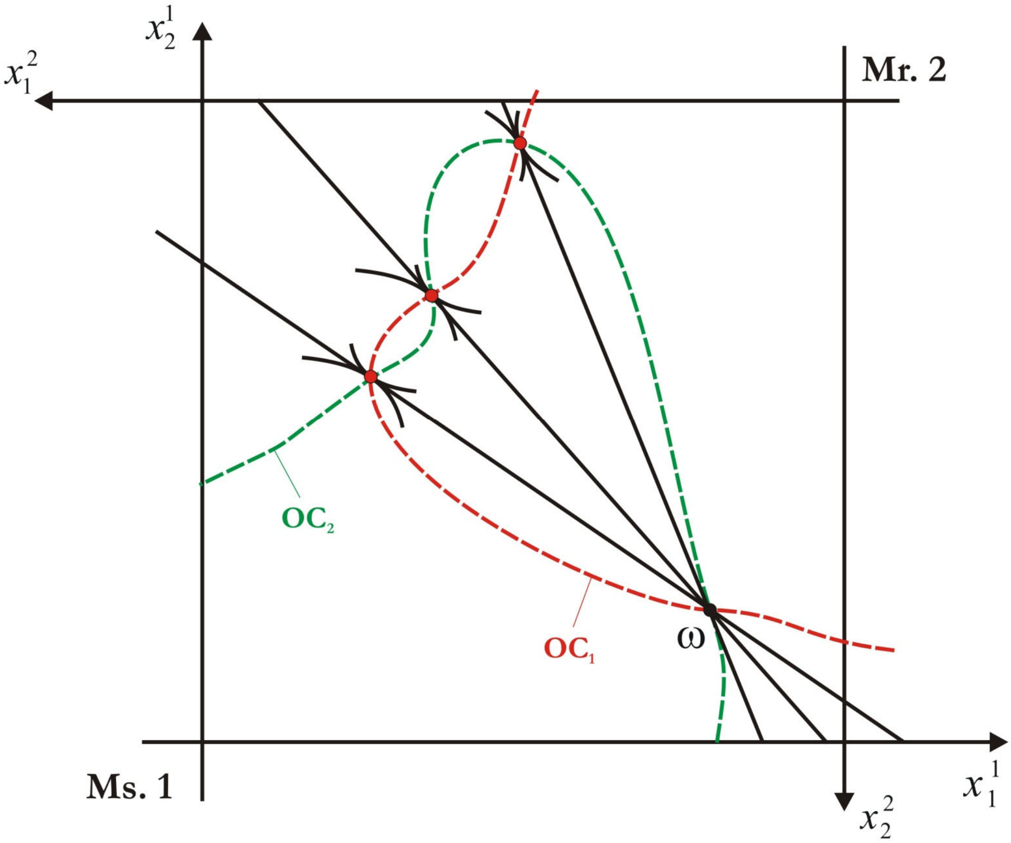
**In other words, *the price ratio in equilibrium is equal to the marginal rate of substitution between the two goods at the equilibrium allocation.*[(1)]**

**Note also that market clearing for good 1 implies market clearing for good 2 (why?)**

(1) The marginal rate of substitution for consumer 1 between goods 1 and 2 is $MRS_{1,2}(\hat{x}^1) = \dfrac{\left.\frac{\partial u_1}{\partial x_1^1}\right|_{(\hat{x}_1^1,\hat{x}_2^1)}}{\left.\frac{\partial u_1}{\partial x_2^1}\right|_{(\hat{x}_1^1,\hat{x}_2^1)}} = \dfrac{\alpha}{1-\alpha}\dfrac{\hat{x}_1^1}{\hat{x}_2^1} = \dfrac{\alpha}{1-\alpha}\dfrac{2-\alpha}{2-\alpha} = \dfrac{\alpha}{1-\alpha}$ .

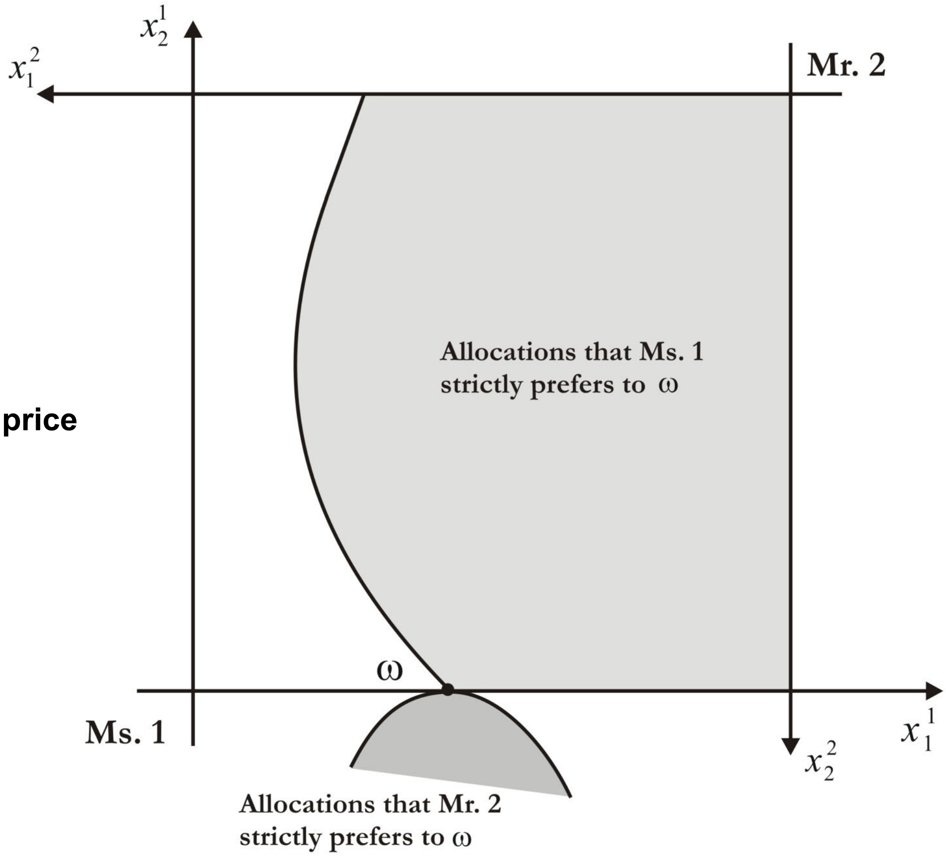


# THERE MAY BE DIFFERENT WALRASIAN EQUILIBRIA
## Nonuniqueness



# A WALRASIAN EQUILIBRIUM MAY NOT EXIST
## Example: Preferences Not Strictly Monotone

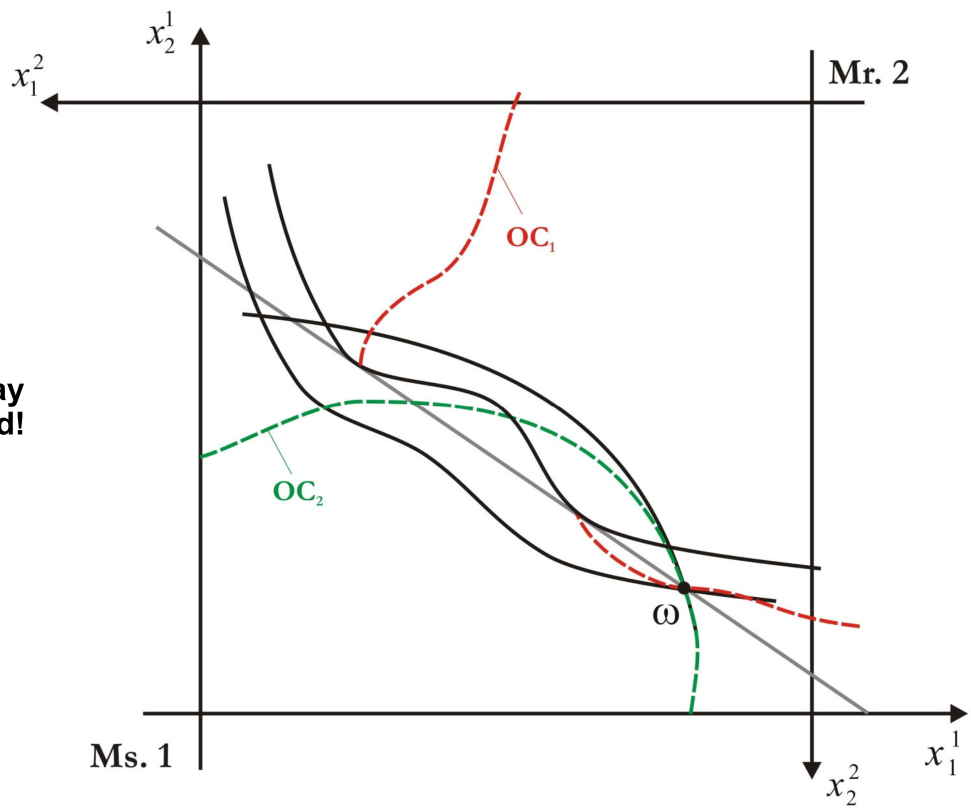
No supporting price vector exists!






# A WALRASIAN EQUILIBRIUM MAY NOT EXIST (cont'd)
## Example: Nonconvex Preferences

Offer curves may be disconnected!



# AGENDA

**General Equilibrium: The Standard Model**

**Pure Exchange**

**Production Economies**

**Key Concepts to Remember**



# ROBINSON CRUSOE ECONOMY

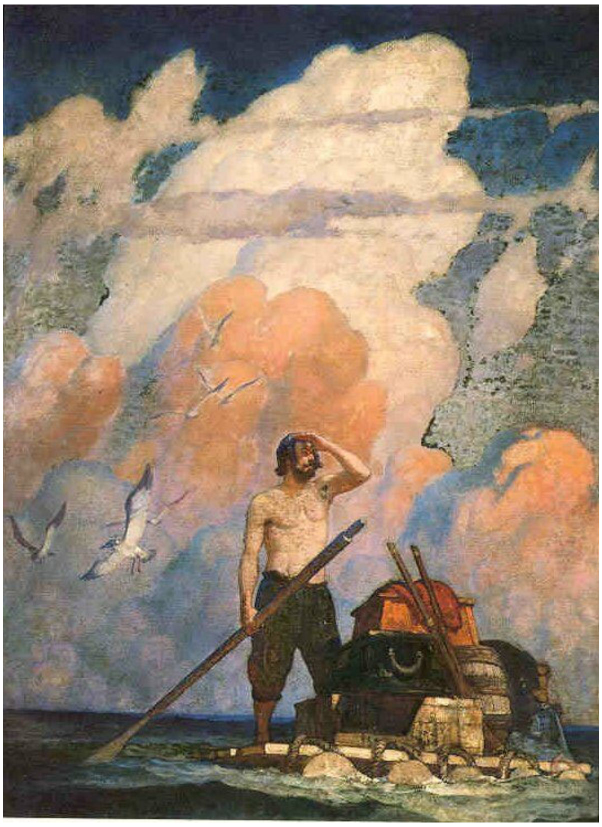
**Robinson is alone on an island. He can *either* rest (i.e., consume leisure $x_1$) *or* use his own labor to pick yummy coconuts $x_2$**

**You can think of the firm "Robinson Crusoe Enterprises" producing coconuts using Robinson's labor $z = \bar{L} - x_1$ as the only production input, i.e., the firm maximizes profits**

$$\pi(w, p) = \max_{z \geq 0} \{ pf(z) - wz \}$$

**where $f(\cdot)$ is the firm's *production function*,[(1)] $w$ is the *wage* Robinson pays himself, $p$ is the price of coconuts, and $\bar{L} > 0$ is a constant**

**Question: If Robinson maximizes his utility $u(x_1, x_2)$, *what is the Walrasian equilibrium of his private economy*?**



(1) For simplicity assume that the production function is strictly concave, increasing, and continuously differentiable; we also assume that Robinson's utility is smooth. In addition it is sometimes useful to add the assumptions that $f(0) = 0$ (to all for the possibility of inaction), and the so-called Inada conditions $f'(0) = \infty, f'(\bar{L}) = 0$ (to guarantee strict interiority of the optimizer).

# ROBINSON CRUSOE ECONOMY (cont'd)

**Answer: Naturally, Robinson owns all of "Robinson Crusoe Enterprises," whence we obtain his utility maximization problem,**

$$x(w,p) = \arg \max_{(x_1,x_2)\in B((p,w),I)} u(x_1,x_2)$$

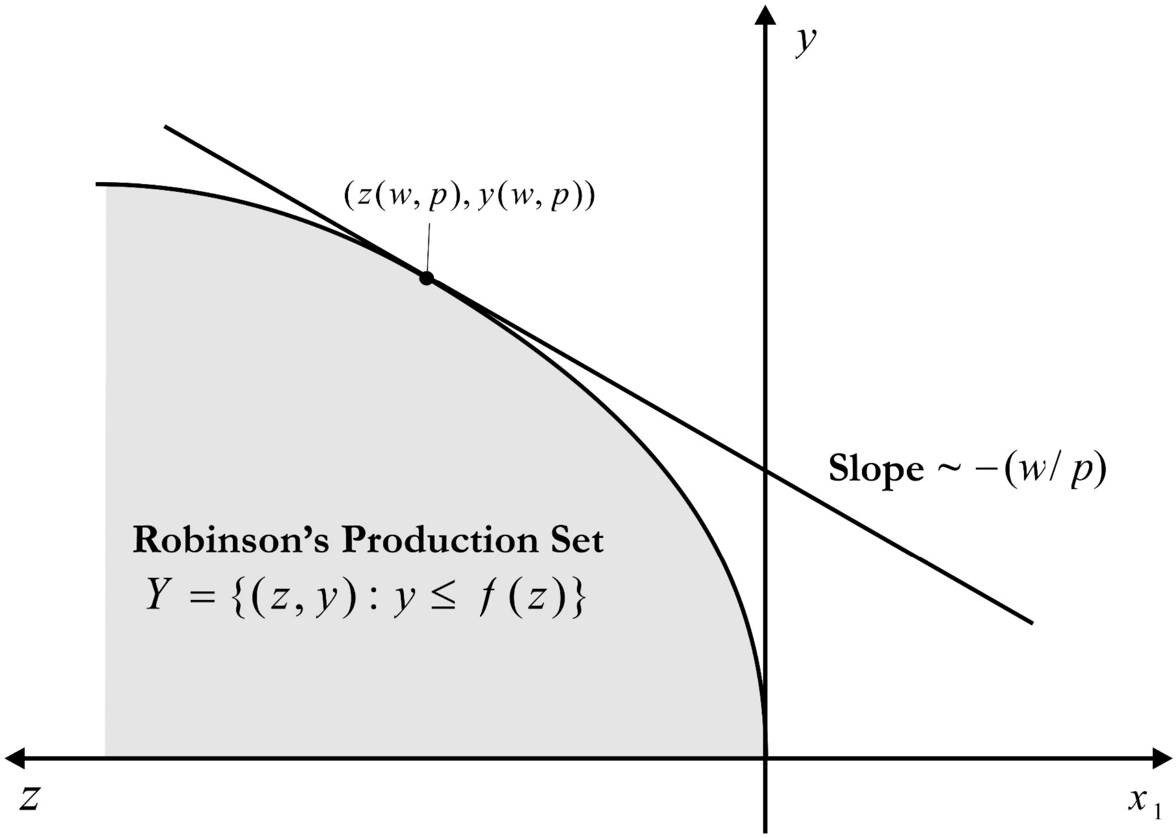
**where** $$B((w,p),I) = \{(x_1,x_2)\in \Re_+^2 : px_2 \le w(\overline{L}-x_1)+\pi(w,p)\}$$

**From the definition of a Walrasian equilibrium, we obtain the following conditions:**

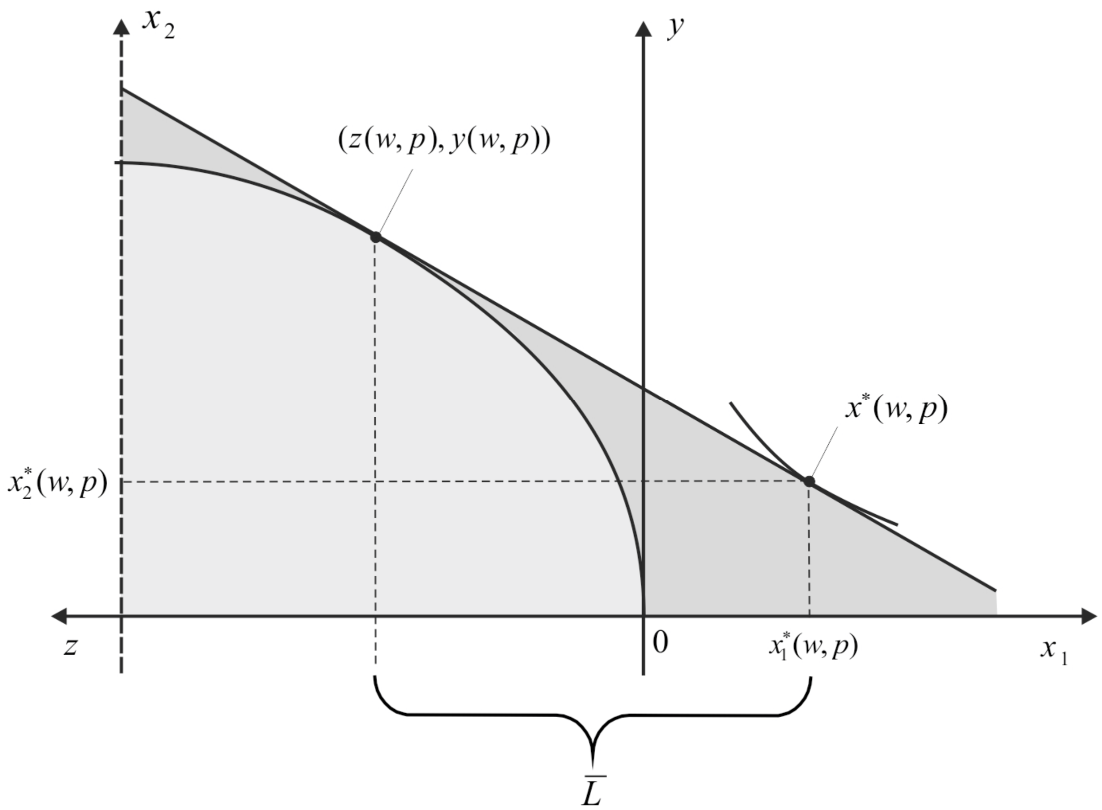
- **Profit maximization:** $$f'(z^*(w,p)) = \frac{w}{p}$$
- **Utility Maximization:** $$\left[\frac{\partial u}{\partial x_1} - \left(\frac{w}{p}\right)\frac{\partial u}{\partial x_2}\right](x_1^*, \frac{w}{p}(\overline{L}-x_1^*) + y^*(w,p) - \frac{w}{p}z^*(w,p)) = 0$$
  **where** $y^*(w,p) = f(z^*(w,p))$ **is the** ***equilibrium production quantity*** **of coconuts**
- **Demand = Supply:** $$x_1^* = \overline{L} - z^*$$ $$x_2^* = y^*$$

 Note: Without loss of generality we can *take coconuts as the numeraire good* (Robinson pays himself in coconuts), i.e., $p=1$. 

# ROBINSON CRUSOE ECONOMY (cont'd)
## Production Problem



# ROBINSON CRUSOE ECONOMY (cont'd)
## Walrasian Equilibrium






# 2X2 PRODUCTION ECONOMY

**Consider now a simple two-input two-output production economy:**

- **2 Outputs, $j \in \{1,2\}$ : there are 2N firms *producing one output each* (1)**
- **2 Inputs, *capital* $K^j$ and *labor* $L^j$**
- **The production function of each firm for *output* $j$ is given by**

$$y^j = f^j(K^j, L^j)$$

**where $f^j(0,0) = 0$ (*possibility of inaction*) and $f^j$ is strictly concave**

- **N consumers *of each of two types*, $c \in \{1,2\}$, with increasing and strictly quasi-concave utility functions $u_c(x_1^c, x_2^c)$, where $x_j^c$ is the amount a consumer of type c consumes of product j**
- **We assume for simplicity that consumers do not want to consume either capital or labor**
- **Consumers start with zero endowments in the production goods $j \in \{1,2\}$ and with $\overline{K}_c$ units of capital and $\overline{L}_c$ units of labor. In addition, each consumer of type c owns the fraction $\theta_j^c$ of the outstanding shares in the firms producing output j**

(1) More precisely, we assume that there are N firms producing output 1 and N firms producing output 2.

# 2X2 PRODUCTION ECONOMY (cont'd)

**Definition**: **A symmetric allocation is one in which *all consumers of type c* receive the *same consumption vector* $x^c = (x_1^c, x_2^c)$ and all firms of type j produce the same output level $y^j$ using the same input vector $(K^j, L^j)$.**

**A *symmetric allocation* $((x^1, x^2), (y^1, K^1, L^1), (y^2, K^2, L^2))$ is feasible, if demand = supply, i.e.,**

$$y^j = x_j^1 + x_j^2 \quad (1)$$

$$\overline{K}_c = K_c^1 + K_c^2 \quad (2)$$

$$K^j = K_1^j + K_2^j \quad (3)$$

$$\overline{L}_c = L_c^1 + L_c^2 \quad (4)$$

$$L^j = L_1^j + L_2^j \quad (5)$$

$$y^j = f^j(K^j, L^j) \quad (6)$$

**Let $F$ be the set of feasible allocations.**



# 2X2 PRODUCTION ECONOMY (cont'd)

**Definition**: **A symmetric Walrasian equilibrium is a specification of a price $p_j$ for each output $j$, a *price $r$ of capital*, a *wage $w$ for labor*, a *consumption vector* $\hat{x}^c$ for each consumer type c, and a *production vector* $(\hat{y}^j, \hat{K}^j, \hat{L}^j)$ for each type of firm, such that the following three conditions are satisfied.**

- **Utility maximization** $\hat{x}^c = \arg\max_{x \in B_c} \{u_c(x)\}$

$$B_c = \{(x_1^c, x_2^c) \in \Re_+^2 : p_1 x_1^c + p_2 x_2^c \le r\overline{K}_c + w\overline{L}_c + \theta_1^c \pi_1 + \theta_2^c \pi_2\}$$

- **Profit maximization** $(\hat{y}^j, \hat{K}^j, \hat{L}^j) = \arg\max_{y^j = f^j(K,L)} \{p_j y^j - rK^j - wL^j\}$

- **Demand = supply, i.e.,** $((\hat{x}^1, \hat{x}^2), (\hat{y}^1, \hat{K}^1, \hat{L}^1), (\hat{y}^2, \hat{K}^2, \hat{L}^2))$ **is a *symmetric feasible allocation* for each $j \in \{1,2\}$, where**

$$\pi_j = p\hat{y}^j - r\hat{K}^j - w\hat{L}^j$$

**is firm j's equilibrium profit**

## 2X2 PRODUCTION ECONOMY (cont'd)

***First-order necessary optimality conditions*** **hold in a symmetric WE:**

- **Utility maximization**

$$\frac{\partial u_1(\hat{x}^1)/\partial x_1^1}{\partial u_1(\hat{x}^1)/\partial x_2^1} = \frac{\partial u_2(\hat{x}^2)/\partial x_1^2}{\partial u_2(\hat{x}^2)/\partial x_2^2} = \frac{p_1}{p_2} \quad \textbf{(7)}$$

- **Profit maximization**

$$p_1 \frac{\partial f^1(\hat{K}^1, \hat{L}^1)}{\partial K^1} = p_2 \frac{\partial f^2(\hat{K}^2, \hat{L}^2)}{\partial K^2} = r \quad \textbf{(8)}$$

$$p_1 \frac{\partial f^1(\hat{K}^1, \hat{L}^1)}{\partial L^1} = p_2 \frac{\partial f^2(\hat{K}^2, \hat{L}^2)}{\partial L^2} = w \quad \textbf{(9)}$$

**From (7)—(9) we obtain**

$$\boxed{\frac{\partial u_c(\hat{x}^c)/\partial x_1^c}{\partial u_c(\hat{x}^c)/\partial x_2^c} = \frac{p_1}{p_2} = \frac{\partial f^2(\hat{K}^2, \hat{L}^2)/\partial K^2}{\partial f^1(\hat{K}^1, \hat{L}^1)/\partial K^1} = \frac{\partial f^2(\hat{K}^2, \hat{L}^2)/\partial L^2}{\partial f^1(\hat{K}^1, \hat{L}^1)/\partial L^1}}$$

**All marginal rates of substitution are determined by the equilibrium price ratio of the traded market commodities.**



## AGENDA

**General Equilibrium: The Standard Model**

**Pure Exchange**

**Production Economies**

**Key Concepts to Remember**

# KEY CONCEPTS TO REMEMBER

- Edgeworth Box
- Pareto Optimality
- Budget Line
- Endowment
- Numeraire Good
- Offer Curve
- Walrasian Equilibrium (Competitive Equilibrium) (w/ or w/o transfers)
- Pareto Set
- Contract Curve/Core
- Pure Exchange / Production Economy
- Private Ownership Economy
- Price-Taking Behavior
- Walras' Law

# MGT 621 – MICROECONOMICS

Thomas A. Weber

## *8.* (Optional) *General Equilibrium, Part II*

Autumn 2023

École Polytechnique Fédérale de Lausanne
College of Management of Technology



# AGENDA

**Some Preliminaries**

**Fundamental Welfare Theorems**

**Existence of a Competitive Equilibrium**

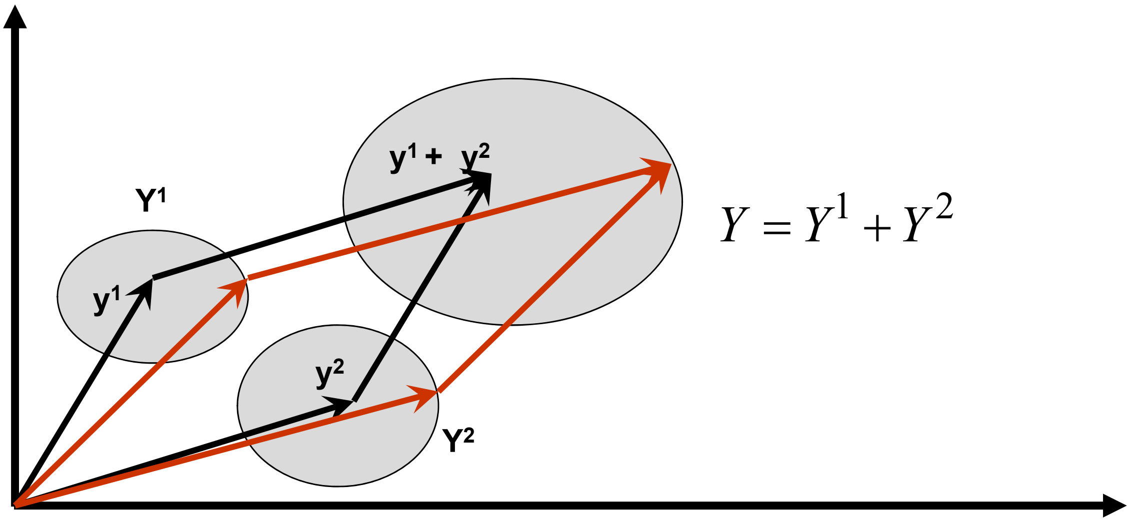
**General Equilibrium vs. Partial Equilibrium**

**Key Concepts to Remember**

# CONCEPT OF SET SUMMATION

**Set summation of set $Y^1$ and set $Y^2$:** $Y = Y^1 + Y^2 = \{y : y = y^1 + y^2, y^1 \in Y^1, y^2 \in Y^2\}$

**Intuition. Choose any point, $y^1$ from set $Y^1$ and any point $y^2$ from set $Y^2$; the set Y consists of the set of all points $y^1 + y^2$.**




$$Y = Y^1 + Y^2$$



# FEASIBLE TOTAL OUTPUT IN THE ECONOMY

**Initial Endowment of commodities by consumer c:** $\omega^c$

**Net output by firm f:** $y^f \in Y^f$

**→ Total Supply:** $\sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} y^f$

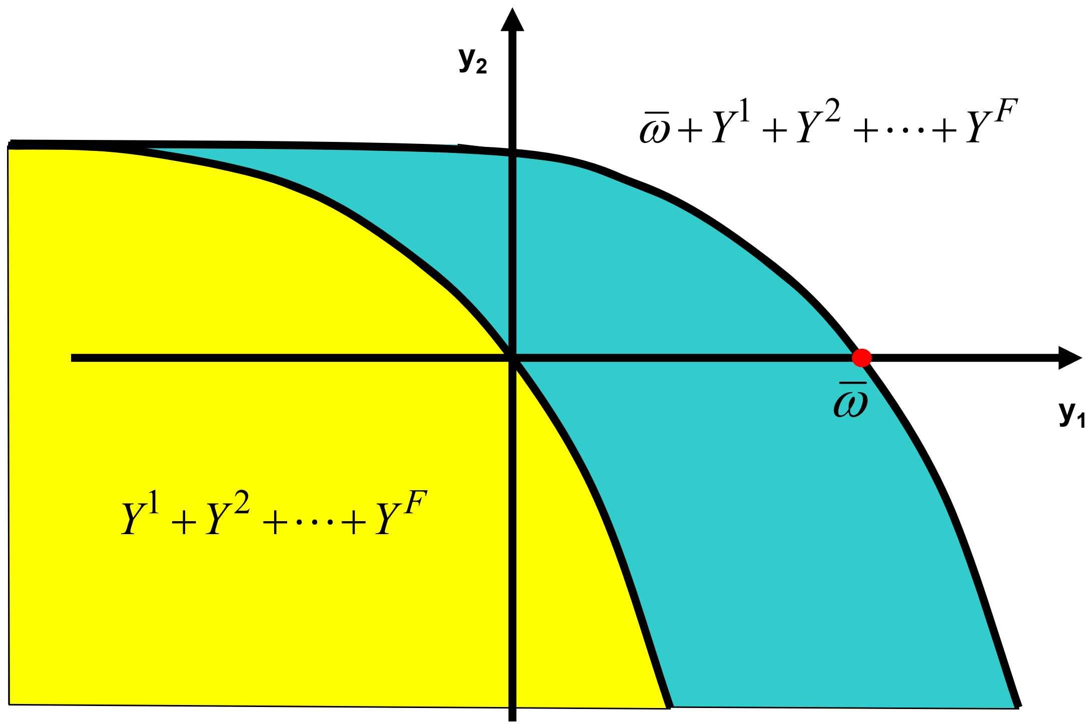
**Define total Initial Endowments** $\overline{\omega} = \sum_{c=1}^{C} \omega^c$

**Then, feasible set of total outputs is**

$$Y = \overline{\omega} + Y^1 + Y^2 + \cdots + Y^F$$

Remark. Note that this depends on the assumption that there are no externalities in production. If there are externalities, then the feasible set of total outputs is not set summation of individual production sets.

## TOTAL FEASIBLE OUTPUT IN THE ECONOMY (Cont'd)



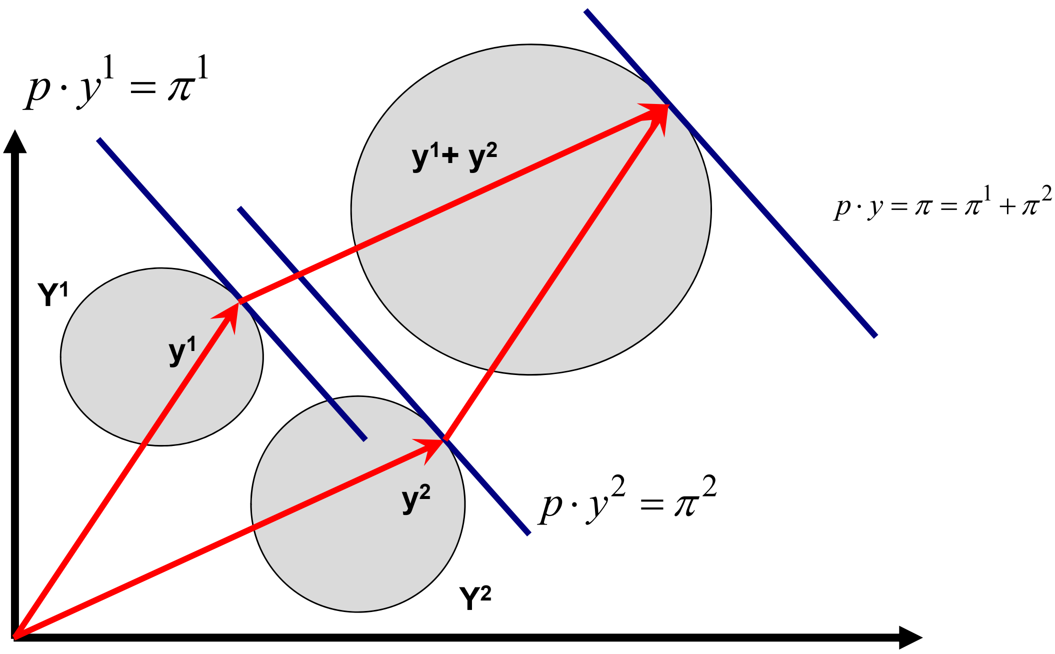



## VALUING TOTAL OUTPUT AT MARKET PRICES

$$p \cdot y = \sum_{c=1}^{C} p \cdot \omega^c + \sum_{f=1}^{F} p \cdot y^f$$

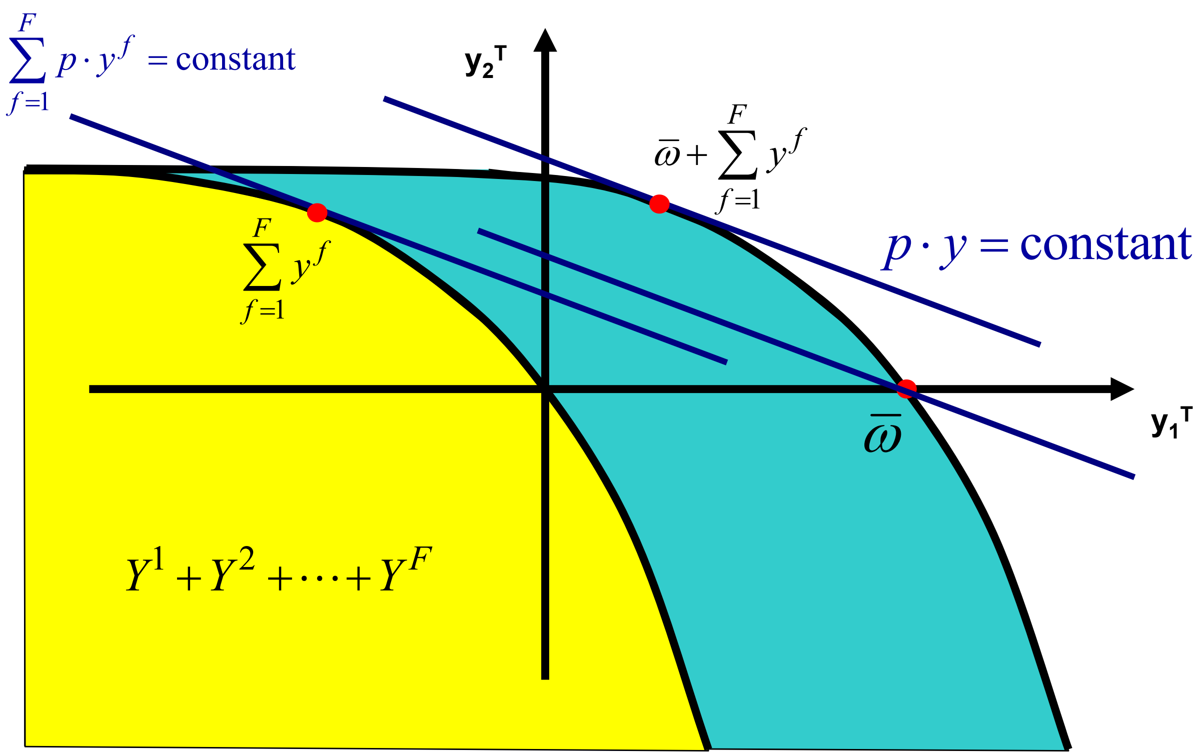
**Value of Total Output in Economy = Value of Initial Endowments + Sum of Firms' Profits**

# INDIVIDUAL MAXIMIZATION IMPLIES GLOBAL MAXIMIZATION






# MAXIMUM VALUE OF TOTAL OUTPUT



## CONSUMER CHOICE

**Consumer maximizes utility, subject to budget constraint**

$$u^{c*} = \max \; u^c(x^c)$$
$$\text{s.t. } p \cdot x^c \le w^c$$

**Equivalently, consumer minimizes expenditure for achieving a certain utility level**

$$w^{c*} = \min \; p \cdot x^c$$
$$\text{s.t. } u^c(x^c) \ge U^{c*}$$



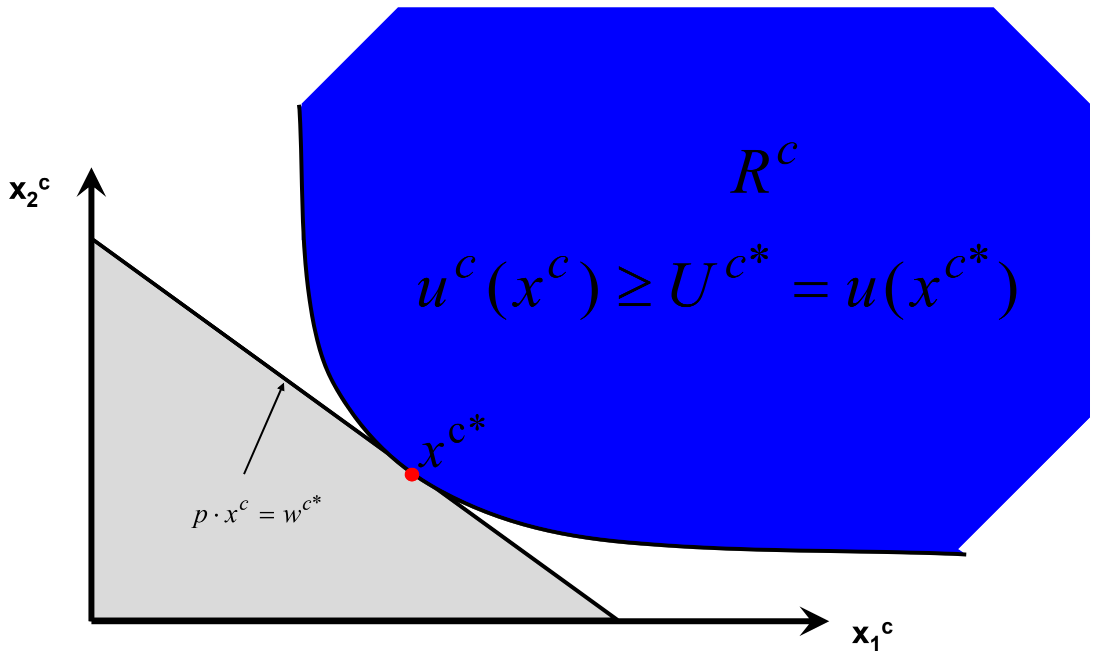
## CONSUMER CHOICE (Cont'd)

**Define set of consumption bundles weakly preferred to optimal choice as preference set for consumer c: $R^c$**

**Then consumer minimizes expenditure, given $x^c$ in $R^c$**

$$\min_{x^c \in R^c} \; p \cdot x^c$$

# CONSUMER CHOICE (Cont'd)

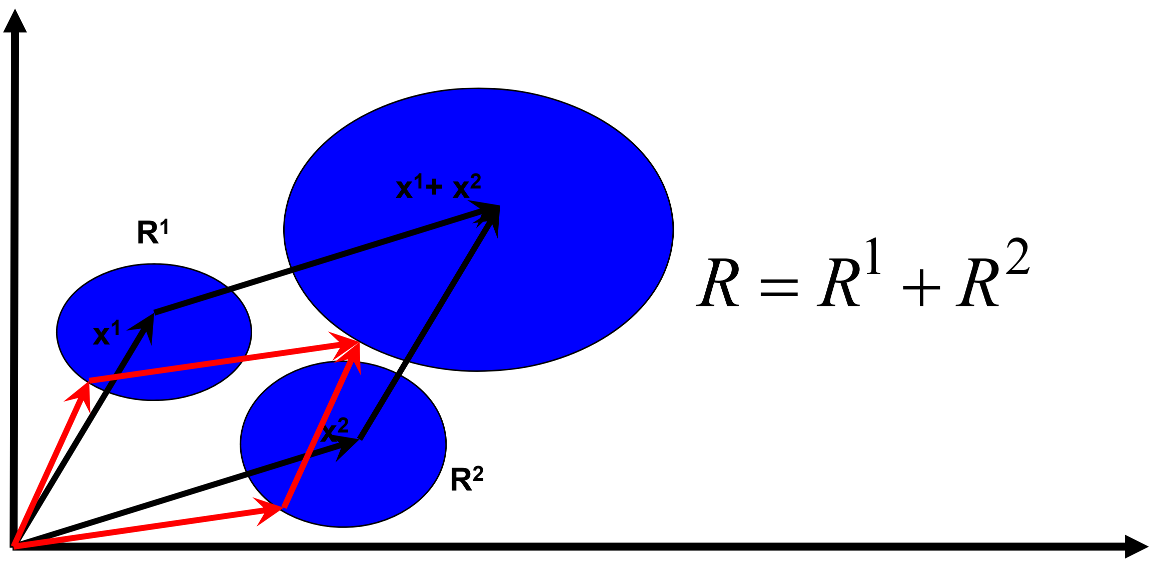





# SET SUMMATION OF INDIVIDUAL PREFERENCE SETS

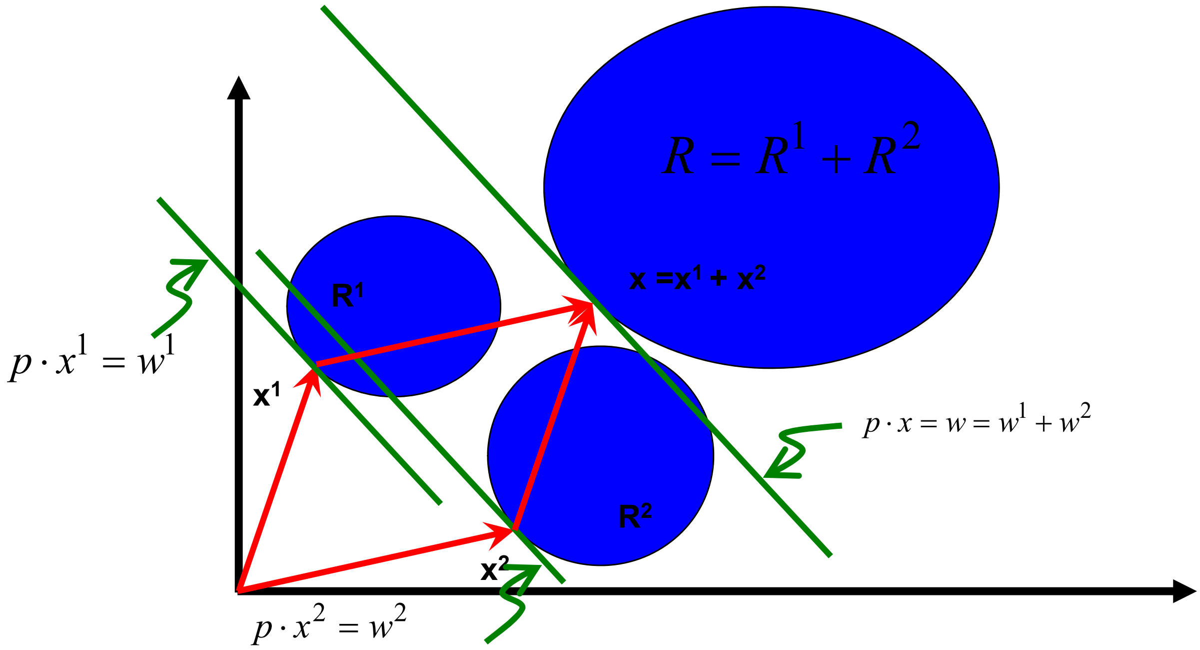
**Set summation of individual preference sets is the set of total consumption bundles that allows each consumer to have utility at least as high as his/her $U^{c*}$. R is the aggregate preference set.**

**Total consumption in interior of R could allow Pareto-superior allocations to consumers.**



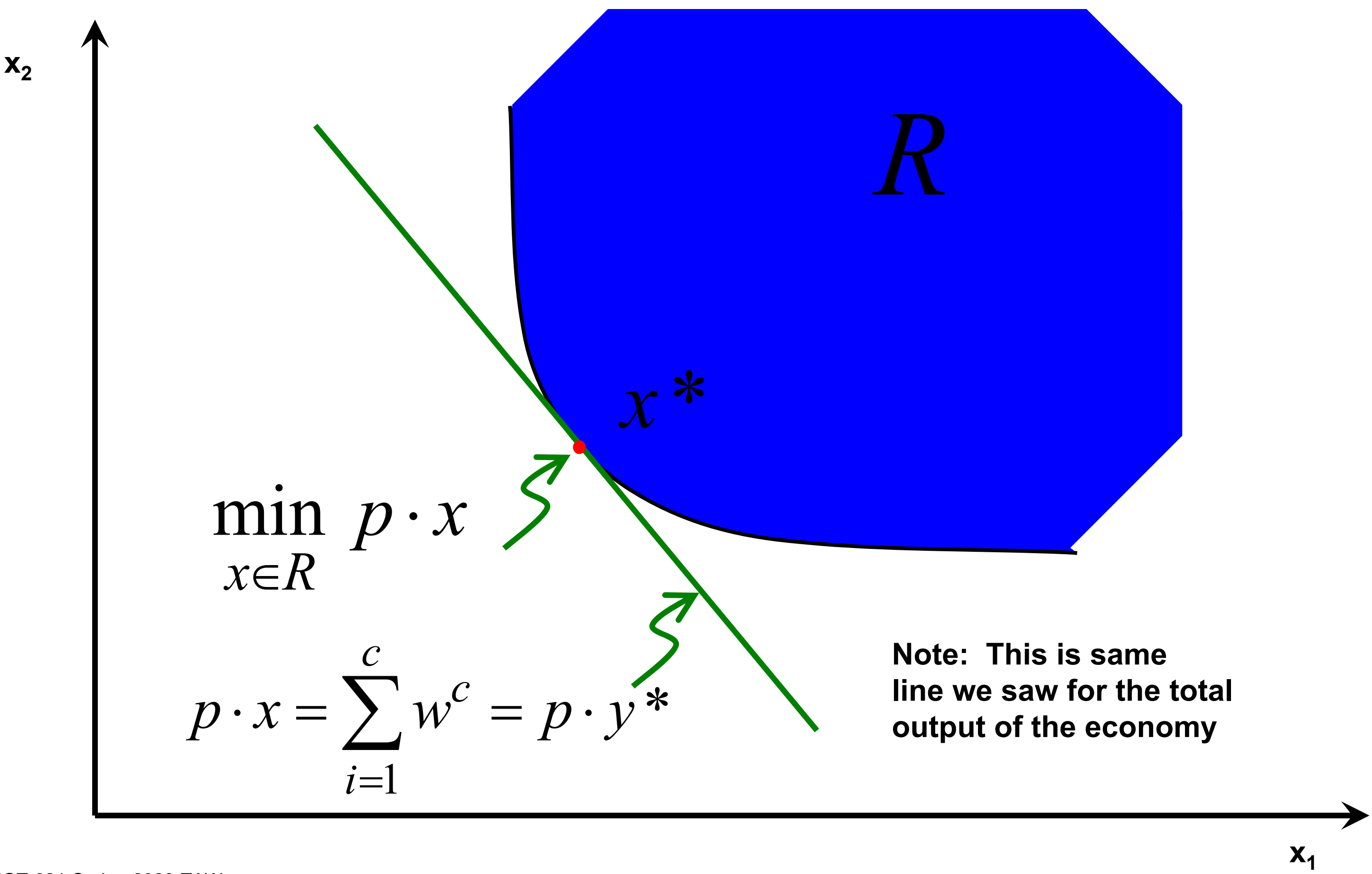

Remark. Note that this depends on the assumption that there are no externalities in consumption. If there are externalities, then the aggregate preference set is not set summation of individual preference sets.

# INDIVIDUAL MINIMIZATION IMPLIES GLOBAL MINIMIZATION






# MINIMIZATION OF TOTAL EXPENDITURE




**Note: This is same line we saw for the total output of the economy**

# AGENDA

Some Preliminaries

**Fundamental Welfare Theorems**

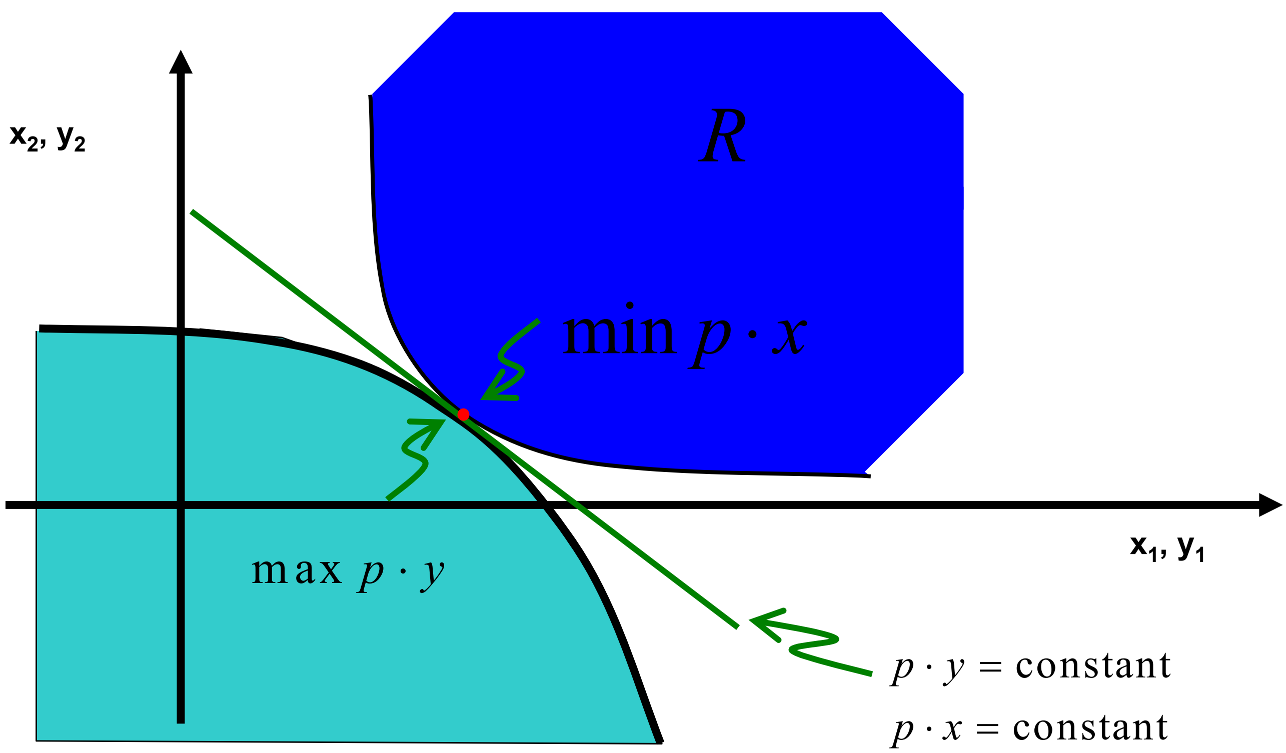
Existence of a Competitive Equilibrium

General Equilibrium vs. Partial Equilibrium

Key Concepts to Remember

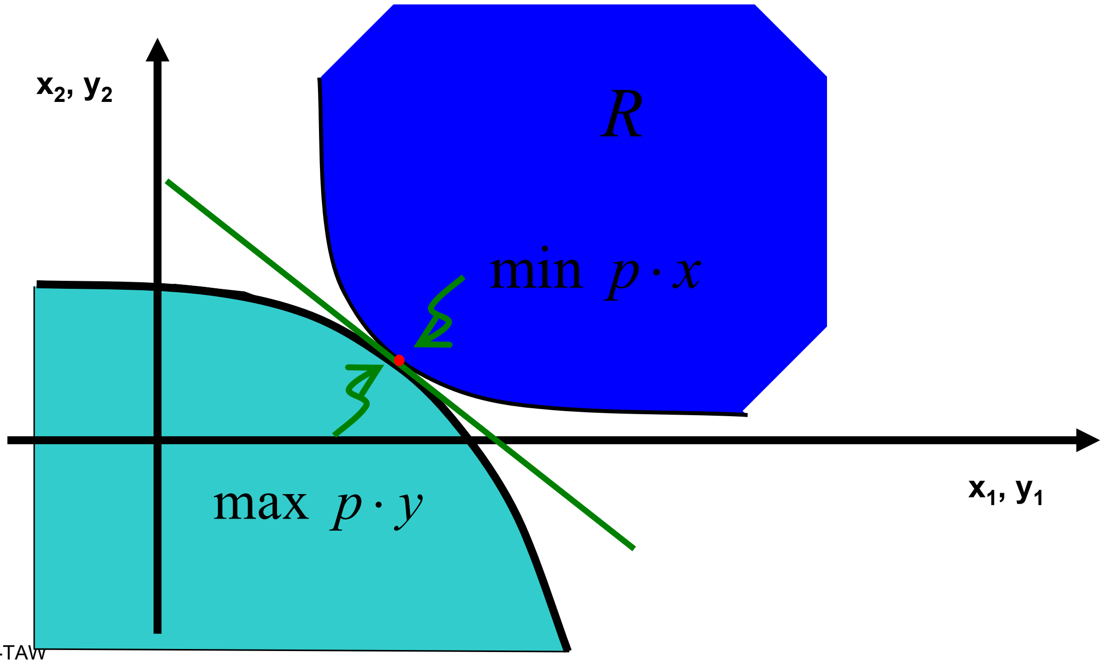


# COMPETITIVE EQUILIBRIUM MATCHES SUPPLY AND DEMAND



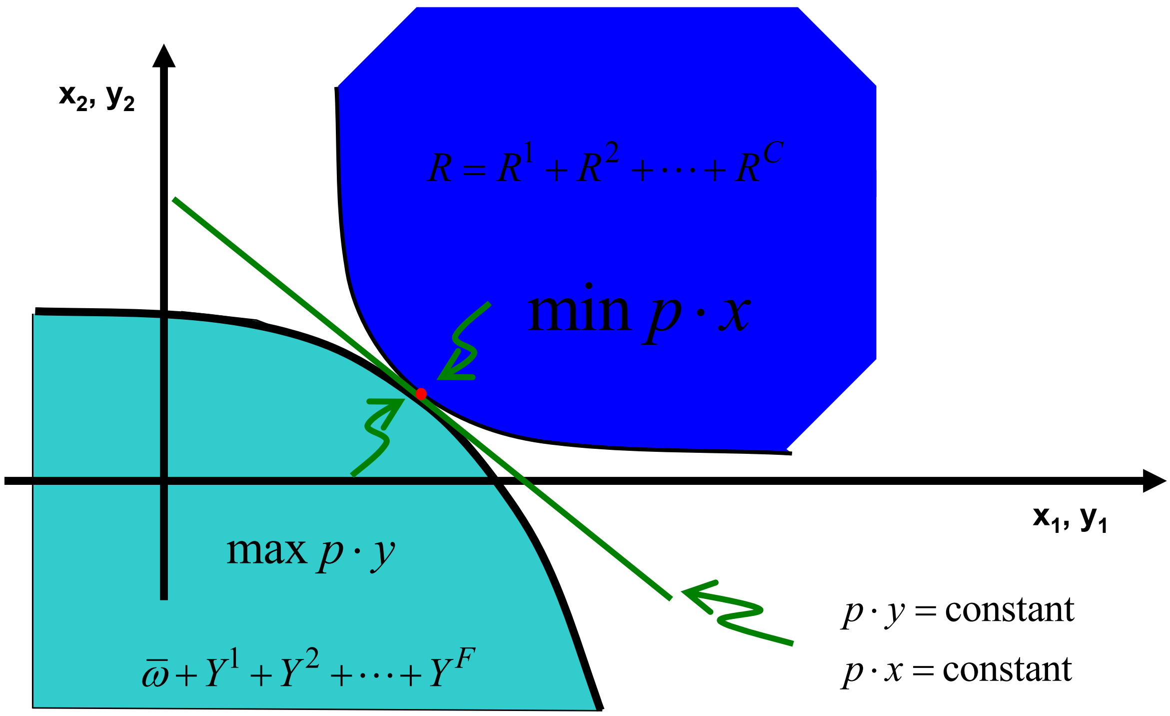
# FIRST FUNDAMENTAL WELFARE THEOREM

**Competitive equilibrium implies set of economy-wide feasible outputs is separated from aggregate preference set, the set of points that allow Pareto-dominant allocation (neither set includes interior point of other set). Therefore, the competitive market equilibrium must be a Pareto-optimal allocation**






# COMPETITIVE EQUILIBRIUM IS PARETO OPTIMAL



# FIRST FUNDAMENTAL WELFARE THEOREM

**Definition: Assume that consumer c's preferences are representable by a continuous utility function $u_c(\cdot)$. His preferences are locally nonsatiated *if for any feasible consumption vector* $x^c \in \Re_+^N$ *and any* $\varepsilon > 0$ *there exists another feasible consumption vector* $\hat{x}^c \in U_\varepsilon(x^c) = \{y \in \Re_+^N : \| y - x^c \| < \varepsilon\}$ *such that* $u_c(\hat{x}^c) > u_c(x^c)$.(1)**

**Theorem (1st FWT): Assume that for all consumers $c \in \{1,\dots,C\}$ the utility function is locally nonsatiated. If $(p, (\hat{x}^1,\dots,\hat{x}^C), (\hat{y}^1,\dots,\hat{y}^F))$ is a Walrasian equilibrium, then the allocation $((\hat{x}^1,\dots,\hat{x}^C), (\hat{y}^1,\dots,\hat{y}^F))$ is Pareto optimal.**

(1) *Local nonsatiation is implied by strict monotonicity of consumer c's utility function*. The converse is not true since some of the components of the consumption vector may not be desirable, i.e., it may contain "bads" instead of "goods". (However, it is not possible that all consumption goods are "bads," since then at 0 would become a (global) *satiation point*.)



# FIRST FUNDAMENTAL WELFARE THEOREM
## Proof

**Proof: [*by contradiction*]**

**Suppose that $((x^1,\dots,x^C),(y^1,\dots,y^F))$ is a feasible allocation, such that for all $c \in \{1,\dots,C\}$:**

$$u_c(x^c) \geq u_c(\hat{x}^c) \tag{1}$$

**and for some $c$, say $c = c'$, we have a strict inequality. Then, necessarily (by utility maximization), it is**

$$p \cdot x^{c'} > p \cdot \hat{x}^{c'} \tag{2}$$

**and local nonsatiation implies that as a consequence of (1), for all $c \in \{1,\dots,C\}$:(1)**

$$p \cdot x^c \geq p \cdot \hat{x}^c$$

**Hence, using (2),**

$$p \cdot \left( \sum_{c=1}^{C} x^c \right) = \sum_{c=1}^{C} (p \cdot x^c) > \sum_{c=1}^{C} (p \cdot \hat{x}^c) = p \cdot \left( \sum_{c=1}^{C} \hat{x}^c \right) \tag{3}$$

(1) Otherwise, any $\breve{x}^c \in \Re_+^N$ sufficiently close to $x^c$ must satisfy $p \cdot \breve{x}^c < p \cdot \hat{x}^c$. But by local nonsatiation, there must exist at least one such $\breve{x}^c$ for which also $u_c(\breve{x}^c) > u_c(x^c)$. By transitivity this implies $u_c(\breve{x}^c) > u_c(\hat{x}^c)$, which contradicts the assumption that $\hat{x}^{c'}$ solves the utility maximization problem as part of a Walrasian equilibrium. If some consumers were local satiated one may be able to transfer small amounts of money from consumers that are locally indifferent to a consumer that cares at the margin.

## FIRST FUNDAMENTAL WELFARE THEOREM
### Proof (cont'd)

**Feasibility of the WE (i.e., demand = supply) implies**

$$\sum_{c=1}^{C} x^c = \sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} y^f \qquad \textbf{(4)}$$

**and**

$$\sum_{c=1}^{C} \hat{x}^c = \sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} \hat{y}^f \qquad \textbf{(5)}$$

**Combining (3)—(5) we obtain**

$$p \cdot \left( \sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} y^f \right) > p \cdot \left( \sum_{c=1}^{C} \omega^c + \sum_{f=1}^{F} \hat{y}^f \right)$$

**whence**

$$p \cdot \left( \sum_{f=1}^{F} y^f \right) > p \cdot \left( \sum_{f=1}^{F} \hat{y}^f \right) \qquad \textbf{(6)}$$



## FIRST FUNDAMENTAL WELFARE THEOREM
### Proof (cont'd)

**Since the allocation $((x^1,\ldots,x^C),(y^1,\ldots,y^F))$ is by assumption feasible, we have that $y^f \in Y_f$ for all $f \in \{1,\ldots,F\}$. Profit maximization implies that for all $f \in \{1,\ldots,F\}$:**

$$p \cdot \hat{y}^f \geq p \cdot y^f$$

**But then it must be true that**

$$p \cdot \left( \sum_{f=1}^{F} \hat{y}^f \right) \geq p \cdot \left( \sum_{f=1}^{F} y^f \right)$$

**which contradicts (6). QED**

# SECOND FUNDAMENTAL WELFARE THEOREM

**The common point is the competitive equilibrium, since**

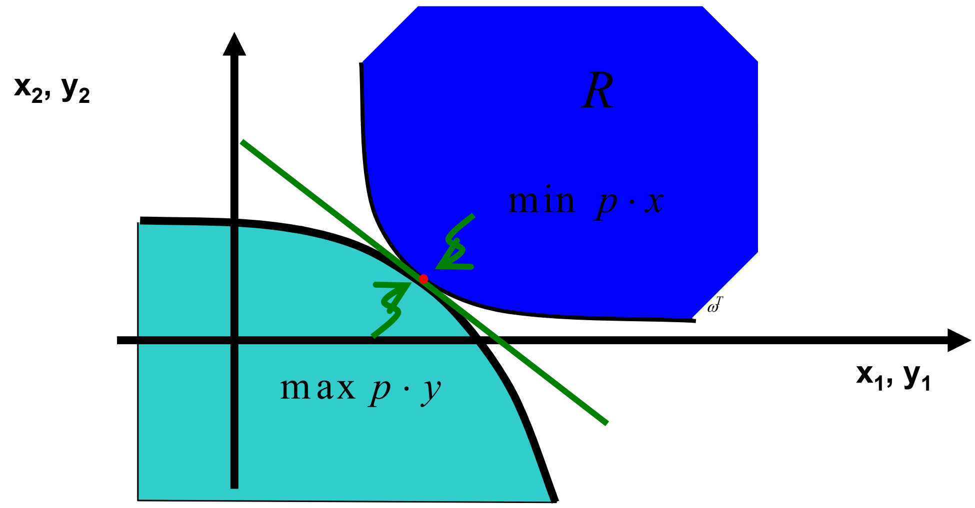
**(1) it minimizes expenditure,**

**(2) it maximizes profits,**

**(3) it has all supplies equal to all demands, and**

**(4) it has all profits allocated to consumers.**

**However, wealth is not necessarily consistent with initial endowments. Thus, a lump-sum wealth redistribution is likely to be required.**






# SECOND FUNDAMENTAL WELFARE THEOREM

**Theorem (2nd FWT): Assume that for all consumers $c \in \{1,...,C\}$ the utility function is locally nonsatiated, continuous, and has convex upper contour sets. Let $\overline{\omega} \in \Re_+^N$ be some vector of initial resources (endowments). (i) If, *starting from* $\overline{\omega}$ , the allocation $((\hat{x}^1,...,\hat{x}^C),(\hat{y}^1,...,\hat{y}^F))$ is Pareto optimal, then there exists a price vector $p \in \Re_+^N$ such that**

- **for all $c \in \{1,...,C\}$ :** $u_c(x^c) \geq u_c(\hat{x}^c) \Rightarrow p \cdot x^c \geq p \cdot \hat{x}^c$
- **for all $f \in \{1,...,F\}$:** $y^f \in Y_f \Rightarrow p \cdot \hat{y}^f \geq p \cdot y^f$

**(ii) If, *in addition*, for all $c \in \{1,...,C\}$ there exists a vector $\overline{x}_c \in \Re_+^N$ such that $p \cdot \hat{x}^c > p \cdot \overline{x}^c$ , then there is a division of initial resources $\overline{\omega}$, $(\omega^1,...,\omega^C)$ , and of firm ownership shares, $(\theta^1,...,\theta^C)$, such that $(p,(\hat{x}^1,...,\hat{x}^C),(\hat{y}^1,...,\hat{y}^F))$ is a Walrasian equilibrium relative to $(\omega^1,...,\omega^C)$ and $(\theta^1,...,\theta^C)$ .**

# SEPARATING HYPERPLANE THEOREM

**Definition**: **A plane** $P = \{x \in X : f(x) = 1\}$ **separates two sets** $A, B \subset X$**, if**

$$x \in A \Rightarrow f(x) \le 1$$
$$x \in B \Rightarrow f(x) \ge 1$$

**Hahn-Banach Theorem**: **Let** $A$ **and** $B$ **be *two disjoint nonempty convex sets in a vector space*** $X$**. If** $A$ **has an inner point, then there exists a plane** $P$ **separating** $A$ **and** $B$ **.**(1)

**Separating Hyperplane Theorem**: **Let** $A, B \subset \Re^N$ **be *two disjoint nonempty convex sets*. Then there exists a nonzero vector** $p \in \Re^N$ **and a scalar** $\alpha \in \Re$ **such that**

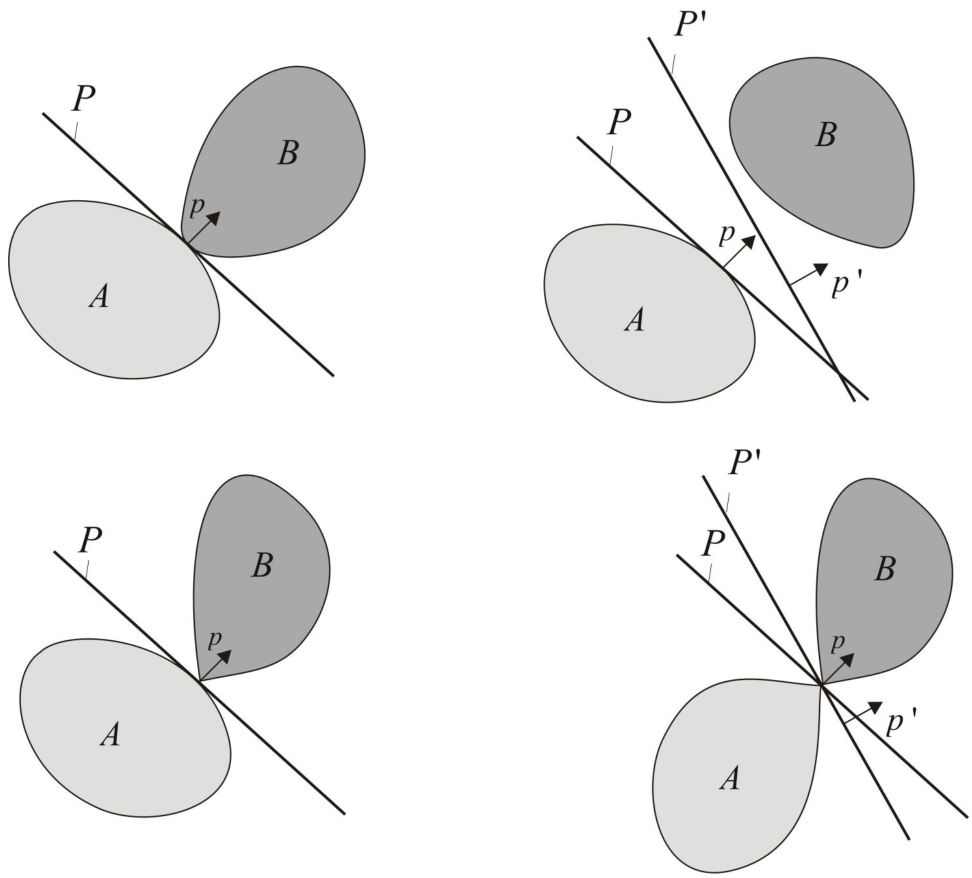
$$p \cdot x \le \alpha \le p \cdot y$$

**for any** $(x, y) \in A \times B$ **.**(2)

(1) For a proof of this theorem, see e.g., Berge, C. (1963), "Topological Spaces," Oliver & Boyd, Edinburgh and London, UK, pp. 154—157. Reprinted by Dover Publications in 1997.
(2) In other words, it is possible to select a *linear form* $f$ in the Hahn-Banach theorem. For a proof of that theorem, see MWG, p. 948.



# SEPARATING HYPERPLANE THEOREM
## Geometric Interpretation



# SECOND FUNDAMENTAL WELFARE THEOREM
## Proof

**Proof: [*proceeds in 7 steps*]**

***Step 1*: Apply the Separating Hyperplane Theorem**

**For all consumers $c \in \{1,...,C\}$, the set of preferred allocations (upper contour set),**

$$V^c(\hat{x}^c) = \{x^c \in \Re_+^N : u_c(x^c) > u_c(\hat{x}^c)\}$$

**is convex. As a result, $V = \sum_{c=1}^{C} V^c(\hat{x}^c)$ is convex. Similarly, convexity of the production set $Y_f$ for all $f \in \{1,...,F\}$ implies that**

$$Y = \sum_{f=1}^{F} Y_f + \{\overline{\omega}\}$$

**is convex. By assumption we know that the allocation $((\hat{x}^1,...,\hat{x}^C),(\hat{y}^1,...,\hat{y}^F))$ is Pareto optimal, i.e.,**

$$\left(\sum_{c=1}^{C} V^c(\hat{x}^c)\right) \cap \left(\sum_{f=1}^{F} Y_f + \{\overline{\omega}\}\right) = V \cap Y = \varnothing$$

**In other words, *there is nothing that the economy can produce that makes everybody better off*.**



# SECOND FUNDAMENTAL WELFARE THEOREM
## Proof (cont'd)

**The separating hyperplane theorem implies that for any $(x,y) \in V \times Y$ there exists a vector $p$ and a scalar $\alpha$, such that $p \cdot y \le \alpha \le p \cdot x$**

***Step 2*: Show that** $p \cdot \left(\sum_{f=1}^{F} \hat{y}^f + \overline{\omega}\right) = p \cdot \left(\sum_{c=1}^{C} \hat{x}^c\right) = \alpha$

**Since $((\hat{x}^1,...,\hat{x}^C),(\hat{y}^1,...,\hat{y}^F))$ is feasible, we have $\sum_{c=1}^{C} \hat{x}^c = \sum_{f=1}^{F} \hat{y}^f + \overline{\omega} \in Y$, so that by Step 1:**

$$\alpha \ge p \cdot \left(\sum_{f=1}^{F} \hat{y}^f + \overline{\omega}\right) = p \cdot \left(\sum_{c=1}^{C} \hat{x}^c\right)$$

**Now, for each $c \in \{1,...,C\}$ and $n \ge 1$, let**

$$\widehat{x}^c(n) = \hat{x}^c + \left(\frac{1}{n},...,\frac{1}{n}\right)$$

**By local nonsatiation(1) it is $\widehat{x}^c(n) \in V^c(\hat{x}^c)$ and thus $\sum_{c=1}^{C} \widehat{x}^c(n) \in V$. Hence by Step 1:**

$$p \cdot \left(\sum_{c=1}^{C} \widehat{x}^c(n)\right) \ge \alpha$$

(1) Actually we are using monotonicity here. For a justification see Step 3, where the same construction is used.

# SECOND FUNDAMENTAL WELFARE THEOREM
## Proof (cont'd)

**Taking the limit for** $n \to \infty$ **gives thus** $\alpha \le \lim_{n\to\infty} p \cdot \left( \sum_{c=1}^{C} \hat{x}^c(n) \right) = p \cdot \left( \sum_{c=1}^{C} \hat{x}^c \right) = p \cdot \left( \sum_{f=1}^{F} \hat{y}^f + \overline{\omega} \right)$

***Step 3*: Show that** $x \in \overline{V} \Rightarrow p \cdot \left( \sum_{c=1}^{C} \hat{x}^c \right) \le p \cdot x$, **where** $\overline{V} = \sum_{c=1}^{C} \underbrace{\{x^c \in \Re_+^N : u_c(x^c) \ge u_c(\hat{x}^c)\}}_{\overline{V}^c(\hat{x}^c)}$

**For simplicity, let us assume here that all the commodities are desirable, so that local nonsatiation is equivalent to monotonicity of the consumers' utility functions. For any** $x^c \in \overline{V}$ **let**

$$x^c(n) = x^c + \left( \frac{1}{n}, \ldots, \frac{1}{n} \right)$$

**so that by monotonicity,** $x^c(n) \in V^c(\hat{x}^c)$ **and** $\sum_{c=1}^{C} x^c(n) \in V$

**Hence, by Step 1,** $p \cdot \left( \sum_{c=1}^{C} x^c(n) \right) \ge \alpha$**, so that after taking the limit for** $n \to \infty$ **we obtain**

$$p \cdot \left( \sum_{c=1}^{C} \hat{x}^c \right) = \alpha \le \lim_{n\to\infty} p \cdot \left( \sum_{c=1}^{C} x^c(n) \right) = p \cdot \left( \sum_{c=1}^{C} x^c \right) = p \cdot x$$



# SECOND FUNDAMENTAL WELFARE THEOREM
## Proof (cont'd)

***Step 4*:** $y \in Y \Rightarrow p \cdot \left( \sum_{f=1}^{F} \hat{y}^f + \overline{\omega} \right) \ge p \cdot y$ **and thus(1)** $y^f \in Y_f \Rightarrow p \cdot \hat{y}^f \ge p \cdot y^f$

***Step 5*:** $x \in \overline{V} \Rightarrow p \cdot \left( \sum_{c=1}^{C} \hat{x}^c \right) \le p \cdot x$ **and thus(1)** $x^c \in \overline{V}^c(\hat{x}^c) \Rightarrow p \cdot \hat{x}^c \le p \cdot x^c$

***Step 6*: Show that: if for all consumers** $c \in \{1, \ldots, C\}$ **there exists a vector** $\overline{x}^c \in \Re_+^N$ **such that** $p \cdot \overline{x}^c < p \cdot \hat{x}^c$**, then** $u_c(x^c) > u_c(\hat{x}^c) \Rightarrow p \cdot \hat{x}^c < p \cdot x^c$ **.**

**By Step 5,** $p \cdot \overline{x}^c < p \cdot \hat{x}^c$ **implies that** $u_c(\hat{x}^c) > u_c(\overline{x}^c)$ **and (since** $u_c(x^c) > u_c(\hat{x}^c)$**) also** $p \cdot \hat{x}^c \le p \cdot x^c$ **. Thus,** $p \cdot \overline{x}^c < p \cdot (\beta x^c + (1-\beta)\overline{x}^c) < p \cdot x^c$ **for any** $\beta \in (0,1)$ **.**

**By the continuity of** $u_c(\cdot)$ **there is a** $\beta \in (0,1)$ **such that** $u_c(\hat{x}^c) = u_c\left(\beta x^c + (1-\beta)\overline{x}^c\right)$**.**

**But** $\beta x^c + (1-\beta)\overline{x}^c \in \overline{V}^c(\hat{x}^c)$ **, so that by Step 5** $p \cdot \hat{x}^c \le p \cdot (\beta x^c + (1-\beta)\overline{x}^c)$**, and thus** $p \cdot \hat{x}^c \le p \cdot (\beta x^c + (1-\beta)\overline{x}^c) < p \cdot x^c$ **as claimed.**

(1) Just consider the inequality for each consumer/producer individually by setting all other components to $\hat{x}^c$ or $\hat{y}^f$ , so that they cancel out.

# SECOND FUNDAMENTAL WELFARE THEOREM
## Proof (cont'd)

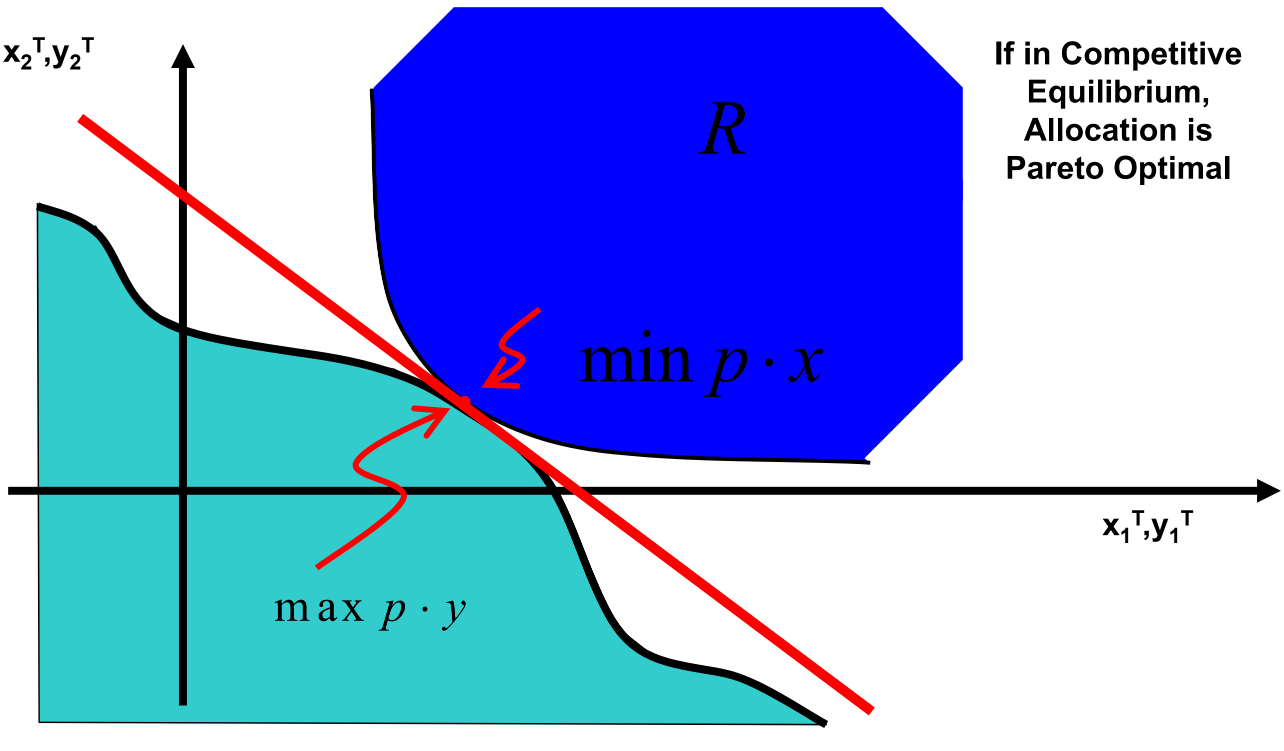
***Step 7***: It is now enough to choose a division of the initial endowment $\omega,(\omega^1,...,\omega^C)$, and of firm ownership shares, $(\theta^1,...,\theta^C)$, such that

$$p\cdot\hat{x}^c = p\cdot\omega^c + \sum_{f=1}^{F}\theta_f^c(p\cdot\hat{y}^f)$$

which completes our proof. QED

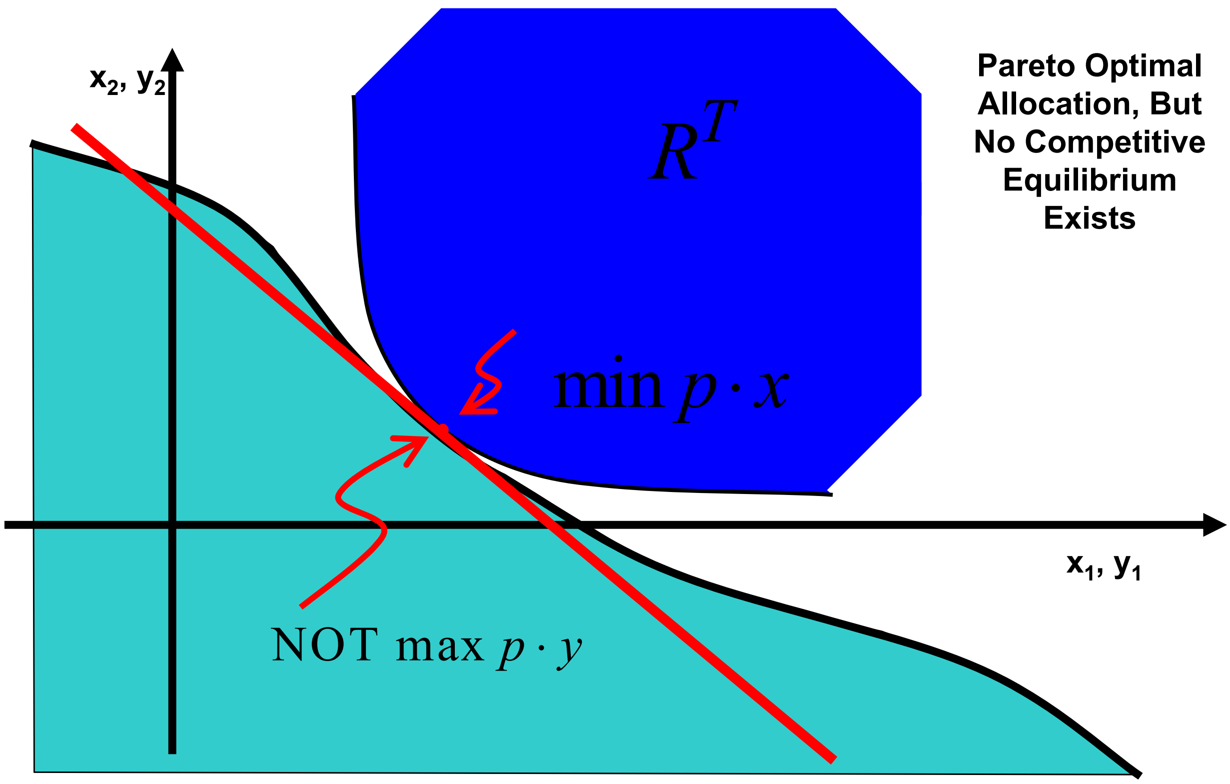


# NONCONVEXITY (1st FWC)




If in Competitive Equilibrium, Allocation is Pareto Optimal

## NONCONVEXITY (2nd FWC)




Pareto Optimal Allocation, But No Competitive Equilibrium Exists



## AGENDA

**Some Preliminaries**

**Fundamental Welfare Theorems**

**Existence of a Competitive Equilibrium**

**General Equilibrium vs. Partial Equilibrium**

**Key Concepts to Remember**

# HOMOGENEOUS FUNCTIONS

**Definition: A function $g:\Re_+^N \to \Re$ is homogeneous of degree $k$, if for any $\lambda > 0$ and $x \in \Re_+^N$ :**

$$g(\lambda x) = \lambda^k g(x)$$

**Examples:**

- **The supply function** $y^f(p) = \arg\max_{y \in Y_f} p \cdot y$

  ***is homogeneous of degree zero*. Indeed, for any $\lambda > 0$ we have that**

  $$y^f(p) = \arg\max_{y \in Y_f} p \cdot y = \arg\max_{y \in Y_f} \{(\lambda p) \cdot y\}$$

- **The profit function** $\pi^f(p) = \max_{y \in Y_f} p \cdot y$ **is also *homogeneous of degree one*, since**

  $$\lambda \pi^f(p) = \lambda \max_{y \in Y_f} p \cdot y = \max_{y \in Y_f} \{(\lambda p) \cdot y\} = \pi^f(\lambda p)$$



# EXCESS DEMAND

**Definition: The excess demand function for consumer $c \in \{1,\dots,C\}$ is**

$$z^c(p) = x^c(p, I^c) - \omega^c$$
$$= x^c(p, p \cdot \omega^c + \sum_{f=1}^{F} \theta_f^c \pi^f(p)) - \omega^c$$

**Summing up over all consumers and subtracting the firms' production, the function**

$$z(p) = \sum_{c=1}^{C} z^c(p) - \sum_{f=1}^{F} y^f(p)$$

**denotes excess market demand (also referred to *aggregate excess demand function*).**

**Exercise: Show that the excess demand function *and* the excess market demand are homogeneous of degree zero.**

# WALRAS' LAW

**Proposition (Walras' Law): For any price vector $p$ the value of excess market demand is zero, i.e.,**

$$\boxed{p \cdot z(p) = 0}$$

**Proof: Consumer c's budget constraint implies that**

$$p \cdot z^c(p) = p \cdot x^c(p, p \cdot \omega^c + \sum_{f=1}^{F} \theta_f^c \pi^f(p)) - p \cdot \omega^c = \sum_{f=1}^{F} \theta_f^c \pi^f(p)$$

**Adding up over all consumers and subtracting the firms' production yields**

$$p \cdot z(p) = p \cdot \left[ \sum_{c=1}^{C} z^c(p) - \sum_{f=1}^{F} y^f(p) \right] = \sum_{f=1}^{F} \sum_{c=1}^{C} \theta_f^c \pi^f(p) - \sum_{f=1}^{F} p \cdot y^f(p)$$

$$= \sum_{f=1}^{F} \pi^f(p) - \sum_{f=1}^{F} p \cdot y^f(p) = \sum_{f=1}^{F} p \cdot y^f(p) - \sum_{f=1}^{F} p \cdot y^f(p) = 0$$

**QED**



# EXISTENCE OF A WALRASIAN EQUILIBRIUM

**Proposition: Assume that the supply function $y^f(p)$ and the (finite) demand function**

$$x^c(p, p \cdot \omega^c + \sum_{f=1}^{F} \theta_f^c \pi^f(p)) < \infty$$

**exist for all $c \in \{1,\ldots,C\}$, $f \in \{1,\ldots,F\}$, and all $p \in \Delta = \{\hat{p} \in [0,1] : \sum_{i=1}^{N} \hat{p}_i = 1\}$ Suppose further that**

- **The production sets $Y_f$ are closed, bounded, and strictly convex**
- **The utility functions $u_c(\cdot)$ are continuous, locally nonsatiated, and with strictly convex upper contour sets $V^c(\cdot)$**

**Then there exists a price vector $p^* \in \Delta$ such that excess market demand is zero, i.e., $z(p^*) = 0$ (*this price supports a WE*)**

# EXISTENCE OF A WALRASIAN EQUILIBRIUM
## Proof

**Proof: The supply function $y^f(p)$ and the (finite) demand function**

$$x^c(p, p\cdot\omega^c + \sum_{f=1}^{F}\theta_f^c \pi^f(p))$$

**exist for all $c \in \{1,...,C\}$, $f \in \{1,...,F\}$, are unique as a consequence of the imposed convexity/concavity assumptions, and are continuous(1) in $p \in \Delta$. Hence the market excess demand function $z(p) = (z_1(p),...,z_n(p))$ is unique and continuous on $\Delta$.**

**Let us now define**

$$z_i^+(p) = \max\{z_i(p), 0\}$$

**and the corresponding vector**

$$z^+(p) = (z_1^+(p),...,z_n^+(p))$$

**Then the mapping $h: \Delta \to \Delta$ with**

$$h(p) = \frac{p + z^+(p)}{\sum_{i=1}^{n}(p_i + z_i^+(p))}$$

**is well-defined and continuous on $\Delta$.**

(1) This can be concluded e.g., from Berge's maximum theorem (cf. an earlier lecture).



# EXISTENCE OF A WALRASIAN EQUILIBRIUM
## Proof (cont'd)

**Brower's fixed-point theorem implies that the mapping h possesses a fixed point p* in $\Delta$, i.e.,**

$$h(p^*) = \frac{p^* + z^+(p^*)}{\sum_{i=1}^{n}(p_i^* + z_i^+(p^*))} = p^*$$

**Using Walras' Law we find**

$$0 = p^* z(p^*) = h(p^*) z(p^*) = \frac{\overbrace{p^* z(p^*)}^{=0} + z^+(p^*) z(p^*)}{\sum_{i=1}^{n}(p_i^* + z_i^+(p^*))} \overset{\sum_{i=1}^{n} p_i^* = 1}{=} \frac{z^+(p^*) z(p^*)}{1 + \sum_{i=1}^{n} z_i^+(p^*)}$$

**and therefore**

$$0 = z^+(p^*) z(p^*) = \sum_{i=1}^{n} z_i(p^*) \max\{0, z_i(p^*)\}$$

**which implies that**

$$z_i(p^*) \le 0$$

(1) This can be concluded e.g., from Berge's maximum theorem (cf. an earlier lecture).

# EXISTENCE OF A WALRASIAN EQUILIBRIUM
## Proof (Cont'd)

**In addition, since** $z^+(p^*) = p^* \sum_{i=1}^{N} z_i^+(p^*)$ **, good i can be in excess supply, i.e.,** $z(p^*) < 0$ **, only if it is worthless, that is to say only if** $p_i^* = 0$ **.**

**In particular,** $0 = p^* \cdot z(p^*) = p_1 z_1(p^*) + \cdots + p_N z_N(p^*)$ **implies that**

$$z_i(p^*) < 0 \;\Rightarrow\; p_i^* = 0$$

**One can also show that the fixed point p* needs to occur in the interior of the simplex Δ, (cf. MWG, p. 586) so that the excess demand must vanish in equilibrium,**

$$z(p^*) = 0$$

**QED**



# AGENDA

**Some Preliminaries**

**Fundamental Welfare Theorems**

**Existence of a Competitive Equilibrium**

**General Equilibrium vs. Partial Equilibrium**

**Key Concepts to Remember**

# GENERAL VS. PARTIAL EQUILIBRIUM ANALYSIS

**To see how General Equilibrium Theory can yield predictions that are radically different from Partial Equilibrium Theory, consider the following example.**

**Example: Tax Incidence**

**Consider an economy with N cities (where N is a large number).**

- **In each city there is a single price-taking firm that produces a single consumption good using the increasing, *strictly concave production function*** $f(\cdot)$
- **There are M identical workers. *Each worker is free to move between cities to be paid the highest wage*.**
- ***Each worker derives utility from the single consumption good that is available*. Without loss of generality the price of the consumption good can be normalized to 1.**

**Question: If a tax on labor is levied in city 1, who bears the cost (firms or workers)?**



# GENERAL VS. PARTIAL EQUILIBRIUM (cont'd)

**Analysis I (Partial Equilibrium) – Consider only City 1**

- ***Before the tax is introduced*, given that workers can move freely, wages must be equal in each city, i.e.,**

$$w_1 = \cdots = w_N = \overline{w} = f'(M/N)$$

**which yields each firm's equilibrium profit,** $\overline{\pi} = f(M/N) - \overline{w}(M/N)$

- **The supply of workers in city 1 must be completely elastic, and thus the equilibrium wage *after the tax*** $t \geq 0$ ***is introduced* must still be equal to** $\overline{w}$
- **Hence, we find that *in city 1*, output drops to**

$$\pi_1 = f(L_1) - (\overline{w} + t)L_1 < \overline{\pi}$$

**where the labor used in city 1,** $L_1$**, is such that** $f'(L_1) = \overline{w} + t$

- **As a result, since** $L_1 < M/N$**, *some labor moves away from city 1*, but all the tax is borne by producers!**

# GENERAL VS. PARTIAL EQUILIBRIUM (cont'd)

**Analysis II (General Equilibrium)**

- ***Before the tax is introduced*, we obtain the same analysis as before.**
- **Let $w(t) = w_1 = \cdots = w_N$ be the common equilibrium wage in all cities *after the tax* $t \geq 0$ *is introduced***
- **Demand = Supply yields $(N-1)L(t) + L_1(t) = M$, where $L(t)$ is the equilibrium labor demand in cities 2,…,N, and $L_1(t)$ is the equilibrium labor demand in city 1**
- **Profit maximization yields $f'(L_1(t)) = w(t) + t$ and $f'(L(t)) = w(t)$**
- **Using the boundary condition for $t = 0$, when $L(0) = L_1(0) = M/N$, we find by differentiating the optimality conditions and evaluating at $t = 0$:**

$$f''(L_1(0))L_1'(0) = -f''(M/N)(N-1)L'(0) = w'(0) + 1$$

$$f''(M/N)L'(0) = w'(0)$$

**so that $w'(0) = -1/N$. In other words, the wage rate in all cities declines with an imposition of a tax on labor**



# GENERAL VS. PARTIAL EQUILIBRIUM (cont'd)

- **Let us now consider the change of firm profits, which again can be done by differentiating profits with respect to t and evaluating at t=0:(1)**

$$(N-1)\overline{\pi}'(\overline{w})w'(0) + \overline{\pi}'(\overline{w})(w'(0)+1) = \overline{\pi}'(\overline{w})\left(-\frac{N-1}{N} + \frac{N-1}{N}\right) = 0$$

**In other words, aggregate profits are very little affected (and in the limit unaffected) by a (small) tax.**

- **We therefore find that (at least for small taxes) virtually all of the tax in city 1 is incurred by the workers, which is the *opposite* conclusion of what we obtained using partial equilibrium analysis!**

(1) A more detailed proof of this statement can be given as follows. Let $S(t) = (f(z_1) - (w+t)z_1) + (N-1)(f(z) - wz) = f(z_1) + (N-1)f(z) - w(z_1 + (N-1)z) - tz_1$ be the sum of the firms' profits at the tax rate t. Note that (supply = demand) implies that $z_1 + (N-1)z = M$, so that $S(t) = f(z_1) + (N-1)f(z) - wM - tz_1$. We obtained earlier that $w'(0) = -1/N$, and $z_1(0) = z(0) = M/N$. We can therefore differentiate $S(t)$ with respect to $t$ and evaluate at $t = 0$:

$$S'(0) = f'(M/N)(z_1'(0) + (N-1)z'(0)) - w'(0)M - z_1(0) = f'(M/N)\left.\frac{d}{dt}\right|_{t=0}(z_1(t) + (N-1)z(t)) - \left(-(M/N) + (M/N)\right) = f'(M/N)(M)' - 0 = 0$$

The significance of this result is that small deviations from zero in the tax level have an arbitrarily small impact on aggregate profit, whereas the impact on the workers' utilities has a strictly positive slope. In other words, in the limit a very small positive tax is borne entirely by the workers.

# AGENDA

**Some Preliminaries**

**Fundamental Welfare Theorems**

**Existence of a Competitive Equilibrium**

**General Equilibrium vs. Partial Equilibrium**

**Key Concepts to Remember**



# KEY CONCEPTS TO REMEMBER

- **Set Summation**
- **Walrasian Equilibrium (Competitive Equilibrium) (w/ or w/o transfers)**
- **Fundamental Welfare Theorems**
- **Separating Hyperplane Theorem**
- **Walras' Law**
- **General Equilibrium vs. Partial Equilibrium**

# MGT 621 – MICROECONOMICS

## Thomas A. Weber

## *9. Markets and Intermediaries*

## Autumn 2023

## École Polytechnique Fédérale de Lausanne
## College of Management of Technology



# AGENDA

**What is an intermediary?**

**Intermediary's Value Proposition**

**Key Concepts to Remember**

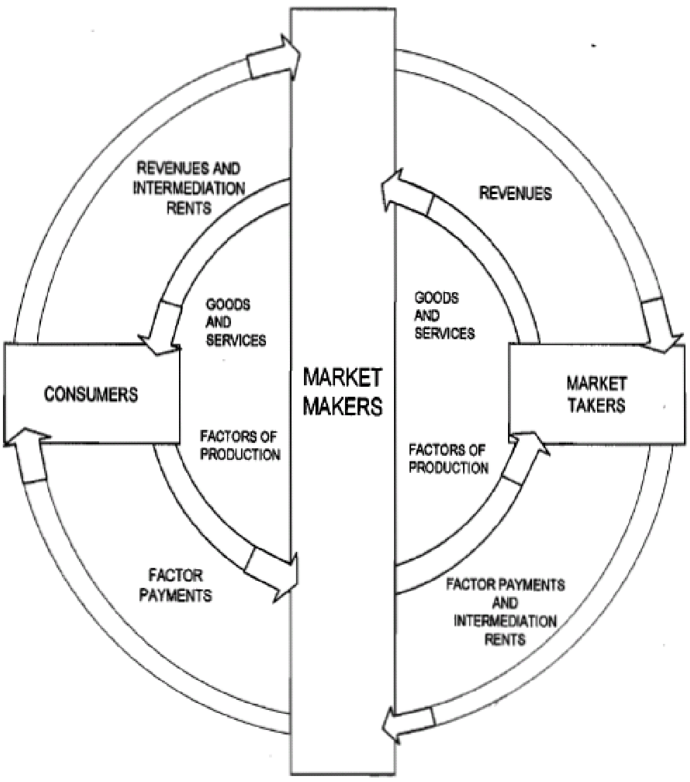
# WHAT IS AN INTERMEDIARY

**Definition. An intermediary offers intermediation services between two trading parties by acting as a conduit for goods and services offered by a supplier to a consumer. Typically the intermediary offers an added value to the transaction that is not available in a direct exchange between the two trading parties.**



# INTERMEDIARIES ARE MARKET MAKERS
## … and Create Two-Sided Markets …

**Consumption Side**




**Supply Side**

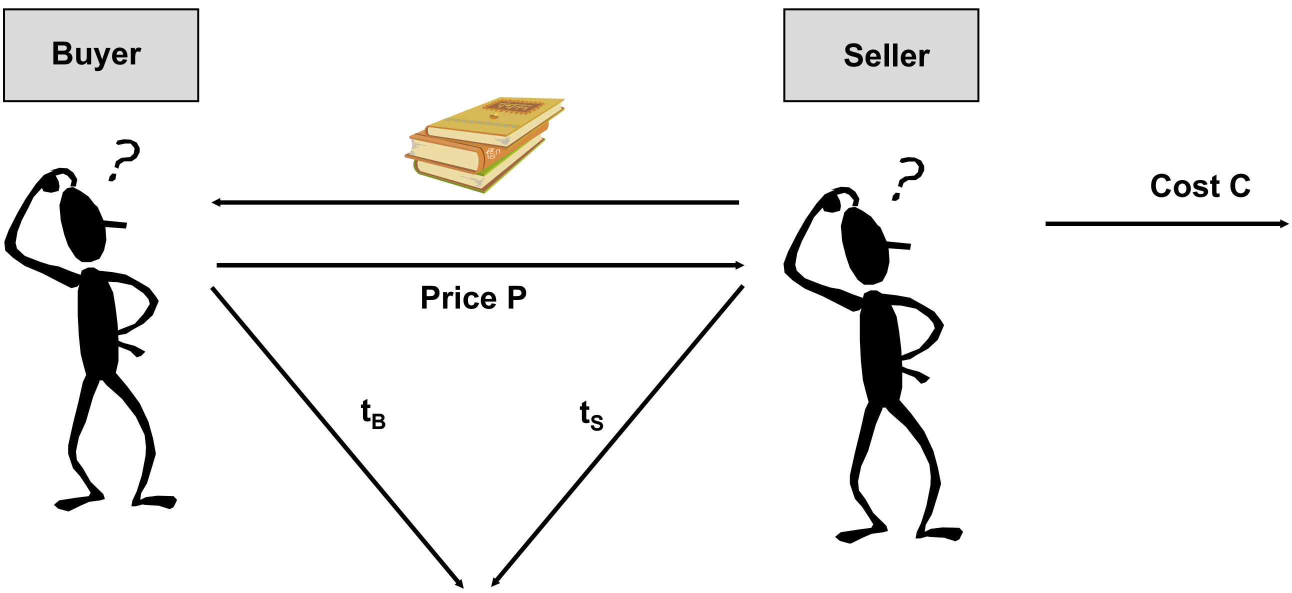
Source: Spulber (1999)

# EXAMPLES OF INTERMEDIARIES

**There are plenty …**



# DIRECT EXCHANGE




**Transaction Cost** $T = t_B + t_S$

$U_B = V - t_B - P$

$U_S = P - t_S - C$

**Direct exchange takes place** $\Leftrightarrow \quad V - C \geq T$

# AGENDA

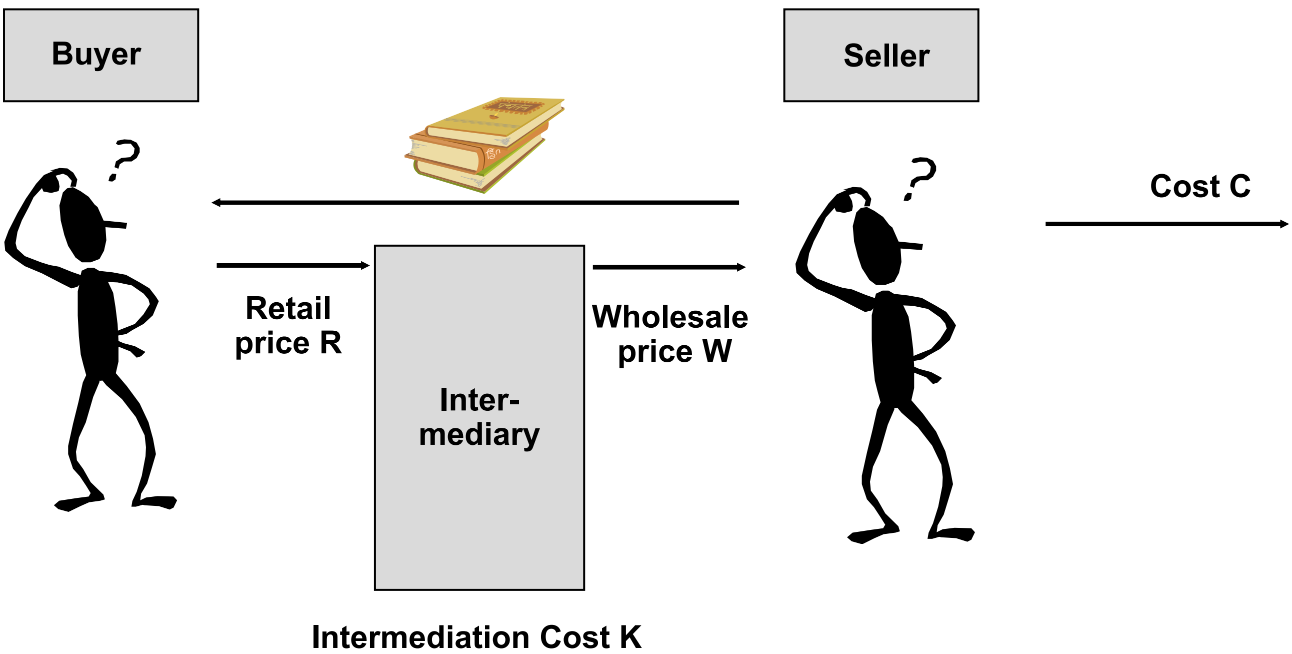
**What is an intermediary?**

**Intermediary's Value Proposition**

**Key Concepts to Remember**



# INTERMEDIATED EXCHANGE




$U_B = V - R$

$U_S = W - C$

**Intermediated exchange takes place** $\Leftrightarrow$ $V - C \geq K$ **and** $T \geq K$

# REASONS FOR LOWER TRANSACTION COSTS

- Intermediary trades larger volume → Economies of scale
- Commitment power →Intermediary can guarantee prices
- Longevity of Intermediary → Reputation
- Information aggregation → Intermediary knows more
- Inventory → Intermediary can achieve immediacy by keeping an inventory



# MONOPOLISTIC INTERMEDIARY

The profit-maximizing prices (R,W) satisfy

$V - R = (V - C - T)/2 = W - C$

The intermediary's "markup" is therefore

$R - W = T.$

Hence, the intermediary is viable if and only if

$K \leq T.$

# PRICE COMPETITION BETWEEN INTERMEDIARIES

**"Bertrand Price Competition"**

**Assume two intermediaries have identical intermediation cost K. Then in a simultaneous-move price-setting game their gains are dissipated fully since undercutting the opponent is a dominant strategy, as long as payoffs are positive.**

**Hence, at the unique Nash equilibrium, both intermediaries charge (P,W) such that**

$$V - P = (V - C - K)/2 = W - C$$

**The intermediaries' equilibrium payoffs are zero, while the seller's and the buyer's payoffs are (V – C – K)/2, respectively.**

**Intermediaries can enable social gains by lowering transaction cost.**



# TRANSACTION COST DECREASE: IMMEDIACY

**Assume that the buyer and seller are equally sensitive to the time value of money and have a common per-period discount factor $\delta \in (0,1)$. When conducting a direct exchange, the gains from trade V – C are realized in the following period, so that the surplus to be divided between the trading parties becomes**

$$S = \delta\,(V - C).$$

**An intermediary, e.g., by keeping an inventory of the items to be traded, can provide immediacy of the exchange.**

**It is viable if and only if the intermediation cost K is such that**

$$K \leq (1 - \delta)(V - C)$$

## TRANSACTION COST DECREASE: ELIMINATION OF SEARCH

**Assume that the buyer and seller have a probability $\beta \in (0,1)$ of meeting in a direct exchange. Then with probability $(1 – \beta)$ no exchange takes place, resulting in a transaction cost**

**$T = (1 – \beta)(V – C)$**

**Hence, an well-known intermediary that provides a trading platform can be viable if the intermediation cost K is such that**

**$K \leq (1 – \beta)(V – C)$**

**The lower the probability of matching between buyers and sellers, the higher the likelihood that an intermediary emerges.**



## INTERMEDIARY CAN ENABLE TRADE WHEN MARKETS FAIL

**Consider a buyer whose value for an item is either high ($V_H$) or low ($V_L$), with equal probability (where $V_H > V_L > 0$), so that in expectation**

**$V = (V_H + V_L)/2.$**

**Suppose further that a seller has either high opportunity cost ($C_H$) or low opportunity cost ($C_L$), with equal probability (where $C_H > C_L > 0$), so that in expectation**

**$C = (C_H + C_L)/2.$**

**Assume that $V_H > C_H > V_L > C_L$ . After meeting and learning each other's type they decide to transact or not. Thus, with probability 1/4 there is no trade.**
**The expected gains from direct transaction are therefore**

- **Buyer L: $(V_L – C_L)/4$**
- **Buyer H: $(V_H – C)/2$**
- **Seller L: $(V – C_L)/2$**
- **Seller H: $(V_H – C_H)/4$**

**Total ex-ante expected surplus:**

**$V_H – C_L – (C_H – V_L)/2 \quad (< V_H – C_L)$**

# ENABLE TRADE WHEN MARKETS FAIL (Cont'd)

**If a monopolist intermediary offers prices $R = V_H - (V_H - C)/2$ and $W = C_L + (V - C_L)/2$, then buyer H's and seller L's expected surplus from using the intermediary are equal to the expected surplus from direct exchange, since**

$$V_H - R = V_H - V_H + (V_H - C)/2 = (V_H - C)/2$$

**and**

$$W - C_L = C_L + (V - C_L)/2 - C_L = (V - C_L)/2.$$

**However, buyer L's and seller H's surplus are negative, preventing them from using the intermediary. Hence, they will be inactive in equilibrium, while the intermediary is viable if its intermediation cost K is such that**

$$R - W = V_H - (V_H - C)/2 - C_L - (V - C_L)/2 = (V_H - V_L + C_H - C_L)/4 \geq K.$$

**Thus, an intermediary may produce a separating equilibrium in a market that has a positive probability of failing if intermediation costs are low enough.**



# INTERMEDIARY MAY ALLEVIATE ADVERSE SELECTION

**Consider a seller whose product is of either high (H) or low quality (L). The seller's opportunity cost increases with the quality of the good supplied, $C_H > C_L > 0$. The buyer's willingness to pay is increasing in the product quality, $V_H > V_L > 0$.**

**Let $\lambda \in (0,1)$ be the probability that the good is a "lemon", i.e., is of low quality, such that the following "lemons condition" is satisfied:**

$$V = \lambda V_L + (1 - \lambda) V_H < C_H.$$

**Hence, if the buyer cannot distinguish between the two product qualities in equilibrium, the high-quality seller will leave the market, as the buyer's willingness to pay does not cover the cost of providing the good.**

**Therefore, the lemons condition implies that only lemons are directly exchanged in the market. The payoff for the remaining buyer and seller type (L,L) is $(V_L - C_L)/2$. Trade occurs with probability $\lambda$.**

# ALLEVIATE ADVERSE SELECTION (Cont'd)

**Assume that a trusted intermediary is able to observe the quality of the seller's product at a cost K and then to communicate that information to the buyer.**

$V = \lambda V_L + (1 - \lambda) V_H < C_H$.

**The intermediary can then make the prices (R,W) contingent on the observed quality (L or H). The optimal intermediation prices are such that buyer's and sellers are just as well off as under direct exchange, so that**

$(R_H,W_H) = (V_H,C_H)$

**and**

$(R_L,W_L)$: $W_L - C_L = (V_L - C_L)/2$ **and** $V_L - R_L = \lambda (V_L - C_L)/2$.

**The intermediary is viable if** $(1 - \lambda)(V_H - C_H) + \lambda(V_L - (1 + \lambda)(V_L - C_L)/2 - C_L) \geq K$.



# INTERMEDIARIES CAN MITIGATE MORAL HAZARD

**Suppose that a buyer can enhance the default value $V_L$ obtained from a certain good or service to $V_H$ by making a relationship-specific investment I. This investment is non-contractable. The seller's cost is $C < V_L$, and the surplus is evenly divided such that both parties obtain $(V_i - C)/2$ for i = L,H.**

**Assume that the required investment is "substantial", i.e.,** $V_H - V_L > I > (V_H - V_L)/2$.

**Then the buyer will not find it worthwhile to make the relationship-specific investment, since**

$(V_H - C)/2 - I < (V_L - C)/2$.

## MITIGATE MORAL HAZARD (Cont'd)

**An intermediary can set prices (R,W) such that**

**$V_H - I - R = (V_L - C)/2 = W - C$.**

**The intermediary is viable if**

**$R - W = V_H - V_L - I \geq K$.**



## INTERMEDIARIES AND TRANSACTION COST
### Summary

- **Provide Immediacy**
- **Reduce Search & Matching Cost**
- **Enable Trade when Markets Fail (Bilateral Asymmetric Information)**
- **Alleviate Adverse Selection**
- **Mitigate Moral Hazard**

**… and reduce "coordination problems"**

# CONCLUSION

**Public-Policy Implications with respect to Intermediaries**

- **Should the government encourage the entrance of intermediaries?**
- **Should the government act as an intermediary?**
- **Should the government encourage the competition of intermediaries?**



# AGENDA

**What is an intermediary?**

**Intermediary's Value Proposition**

**Key Concepts to Remember**

# KEY CONCEPTS TO REMEMBER

- **Intermediary / Market Maker / Two-Sided Market**
- **Transaction Cost**
- **Intermediation Cost**
- **Immediacy**
- **Search Cost**
- **Market Failure**
- **Adverse Selection**
- **Moral Hazard**