École Polytechnique Fédérale de Lausanne

College of Management of Technology

MGT-621 Microeconomics (Prof. Weber)

Final Exam – Solutions

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Problem 1. (35 Points)

(i) Suppose Claude is offered one share of the company at the price p. Claude believes that the random share price $\tilde{x}$ can be described as a binary money lottery of the form

$$\tilde{x} = [0.5, 90; 0.5, 120].$$

He accepts the offer if and only if his expected utility of his random ex-post wealth $\tilde{x} - p$ three months from now weakly exceeds the utility of his current wealth, i.e., if and only if

$$EU[\tilde{x}-p] = \left(u(y+90-p) + u(y+120-p)\right)/2 = 1-e^{-\rho(y-p)}\left(e^{-90\rho} + e^{-120\rho}\right)/2 \geq 1-e^{-\rho y} = u(y).$$

This inequality is satisfied if and only if

$$p \leq \frac{\ln 2 - \ln\left(e^{-90\rho} + e^{-120\rho}\right)}{\rho} \equiv p_b,$$

so that p_b is Claude's willingness to pay for a share. Note that p_b is independent of y because Claude has constant absolute risk aversion, i.e., his risk attitude is unaffected by his wealth.

(ii) If Claude buys q shares at the price of $p_0 = 100$ per share, his expected utility is

$$EU\left[q\left(\tilde{x} - p_0\right)\right] = \left(u(y-10q) + u(y+20q)\right)/2 = 1 - e^{-\rho y}\left(e^{10\rho q} + e^{-20\rho q}\right).$$

Ignoring the budget constraint (i.e., that $qp_0 \in [0, y]$) for a moment, we obtain the first-order necessary optimality condition

$$-5\rho e^{10\rho q} + 10\rho e^{-20\rho q} = 0,$$

so that

$$q = \frac{\ln 2}{30\rho}.$$

Introducing the budget constraint, we find that Claude's optimal number of shares to buy (at the price $p_0 = 100$) is

$$q^*(y) = \min\left\{\frac{\ln 2}{30\rho}, \frac{y}{100}\right\}.$$

(iii) Given that Claude has bought z shares at the price $p_0 = 100$, his expected utility is (cf. part (ii))

$$EU\left[z\left(\tilde{x} - p_0\right)\right] = 1 - e^{-\rho y}\left(e^{10\rho z} + e^{-20\rho z}\right)/2.$$

Claude is willing to sell the z shares at a price p per share if and only if

$$EU\left[z\left(\tilde{x} - p_0\right)\right] \leq u(y - (p_0 - p)z).$$

This inequality is equivalent to

$$-\left(e^{10\rho z} + e^{-20\rho z}\right)/2 \leq -e^{\rho(p_0 - p)z},$$

so that

$$p \geq 100 - \frac{\ln\left(\frac{e^{10\rho z} + e^{-20\rho z}}{2}\right)}{\rho z} \equiv p_s(z).$$

The minimum price per share $p_s(z)$ that Claude would accept to sell all of his shares depends on the number of shares z Claude possesses. For example, for $z = 0$ we have that $p_s(0) = p_0 = 100$ (of course Claude has nothing to sell in this case). If Claude bought the optimal quantity $z = q^*(y)$ of shares (cf. part (ii)), then his willingness to accept becomes

$$p_s(q^*(y)) = 100 - \frac{\ln\left(\min\left\{\frac{3\sqrt[3]{2}}{4}, \frac{e^{\rho y/10} + e^{-\rho y/5}}{2}\right\}\right)}{\min\left\{(\ln 2)/30, y\rho/100\right\}} > 100.$$

Note that if Claude invested his entire wealth into shares, then his minimum selling price (willingness to accept) per share depends on the magnitude of his former wealth (i.e., on the number of shares bought, as obtained earlier). It is also interesting to note that if at the purchase date Claude's budget constraint was not binding, then his willingness to accept is independent of his risk aversion parameter. Hence, we obtain the interesting conclusion that if different CARA investors have the same beliefs over the returns of an asset and each buys an optimal amount without hitting the budget constraint, then they all are willing to sell their portfolio at the same price per share, even though the sizes of their optimal holdings and their CARA parameters might vary across individuals.

(iv) At a price p for the call option on the company's stock, Claude's expected utility of his ex-post wealth becomes

$$EU\left[\max\{\tilde{x} - s, 0\} - p\right] = 1 - e^{-\rho(y-p)}\left(1 + e^{-15\rho}\right)/2.$$

As above, by comparing this expected utility with Claude's status-quo utility $u(y)$, we obtain his willingness to pay for the call option of

$$p_c = \frac{1}{\rho}\ln\left(\frac{2}{1 + \exp(-15\rho)}\right).$$

(v) After the change in the payoff distribution, the random variable associated with the share price in three months becomes $\tilde{y} = [0.5, 70; 0.5, 140]$. It can be obtained by a mean-preserving spread from the original lottery $\tilde{x} = [0.5, 90; 0.5, 120]$, so that $\tilde{x}$ second-order stochastically dominates $\tilde{y}$, i.e.,

$$\tilde{y} \preceq_{\text{SOSD}} \tilde{x}.$$

Claude's willingness to pay for the new share is

$$\hat{p}_b = \frac{\ln 2 - \ln\left(e^{-70\rho} + e^{-140\rho}\right)}{\rho} < p_b;$$

the increase in risk reduces the stock's value for Claude. He is risk-averse after all. Yet, despite his risk aversion, the price $\hat{p}_c$ of a call option with identical strike price $s = 105$ on the new stock increases, since

$$\hat{p}_c = \frac{1}{\rho} \ln\left(\frac{2}{1 + \exp(-35\rho)}\right) > p_c.$$

The reason for this increase in the value of the call option is that the holder cannot experience negative payoffs and therefore appreciates any increase in the volatility of the the underlying stock.

Problem 2. (25 Points)

(i) The production possibilities set Y is depicted in Figure 1.

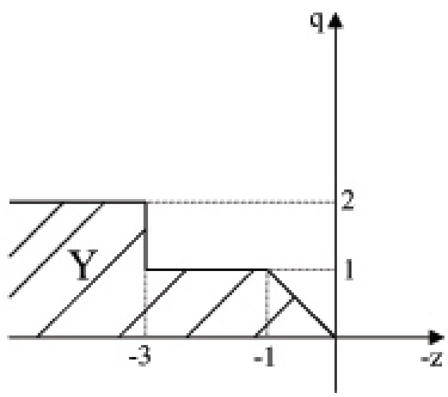

Figure 1: The production possibilities set Y.

(ii) Which of the following statements about the firm's production possibilities set are correct? Explain.

1. Y is closed.
   Answer: True. Any convergent sequence in Y has a limit in Y.
2. Y exhibits the no-free-lunch property.
   Answer: True. $Y \cap \mathbb{R}^2_+ = \{0\}$
3. Y allows for the possibility of inaction.
   Answer: True. $0 \in Y$
4. Y allows for free disposal.
   Answer: False. $Y - \mathbb{R}^2_+ \nsubseteq Y$.
5. The firm's production is irreversible.
   Answer: True. If $y \in Y$ and $y \neq 0$, then $-y \notin Y$.
6. Y exhibits nonincreasing/nondecreasing/constant returns to scale.
   Answer: Neither nonincreasing nor nondecreasing returns to scale holds. As a result, constant returns to scale does not hold. Let $y = (-3, 2)$ and $\alpha = \frac{2}{3}$. Then, $\alpha y = (-2, \frac{4}{3}) \notin Y$. If $\alpha = \frac{4}{3}$, then $\alpha y = (-4, \frac{8}{3}) \notin Y$.

7. Y is additive.
   Answer: False. Let $y_1 = (-1, 1)$ and $y_2 = (-3, 2)$. Then, $y_1 + y_2 = (-4, 3) \notin Y$.
8. Y is convex.
   Answer: False. Let $y_1 = (-1, 1)$ and $y_2 = (-3, 2)$. Then, for any $\alpha \in (0, 1)$, $\alpha y_1 + (1-\alpha) y_2 \notin Y$.

(iii) With w being the unit price of the firm's input, its cost function $C(q)$ is

$$C(q) = \begin{cases} wq, & \text{if } 0 \leq q \leq 1, \\ 3w, & \text{if } 1 < q \leq 2, \\ \infty, & \text{if } q > 2. \end{cases}$$

As an example, we can think of a small business such as home-made food service or a small restaurant that needs some capital investment to expand its business significantly. The restriction that the firm cannot produce more than 2 units could be the result of feasibility constraints or of regulation limiting the size of businesses (e.g., due to zoning laws).

(iv) The firm's average cost $AC(q)$ is nonmonotonic on the interval $(0, 2)$,

$$AC(q) = \begin{cases} w, & \text{if } 0 < q \leq 1, \\ \frac{3w}{q}, & \text{if } 1 < q \leq 2, \\ \infty & \text{if } q > 2, \end{cases}$$

More specifically, $AC(q)$ is constant on $(0, 1)$ and decreasing on $(1, 2)$. Note that as q increases from 1^- to 1^+ the average cost jumps up.

(v) The firm's profit maximization problem is

$$\begin{array}{rl} \max & \{wz + pq\} \\ \text{s.t.} & (-z, q) \in Y \end{array}$$

We let $-wz + pq = k$ for some profit level $\pi > 0$ and try to find a point $y = (-z, q) \in Y$ that maximizes π. That is, π is maximized when the line $q = \frac{k}{p} - \frac{w}{p} z$ meets the set Y in figure 1 with the largest q-intercept.

$$y^*(w, p) \in \begin{cases} \{(-3, 2)\}, & \text{if } 0 < \frac{w}{p} < \frac{1}{2}, \\ \{(-1, 1), (-3, 2)\}, & \text{if } \frac{w}{p} = \frac{1}{2}, \\ \{(-1, 1)\}, & \text{if } \frac{1}{2} < \frac{w}{p} < 1, \\ \{(-z, q) : z = q, 0 \leq q \leq 1\}, & \text{if } \frac{w}{p} = 1, \\ \{(0, 0)\}, & \text{if } \frac{w}{p} > 1. \end{cases}$$

Problem 3. (35 Points)

Part I. (i) Let $j \in \{A, B\}$. Firm j's payoff is given by

$$\pi_j(p_j, p_{-j}) = p_j(24 - 2p_j + p_{-j}).$$

(ii) Maximimizing with respect to p_j, we obtain firm j's best response from the first-order necessary optimality condition,

$$p_j^*(p_{-j}) = 6 + \frac{p_{-j}}{4}.$$

(iii) Since $p_j^*(p_{-j})$ is increasing in p_{-j}, the firms' actions (p_j, p_{-j}) are *strategic complements.* (iv) Firm j's profit has increasing differences in (p_j, p_{-j}), since

$$\frac{\partial^2 \pi_j}{\partial p_j \partial p_{-j}} = 1 > 0.$$

As a result, the game is supermodular. (v) Assuming a symmetric solution where $p^* = p_A^* = p_B^*$, the firms' best responses intersect exactly once at p^*, where p^* solves $p_A^*(p) = 6 + p/4 = p$. The symmetric NE strategies are therefore $p_A^* = p_B^* = 8$. The corresponding equilibrium payoffs are $\pi_A^* = \pi_B^* = 128$.

Part II. (vi) At time $t = 1$, firm B's best response has already been computed: $p_B^*(p_A) = 6 + p_A/4$. At time 0, firm A therefore solves

$$p_A^* = \arg\max_{p_A \geq 0} \left\{ p_A(24 - 2p_A + 6 + \frac{p_A}{4}) \right\} = \frac{60}{7}.$$

Correspondingly, firm B's price is given by $p_B^* = 57/7$, and the firms' equilibrium profits are consequently $\pi_A^* = 128.57$ and $\pi_B^* = 132.6$ respectively. We see that firm A's profit is lower than firm B's profit, even though firm A is the first to move. This is surprising: one might imagine that moving first is an advantage as in a Cournot duopoly. However, in this Bertrand duopoly context, firm B benefits from delaying commitment in which price to choose.

Part III. (vii) Since there is no advantage to move first, both firms have no incentive whatsoever to invest and win the R&D race. At the unique NE both firms spend zero.

Problem 4. (25 Points)

(i) From the firm's profit maximization, we obtain directly that

$$\lambda L_1 = L_2. \tag{1}$$

The agents' budget constraints can be written as,

$$\begin{aligned} x_{1s} + w_s(1 - L_{1s}) &= w_s, \\ x_{2s} + w_s(1 - L_{2s}) &= w_s + \pi_s, \end{aligned}$$

or equivalently

$$x_{1s} = w_s L_{1s}, \tag{2}$$

$$x_{2s} = w_s L_{2s} + \pi_s. \tag{3}$$

Taking into account his budget constraint (2), agent 1's utility maximization problem yields

$$L_{1s} = \arg\max_{L_1 \in [0,1]} \{\alpha_{1s} \log(w_s L_1) + \log(1 - L_1)\} = \left(1 + \frac{1}{\alpha_{1s}}\right)^{-1} = \frac{1}{2}. \tag{4}$$

Similarly, taking into account budget constraint (3), maximizing agent 2's utility yields

$$\begin{aligned} L_{2s} &= \arg\max_{L_2 \in [0,1]} \{\alpha_{2s} \log(w_s L_2 + \pi_s) + \log(1 - L_2)\} = \left(1 + \frac{1}{\alpha_{2s}}\right)^{-1} \left(1 - \frac{\pi_s}{\alpha_{2s} w_s}\right) \\ &= \begin{cases} (1 - (\pi_s/w_s))/2, & \text{if } s = 0, \\ (2 - (\pi_s/w_s))/3, & \text{if } s = 1. \end{cases} \end{aligned} \tag{5}$$

Combining (1) with (4)–(5) we find that the state-contingent firm profit π_s relative to the market wage w_s in a Walrasian equilibrium is given by

$$\frac{\pi_s}{w_s} = \begin{cases} 1-\lambda, & \text{if } s=0, \\ 2-3\lambda/2, & \text{if } s=1. \end{cases} \tag{6}$$

Using (1) and (4) the firm's equilibrium profit can be written as

$$\pi_s = \lambda L_{1s} - w_s(1+\lambda)L_{1s} = (\lambda - w_s(1+\lambda))/2, \tag{7}$$

so that with relation (6) we obtain the state-contingent equilibrium wage,

$$w_s = \begin{cases} \lambda/(3-\lambda) & = & 1/5, & \text{if } s=0, \\ \lambda/(5-2\lambda) & = & 1/8, & \text{if } s=1. \end{cases} \tag{8}$$

From (1) and (4) we obtain that

$$(\hat{L}_{1s}, \hat{L}_{2s}) = \left(\frac{1}{2}, \frac{\lambda}{2}\right) = \left(\frac{1}{2}, \frac{1}{4}\right), \tag{9}$$

independent of s. Relations (2)–(3) together with (6) and (9) imply that

$$(\hat{x}_{1s}, \hat{x}_{2s}) = \left(\frac{w_s}{2}, w_s\left(\frac{\lambda}{2} + \frac{\pi_s}{w_s}\right)\right) = \begin{cases} (\lambda/(6-2\lambda), \lambda(1-\lambda/2)/(3-\lambda)) & = & (1/10, 3/20), & \text{if } s=0, \\ (\lambda/(10-4\lambda), \lambda(2-\lambda)/(5-2\lambda)) & = & (1/16, 3/16), & \text{if } s=1. \end{cases}$$

(ii) From (6) and (8) we obtain that in equilibrium the firm's profits are

$$\hat{\pi}_s = \begin{cases} \lambda(1-\lambda)/(3-\lambda) & = & 1/10, & \text{if } s=0, \\ \lambda(4-5\lambda/2)/(9-4\lambda) & = & 5/32, & \text{if } s=1. \end{cases}$$

Even though labor is constant in equilibrium, the firm's profit vary as a result in agent 2's state-contingent preferences. Agent 2's preference for the consumption good is much larger in state 1 than in state 0, so that his willingness to supply labor for an even lower wage increases. The firm can thus lower overall wages, which decreases agent 1's budget. The latter can thus only afford a smaller share of the firm's output (which does not depend on the state). Overall, since the firm can produce the same output at a lower cost its profits are larger at $s = 1$ than at $s = 0$.

(iii) Based on the results obtained in (i), since the consumers labor (or rather leisure time) does not depend on the state, each consumer prefers the state in which he obtains more of the consumption good, i.e., consumer 1 strictly prefers $s = 0$ (since $x_{10} > x_{11}$) and consumer 2 strictly prefers $s = 1$ (since $x_{20} < x_{21}$)).

(iv) The agents would trade firm shares at $t = 0$, since as a consequence of their opposing state preferences gains from trade can be realized.