École Polytechnique Fédérale de Lausanne

College of Management of Technology

MGT-621 Microeconomics (Prof. Weber)

Final Exam – Solutions

Autumn 2020

Monday, October 5, 2020

Problem 1. (30 Points)

(i) Graph the equation $x_2 = \ln(c) - \ln(x_1)$ for $c \in \{1, 2, 3\}$. One obtains intersection points at $(1, 0)$, $(2, 0)$, and $(3, 0)$ respectively.

(ii) Laura's $\text{MRS}_{12}^L(x)$ is equal to $(\partial u(x)/\partial x_1)/(\partial u(x)/\partial x_2) = 1/x_1$, independent of the consumption level x_2. Jane can have a different marginal rate of substitution at x when her preferences are different. For example, if Jane's preferences are represented by the utility function $u(x) = x_1 + x_2$, then her marginal rate of substitution $\text{MRS}_{12}^J(x)$ is 1 for any consumption bundle $x \in \mathbb{R}_{++}^2$.

(iii) If both agents find it optimal to consume positive amounts of both goods, then their marginal rates of substitution are equal to $p_1/p_2 = p_1$. This is because the dual variables for the nonnegativity constraints vanish. Hence, given that $p_2 = 1$ (i.e., good 2 is the *numéraire*), $\text{MRS}_{12}^L(x^{L*}) = \text{MRS}_{12}^J(x^{J*}) = p_1$, where x^{L*} and x^{J*} are Laura and Jane's respective optimal consumption bundles.

(iv) If Laura buys only good 1, then $x_1 = w/p_1$. In addition, $(\partial u/\partial x_1)/(\partial u/\partial x_2) = 1/x_1 \geq p1$. These relations imply that Laura restricts her consumption to good 1 whenever $w/p_1 \leq 1/p_1$, or in other words, when $w \leq 1$. Interestingly, the price of good 1 has no effect on whether she consumes good 1. Her decision depends only on her wealth. If Laura is only consuming good 1, then the Lagrange multiplier λ_2 associated with the constraint $x_2 \geq 0$ can be positive, and thus $\text{MRS}_{12}^L(x^{L*}) = \lambda_0 p_1/(\lambda_0 - \lambda_2) \geq p_1$, generally different from what was obtained in part (iii).

(v) From problem (iv), for $w \leq 1$ Laura only consumes good 1, whence $v(p, w) = w/p_1$. For $w > 1$ Laura consumes both goods. From part (iii), $1/x_1 = p_1$ or $x_1 = 1/p_1$. By virtue of the (binding) budget constraint, it is therefore $x_2 = w - 1$. This leads to an indirect utility of $v(p, w) = 1/p_1 \exp(w - 1)$.

(vi) Since the transition is from a state of lower prices to a state of higher prices, $C(w)$ and $E(w)$ must be negative.

By definition, the compensating variation $C(w)$ is implicitly determined by the following equation:

$$v(p, w) = v(\hat{p}, w - C(w)).$$

It is important to use the correct form of v, depending on whether the wealth is greater or less than 1.

(a) $w \leq 1$, $w - C(w) \leq 1$: The valid range on w is defined implicitly. Using $v(p, w) = w/p_1$ on both sides of the equation yields $w/2 = w - C(w)$, so that via the definition of $C(w)$, one obtains $C(w) = -w$ for $w \leq 1/2$.

(b) $1/2 < w \leq 1$: Using $v(p,w) = w/p_1$ for the left-hand side of the equation and $v(p,w) = 1/p_1 \exp(w-1)$ for its right-hand side one finds $w = 1/2\exp(w - C(w) - 1)$, so $C(w) = w - \ln(2w) - 1$.

(c) $w > 1$: Using $v(p,w) = 1/p_1 \exp(w-1)$ on both sides of the equation one finds that $\exp(w - 1) = 1/2 exp(w - C(w) - 1)$. Thus, $C(w) = -\ln(2)$.

To find the equivalent variation, one again uses the defining implicit relation, in this case $v(\hat{p}, w) = v(p, w + E(w))$. Again, one needs to take into account the correct version of v, depending on whether the wealth exceeds 1 or not.

(a) $w < 1$: Using $v(p,w) = w/p_1$ on both sides of the equation yields $w/2 = w + E(w)$, or equivalently, $E(w) = -w/2$.

(b) $w > 1$, $w + E(w) \leq 1$: The valid range on w is defined implicitly. Use $v(p,w) = 1/p_1 \exp(w - 1)$ on the left-hand side of the equation and $v(p,w) = w/p_1$ on the right-hide side to find $1/2\exp(w-1) = w + E(w)$. Plugging the result into the implicit constraint yields $E(w) = 1/2 exp(w-1) - w$ for $1 < w \leq \ln(2) + 1$.

(c) $w > \ln(2) + 1$: Now use $v(p,w) = 1/p_1 \exp(w-1)$ on both sides of the equation, and obtain $1/2\exp(w-1) = \exp(w + E(w) - 1)$. Thus, $E(w) = -\ln(2)$.

Problem 2. (20 Points)

(i) Charlie's expected-utility maximization problem is

$$\begin{aligned} \max_{(x_1,x_2)\geq 0} \quad & \mathrm{EU}(x) = r(x_1 x_1 + 10x_2) + (1-r)(x_1 x_2 + 10) \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 \leq w. \end{aligned}$$

The Lagrangian associated with this optimization problem is

$$\mathcal{L} = x_1 x_2 + 10 r x_2 + 10(1-r) - \lambda(p_1 x_1 + p_2 x_2 - w).$$

From the first-order necessary optimality conditions, we obtain $\dfrac{x_2}{x_1 + 10r} = \dfrac{p_1}{p_2}$. Combining this with the budget constraint gives the optimal consumption bundle

$$x^*(p,w) = (x_1^*(p,w), x_2^*(p,w)) = \left(\frac{w - 10p_1 r}{2p_1}, \frac{w + 10p_1 r}{2p_2} \right).$$

The corresponding maximum expected utility is then

$$\mathrm{EU}(x^*) = x_1^* x_2^* + 10 r x_2^* + 10(1-r) = \frac{w^2 - 100p_1^2 r^2}{4p_1 p_2} + 10r\frac{w + 10p_1 r}{2p_2} + 10(1-r).$$

(ii) The demand for cocoa powder decreases in r since $\partial x_1^*/\partial r = -5 < 0$. Charlie's expected utility increases in r since

$$\frac{\partial EU(x^*)}{\partial r} = 50\frac{p_1}{p_2} r + 5\frac{w}{p_2} - 10 > 0,$$

provided that $w \gg 2p_2$ (satisfied here).

(iii) When $u(x;q) = x_1 + (1+q)x_2$, the expected-utility maximization problem yields the corner solution $x^* = (30, 0)$. However, the optimal consumption bundle changes to $x^* = (0, 20)$ when $u(x;q) = (x_1 + (1+q)x_2)^2$. Although the ordinal characteristics of a utility function do not change after an increasing transformation when no uncertainty is present, this example shows that it may not hold under uncertainty. A nonlinear transformation of a utility function may change the iso-utility curves and thereby the optimal solution when the expected utility is maximized.

(iv) Let c be Charlie's willingness to pay for shopping at Poots instead of shopping at Mr. Wonka's. Then shopping at Poots with the budget $w - c$ must give at least the same utility as shopping at Mr. Wonka's store with the budget w. In other words, c must satisfy $\mathrm{EU}(x^*(p,w); r = 0.6) = \mathrm{EU}(x^*(p, w-c); r = 1)$.

Plugging $p_1 = 1, p_2 = 1.5, w = 30$, and $r = 0.6$ into $\mathrm{EU}(x^*)$ in part (i), we obtain the expected utility from shopping at Mr. Wonka's store $\mathrm{EU}_{\mathrm{Wonka}} = \mathrm{EU}(x^*(p,w)) = 220$.
The utility from shopping at Poots $\mathrm{U}_{\mathrm{Poots}}$ is

$$\begin{aligned}\mathrm{U}_{\mathrm{Poots}} = E\left[u(x^*(w-c,p);q) \middle| r = 1\right] &= \frac{(w-c)^2 - 100p_1^2}{4p_1p_2} + 10\frac{(w-c) + 10p_1}{2p_2} \\ &= \frac{(30-c)^2}{6} + \frac{10(30-c)}{3} + \frac{100}{6}.\end{aligned}$$

Hence, solving $\mathrm{U}_{\mathrm{Poots}} = 220$, $c^* \approx 3.67$.

Problem 3. (25 Points)

(i) $C(q;w) = \min\limits_{z:F(z)\geq q} \{w \cdot z\}$

(ii) Nondecreasing returns to scale means if $q \leq F(z)$, then $\alpha q \leq F(\alpha z)$ for any $\alpha \geq 1$.

Let $z^* \in \operatorname*{argmin}\limits_{z:F(z)\geq q}\{w \cdot z\}$, then $\alpha z^* \in \{z : F(z) \geq \alpha q\}$ for any $\alpha \geq 1$.

$w \cdot \alpha z^* \geq \min\limits_{z:F(z)\geq \alpha q} \{w \cdot z\}$.

$C(q;w)/q = \min\limits_{z:F(z)\geq q} \{w \cdot z\}/q = w \cdot z^*/q = w \cdot \alpha z^*/\alpha q \geq \min\limits_{z:F(z)\geq \alpha q} \{w \cdot z\}/\alpha q = C(\alpha q; w)/\alpha q$.

Hence, $C(q;w)/q$ is nonincreasing in q.

(iii) $\pi(w,p) = \min\limits_{z\geq 0}\{pF(z) - w \cdot z\}$

Because $F(z) = 0$ if either input is zero, we can ignore the nonnegativity constraints.

$\mathfrak{L} = pF(z) - w \cdot z$

$\partial\mathfrak{L}/\partial z_1 = 1/4(z_2^{1/4}/z_1^{3/4}) - w_1 = 0$, $\partial\mathfrak{L}/\partial z_2 = 1/4(z_1^{1/4}/z_2^{3/4}) - w_2 = 0$

Solving the two above equations, we find:

$$\begin{aligned} z_1^* &= p^2/(16w_1^{3/2}w_2^{1/2}), \\ z_2^* &= p^2/(16w_1^{1/2}w_2^{3/2}). \end{aligned}$$

Thus, $F(z^*) = p/(4\sqrt{w_1w_2})$, and

$$y = (-p^2/(16w_1^{3/2}w_2^{1/2}), -p^2/(16w_1^{1/2}w_2^{3/2}), p/(4\sqrt{w_1w_2})).$$

Furthermore:

$$\pi(w,p) = p(p/(4\sqrt{w_1 w_2})) - w_1(p^2/(16 w_1^{3/2} w_2^{1/2})) - w_2(p^2/(16 w_1^{1/2} w_2^{3/2})) = p^2/8\sqrt{w_1 w_2}.$$

(iv) Nonincreasing returns to scale means if $q \leq F(z)$, then $\alpha q \leq F(\alpha z)$ for any $\alpha \in [0,1]$.

Suppose $q \leq F(z) = z_1^{1/4} z_2^{1/4}$, then $\alpha q \leq \alpha z_1^{1/4} z_2^{1/4} \leq \alpha^{1/4} z_1^{1/4} z_2^{1/4} \leq (\alpha z_1)^{1/4} (\alpha z_2)^{1/4} = F(\alpha z)$. So the production function does have nonincreasing returns to scale.

(v) From $\pi(w,p)$ we can see that if the price of the produced good is doubled, the profit will be multiplied by 4. If the cost of the inputs is cut in half, the profit will be doubled. Thus, they would prefer the price of the produced good to be doubled. Hence, the firm would prefer state A.

Problem 4. (25 Points) As indicated in the problem we let ballet tickets be the numeraire good, so that $p_B = 1$ and the price vector p is equal to $(1, p_F)$.

(i) When the market is competitive, the equilibrium price is $p_F = 1$, since no good is preferred by either student. Therefore, we obtain the Walrasian equilibrium allocation

$$\left(\hat{x}_B^L, \hat{x}_F^L\right) = \left(\frac{\omega_B^L + \omega_F^L}{2}, \frac{\omega_B^L + \omega_F^L}{2}\right) = (5/2, 5/2),$$

$$\left(\hat{x}_B^J, \hat{x}_F^J\right) = \left(\omega_B^J + \omega_B^L - \frac{\omega_B^L + \omega_F^L}{2}, \omega_F^J + \omega_F^L - \frac{\omega_B^L + \omega_F^L}{2}\right) = (7/2, 5/2)\,.$$

(ii) Lina determines her offer curve,

$$x^L(p_F) = \arg\max_{x^L \in \mathcal{B}_L(p_F)} \left\{\log(x_B^L) + \log(x_F^L)\right\},$$

where $\mathcal{B}_L(p_F) = \left\{x \in \mathbb{R}_+^L : p \cdot x = p \cdot \omega^L\right\}$ is the set of allocations satisfying the budget constraint given the price p_F. Lina's offer curve is thus

$$(x_B^L, x_F^L)(p_F) = \left(\frac{\omega_B^L + p_F \omega_F^L}{2}, \frac{\omega_B^L + p_F \omega_F^L}{2 p_F}\right) = \left(\frac{1 + 4p_F}{2}, \frac{1 + 4p_F}{2p_F}\right).$$

Hence, Lina's response to the price p_F offered by Justin is to trade

$$-z_F^L(p_F) = \omega_F^L - x_F^L(p_F) = \frac{\omega_F^L - \omega_B^L/p_F}{2} = \frac{4 - 1/p_F}{2}$$

football tickets in exchange for

$$z_B^L(p_F) = x_B^L(p_F) - \omega_B^L = \frac{p_F \omega_F^L - \omega_B^L}{2} = \frac{4p_F - 1}{2}$$

ballet tickets.

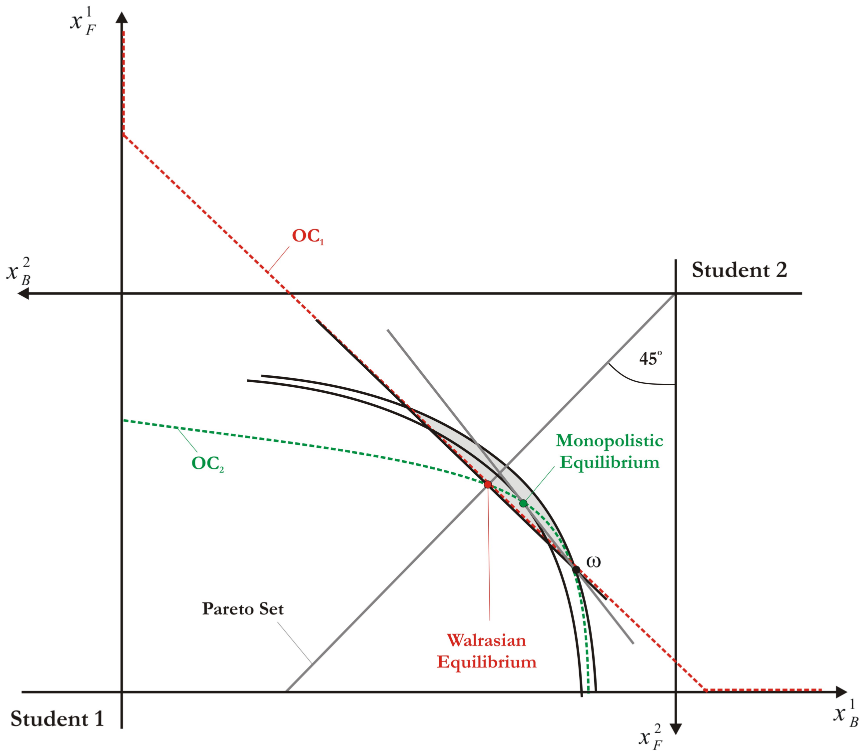

Figure 1: Problem 4.1: Walrasian and Monopolistic Equilibrium in the Edgeworth Box.

(iii) Justin, as a monopolist, chooses p_F so as to maximize his post-trade utility, i.e., he selects the monopoly price

$$p_F^m = \arg\max_{p_F \geq 0} \left\{ \omega_B^J - \frac{p_F \omega_F^L - \omega_B^L}{2} + \omega_F^J + \frac{\omega_F^L - \omega_B^L / p_F}{2} \right\} = \sqrt{\frac{\omega_B^L}{\omega_F^L}} = 1/2.$$

At this price, the consumption bundles are

$$(x_B^L, x_F^L)(p_F^m) = \left(\frac{\omega_B^L + \sqrt{\omega_B^L \omega_F^L}}{2}, \frac{\omega_F^L + \sqrt{\omega_B^L \omega_F^L}}{2} \right) = (3/2, 3),$$

$$(x_B^J, x_F^J)(p_F^m) = \left(\omega_B^J + \omega_B^L - x_B^L, \omega_F^J + \omega_F^L - x_F^L \right) = (9/2, 2),$$

while the trading quantities

$$(z_B^L, z_F^L)(p_F^m) = \left(\frac{\sqrt{\omega_B^L}\left(\sqrt{\omega_F^L} - \sqrt{\omega_B^L}\right)}{2}, \frac{\sqrt{\omega_F^L}\left(\sqrt{\omega_B^L} - \sqrt{\omega_F^L}\right)}{2} \right) = (1/2, -1).$$

(iv) Figure 1 shows both students' offer curves in an Edgeworth-box diagram. The monopolistic equilibrium is such that the budget set $\mathcal{B}_2(p_2^m)$ touches Justin's indifference curve at the equilibrium allocation. Note that the monopoly allocation is not in the Pareto set, because Justin, as a monopolist, charges Lina a higher price for the good that she has the least of (e.g., in Figure 1: ballet tickets).

(v) $p_F^m = \sqrt{\frac{\omega_B^L}{\omega_F^L}} = \sqrt{1/4} = 1/2 < 1 = p_F$. Thus, Justin charges Lina a lower price for the good that Lina has more of. Let us now look at trade in a competitive market versus a monopolistic market. In a competitive regime, Justin trades

$$\omega_F^L - \frac{\omega_B^L + \omega_F^L}{2} = \frac{\omega_F^L - \omega_B^L}{2} = 3/2 \ \ (\geq |z_F^L| = 1/2)$$

football tickets in exchange for

$$-\left(\omega_B^L - \frac{\omega_B^L + \omega_F^L}{2}\right) = \frac{\omega_F^L - \omega_B^L}{2} = 3/2 \ \ (\geq |z_B^L| = 1)$$

ballet tickets. Comparing these quantities with trade under Justin's monopoly, we see that there is more trade in a competitive market (in a monopoly, the exchange does not lead all the way to a symmetric allocation).