École Polytechnique Fédérale de Lausanne

College of Management of Technology

MGT-528 Operations: Economics and Strategy (Prof. Weber)

Final Exam – Solutions

Autumn 2020

Monday, January 25, 2021 (16h15 CET – 19h15 CET)

Problem 1 (30 Points)

(i) The push-pull frontier is the place in the supply chain where the customer order separates the make-to-order (pull) and make-to-stock (push) processes. Traditionally, the latter occur based on demand predictions while the former take place only once the customer signals his interest to purchase a good or service. Industry 4.0 may enable factory processes to receive pull information leading to customization options without significant cost increase. This has a tendency to displace the push-pull frontier upstream, to incorporate customization information as early as possible in the production (and possibly even in the materials sourcing) process.

(ii) The firm's profit as a function of its output $Q > 0$ is:

$$\Pi(Q) = p(Q)Q - C(Q) = \left(p(Q) - \frac{C(Q)}{Q}\right) Q = \big(p(Q) - AC(Q)\big)\, Q,$$

where $AC(Q)$ is the firm's average cost. The firm's viability is therefore ensured for output quantities $Q > 0$ for which

$$p(Q) \geq AC(Q). \tag{1}$$

Assuming differentiability of the inverse demand function $p(\cdot)$ and the cost function $C(\cdot)$, the firm's necessary optimality condition for an optimal production quantity $Q^* > 0$ is:

$$Qp'(Q^*) + p(Q^*) = MC(Q^*), \tag{2}$$

where $MC(Q) = C'(Q)$ is the firm's marginal cost at any positive output Q. Conditions (1) and (2) are different for $Q = Q^*$. Note that condition (1) includes the firm's fixed cost $F = \lim_{Q\to 0^+} C(Q)$, whereas condition (2) does not.

(iii) The correct answer is "no," with the reason being that in general

$$E[\Pi(\tilde{\theta})] \neq \Pi(E[\tilde{\theta}]) = \Pi(1).$$

If the firm's profit function $\Pi(\cdot)$ is strongly convex in θ, then by Jensen's inequality:

$$E[\Pi(\tilde{\theta})] > \Pi(E[\tilde{\theta}]);$$

in that case the firm is "antifragile" and strictly prefers the presence of uncertainty over the deterministic case.

(iv) A missing force in Porter's "five forces" diagram that is captured quite well in Brandenburger and Nalebuff's "value net" is "complementors," i.e., companies that enhance the value of the firm's product. For example, an app developer enhances the value of a mobile phone, so that the latter can be sold more expensively in the presence of app complements than without.

(v) In strategic situations, information may have negative value. For example, in a Cournot duopoly, a firm that knows that (ex-ante uncertain) demand will be low, cannot commit to producing a large quantity, whereas the competitor that knows nothing about the reliable demand forecast can credibly commit to producing a lot. Thus, the firm with the information is worse off because the other firm knows that it is informed. Another example is a used car seller, who typically has trouble selling a second-hand vehicle *because* he knows more about the car than a potential buyer. The buyer cannot trust any price the seller proposes, unless the seller includes a costly signal such as a warranty.[1]

(vi) By communicating information about a user to a firm, that company learns more about the user and can use that information for price discrimination (e.g., through targeted offers for a single product or bundles of different products). In this way, the firm is able to charge different prices to different customers for the exact same product (or product bundle). This amounts to third-degree price discrimination (which is fundamentally based on an observable characteristic). Second-degree price discrimination relies on the self-selection of consumer types to different products (or product bundles), which requires the firms to design a menu of options. The latter is costly and also operates under the assumption that interpersonal arbitrage should not be possible, thus implying the law of one price for any given product (or product bundle). The firm's increased amount of knowledge about a user reduces the information asymmetry and thus reduces the buyer's potential to earn an information rent (i.e., to get a discount over the fully rent-extracting price).

[1] Failure of trade due to asymmetric information about asset values has long been recognized in Finance, giving rise to various "no-trade" theorems.

Problem 2 (30 Points)

(i) The expected total cost $\mathrm{TC}(Q,0)$ (per day) for Q units of the ingredient that arrives in the high-quality state at the restaurant without the possibility of supply failure is [2]

$$\mathrm{TC}(Q,0) = \frac{h}{2}[Q-50]_+ + \frac{KD}{Q} + wD = (0.05)[Q-50]_+ + \frac{500}{Q} + 50.$$

If the ingredient quality is q_L, which happens with probability $p \in [0,1)$, then the firm has to repeat that same order (which is then fulfilled by assumption in zero time). Thus, conditional on getting a high-quality delivery eventually, the firm's total cost (per day) is [3]

$$\mathrm{TC}(Q,p) = (0.05)[Q-50]_+ + \frac{1}{1-p}\left[\frac{500}{Q} + 50\right].$$

(ii) Assuming it reaches at least one day's worth of ingredients, differentiating $\mathrm{TC}(Q,p)$ with respect to Q yields

$$\frac{h}{2} - \frac{KD}{1-p}\left(\frac{1}{Q^2}\right) = (0.05) - \frac{500}{1-p}\left(\frac{1}{Q^2}\right) = 0.$$

This can be used to find the restaurant's optimal ordering quantity,

$$Q^*(p) = \left\lceil \sqrt{\frac{10000}{1-p}} \right\rceil = \left\lceil \frac{100}{\sqrt{1-p}} \right\rceil \geq 100,$$

which is *nondecreasing* in p (as the fraction of the holding cost in the total cost becomes smaller when p goes up). The resulting expected total cost per day is (omitting the integer-rounding)

$$\mathrm{TC}^*(p) = \mathrm{TC}(Q^*(p),p) \approx \frac{10}{\sqrt{1-p}} + \frac{50}{1-p} - \frac{5}{2}.$$

(iii) At the optimal ordering quantity determined in part (ii), the restaurant's daily profit $\Pi(p)$ as a function of p is:

$$\Pi(p) = rD - TC(Q^*(p),p) \approx (3002.5) - \frac{10}{\sqrt{1-p}} - \frac{50}{1-p},$$

which remains nonnegative for failure probabilities $p \in [0,\bar{p}]$, where

$$\bar{p} \approx 0.9829.$$

That is, the restaurant will stay open unless the ingredients arrive successfully less than about 1.71% of the time. At that viability threshold, the optimal order quantity becomes maximal attaining $\bar{Q}^* = Q^*(\bar{p}) = 765$ — which is more than enough supply for 15 days of service.

[2] We are using $h/2$, although all ingredients are really needed in daily batches, complicating the averaging somewhat. Any convex combination of h and $h/2$ is acceptable here.

[3] Another way of deriving this formula is that with probability $1-p$ the firm has to pay $X = wD + (KD/Q)$ (in addition to the holding cost), with probability $p(1-p)$ it pays $2X$, with probability $p^2(1-p)$ it pays $3X$, and so on, which by summation means that the firm pays in expectation $(1-p)X\sum_{k=0}^{\infty}(k+1)p^k$. Since $\sum_{k=0}^{\infty}(k+1)p^k = \sum_{k=0}^{\infty}p^k + p\sum_{k=0}^{\infty}kp^{k-1} = (1-p)^{-1} + p\left(\frac{d}{dp}(1-p)^{-1}\right) = (1-p)^{-2}$, we obtain $X/(1-p)$ as the total expected non-holding-cost-related expenditure.

(iv) The "inventory turnover" corresponds to daily sales (i.e., demand D) divided by the average inventory I. The average inventory (after the first day) is

$$I(p) = \frac{Q^*(p) - 50}{2} \approx 25\left(\frac{2}{\sqrt{1-p}} - 1\right) \geq 25,$$

assuming a continuous depletion of the excess inventory over time (when in reality the inventory is piecewise constant, with ingredients being used in daily batches of size 50; see also footnote 2). The (daily) inventory turnover is therefore

$$\tau(p) = \frac{D}{I(p)} \approx \frac{2\sqrt{1-p}}{2-\sqrt{1-p}} \in (0, 2],$$

for $p \in [0, 1)$, with $\tau(0) = 2$ and $\tau(1^-) = 0$; that is, inventory turns decrease in the failure probability.[4]

(v) If demand is deterministic, but the ingredients are perishable, then the optimal ordering quantity is $\hat{Q}^*(p) = 50$, independent of $p \in [0, 1)$, leading to an expected total cost (per day) of

$$\widehat{TC}^*(p) = \frac{1}{1-p}\left((0.05)[\hat{Q}^*(p) - 50]_+ + \frac{500}{\hat{Q}^*(p)} + 50\right) = \frac{60}{1-p}.$$

The resulting optimal daily profits are

$$\hat{\Pi}^*(p) = r\hat{Q}^*(p) - \widehat{TC}^*(p) = 3000 - \frac{60}{1-p}.$$

The latter are nonnegative, as long as $p \leq 98\%$.

(vi) When the restaurant's demand is stochastic and the ingredients are perishable, then it is best to place a daily order according to the newsvendor logic. Indeed, as a function of the ordering quantity Q, the firm's expected daily profit is

$$\bar{\Pi}(Q) = E\left[r\min\{\tilde{D}, Q\}\Big| Q\right] - \frac{wQ + K}{1-p} = r\left(\int_{40}^{Q} \frac{q\,dq}{20} + Q\int_{Q}^{60} \frac{dq}{20}\right) - \frac{wQ + K}{1-p}.$$

Hence, the necessary optimality condition,

$$\bar{\Pi}'(Q) = 3\,(60 - Q) - \frac{1}{1-p} = 0,$$

implies the optimal ordering quantity of

$$Q^{**}(p) = \left[\left[60 - \left(\frac{1}{3}\right)\frac{1}{1-p}\right]\right]_+,$$

which is positive as long as $p < 179/180 \approx 99.44\%$. An important systematic difference of the newsvendor solution with stochastic demand to the solution in the deterministic case is that the

[4] When the failure p probability tends towards 1, inventory becomes so large that it no longer turns, as each daily service has almost no impact on the total stock.

optimal order quantity $Q^{**}(p)$ is decreasing in p, whereas the optimal order quantity $\hat{Q}^*(p)$ in Part II did not depend on p at all. The expected optimal daily profit (computed ignoring the rounding-operator in the optimal order quantity),

$$\bar{\Pi}^{**}(p) = \bar{\Pi}(Q^{**}(p)) = \frac{18000p^2 - 35580p + 17581}{6(1-p)^2},$$

is nonnegative, as long as $p \leq \left(593 - \sqrt{29}\right)/600 \approx 0.9794$. That is, the viability threshold has become slightly tighter in the stochastic case.

Problem 3 (30 Points)

(i) Given a positive price p, the demand of type-θ_L consumers is

$$D_L(p) \in \arg\max_{q\in[0,1]} \{u(q,\theta_L) - p\cdot q\} = \left\{1 - \frac{p}{\theta_L}\right\}.$$

Similarly, the demand of type-θ_H consumers is

$$D_H(p) \in \arg\max_{q\in[0,1]} \{u(q,\theta_H) - p\cdot q\} = \left\{1 - \frac{p}{\theta_H}\right\},$$

so that the aggregate demand is given by

$$D(p) = \mu D_H(p) + (1-\mu)D_L(p) = 1 - \frac{p}{\bar{\theta}},$$

where

$$\bar{\theta} = \left(\frac{\mu}{\theta_H} + \frac{1-\mu}{\theta_L}\right)^{-1}$$

can be interpreted as an 'average type.' The monopolist's profit maximization problem therefore yields the optimal monopoly price

$$p^m \in \arg\max_{p\geq 0} \{(p-c)D(p)\} = \left\{\frac{\bar{\theta}+c}{2}\right\},$$

resulting in optimal profits of

$$\Pi^m(\mu) = \frac{(\bar{\theta}-c)^2}{4\bar{\theta}}.$$

(ii) If the monopolist can distinguish between the two consumer groups, then he is able to charge a profit-maximizing price schedule separately for each consumer group. In particular, it is best for the monopolist to first maximize the social surplus, and then charge consumers all of their surplus back. Clearly, the competitive outcome when $p = c$ maximizes the social surplus, so that $\hat{q}_L = D(c) = 1 - (c/\theta_L)$ and $\hat{q}_H = 1 - (c/\theta_H)$. Thus, we can immediately conclude that the optimal schedule for type-θ_H consumers is

$$\hat{t}_H(q) = u(\hat{q}_H, \theta_H) + c\,(q - \hat{q}_H) = \theta_H\left(1 - \left(\frac{c}{\theta_H}\right)^2\right) + c\,(q - \hat{q}_H),$$

resulting in a type-θ_H consumption of $\hat{q}_H$, and that the optimal schedule for type-θ_L consumers is

$$\hat{t}_L(q) = u(\hat{q}_L, \theta_L) + c\,(q - \hat{q}_L) = \theta_L\left(1 - \left(\frac{c}{\theta_L}\right)^2\right) + c\,(q - \hat{q}_L),$$

resulting in a type-θ_L consumption of $\hat{q}_L$. The corresponding total transfers paid to the monopolist by each of the two consumer types are

$$\hat{t}_H(\hat{q}_H) = \theta_H\left(1 - \left(\frac{c}{\theta_H}\right)^2\right) \quad \text{and} \quad \hat{t}_L(\hat{q}_L) = \theta_L\left(1 - \left(\frac{c}{\theta_L}\right)^2\right).$$

(iii) If the monopolist cannot distinguish the two consumer types, he can use a nonlinear pricing strategy. By offering the quantities q_H and q_L we know that the type-θ_H consumers' incentive compatibility constraint is binding, i.e.,

$$u(q_H, \theta_H) - t_H = u(q_L, \theta_H) - t_L$$

and that the type-θ_L consumers' individual rationality constraint is binding, i.e.,

$$u(q_L, \theta_L) = t_L.$$

Thus, using these two relations the monopolist's profit maximization problem can be written in the form

$$\max_{(q_L,q_H)\in[0,1]^2} \left\{(1-\mu)(u(q_L,\theta_L) - cq_L) + \mu(u(q_H,\theta_H) - cq_H - (u(q_L,\theta_H) - u(q_L,\theta_L))\right\},$$

so that

$$q_H^* \in \arg\max_{q_H\in[0,1]} \{u(q_H,\theta_H) - cq_H\} = \left\{1 - \frac{c}{\theta_H}\right\} = \{\hat{q}_H\},$$

and

$$q_L^* \in \arg\max_{q_L\in[0,1]} \left\{\theta_L \frac{1-(1-q_L)^2}{2} - cq_L - \frac{\mu}{1-\mu}\frac{(\theta_H - \theta_L)(1-(1-q_L)^2)}{2}\right\},$$

so that

$$q_L^* = \left[1 - \frac{c}{\theta_L - \frac{\mu}{1-\mu}(\theta_H - \theta_L)}\right]_+ < \hat{q}_L.$$

Thus, since $pq_L^* + f = t_L$ and $pq_H^* + f = t_H = t_L + (u(q_H,\theta_H) - u(q_L,\theta_L))$, we obtain the optimal two-part tariff,

$$p^* = \frac{t_H^* - t_L^*}{q_H^* - q_L^*} = \frac{u(q_H^*,\theta_H) - u(q_L^*,\theta_H)}{q_H^* - q_L^*} = 1 - \frac{q_L^* + q_H^*}{2} = \frac{1}{2}\left(\frac{c}{\theta_L - \frac{\mu}{1-\mu}(\theta_H - \theta_L)} + \frac{c}{\theta_H}\right),$$

and

$$f^* = t_L^* - p^* q_L^* = u(q_L^*,\theta_L) - p^* q_L^*.$$

For $\mu \geq \mu_0 = (\theta_L - c)/(\theta_H - c) \in (0,1)$ it is $q_L^* = 0$, in which case the monopolist sells only to type-θ_H consumers, and the optimal price schedule for those consumers becomes the same as in part (ii), i.e., $t^*(q) = \hat{t}_H(q)$ (or, equivalently, $p^* = c$ and $f^* = u(q_H^*,\theta_H) - cq_H^*$).

(iv) With perfect price discrimination, each consumer type is charged his or her full surplus, whereas with second-degree price discrimination type-θ_H consumers may earn a positive surplus, the so-called *information rent*. Type-θ_H consumers consume the "first-best" quantity, identical to the one that would be consumed if a social planner was to determine a welfare-maximizing production and allocation of the goods. However, under the nonlinear pricing scheme type-θ_L consumers consume a quantity q_L^* that is strictly smaller than the corresponding first-best quantity $\hat{q}_L$.

(v) The monopolist cannot do any better by using a fully nonlinear schedule, since the nonlinear pricing model maximizes profits, subject to the fact that consumers are revealing their respective types. When there are more than two different types, it will be generally optimal to have a piecewise affine transfer schedule.

Problem 4 (30+10 Points)

(i) Let $i \in \{1, 2\}$. Firm i's payoff function is given by

$$\pi_i(q_i, q_{-i}) = (p(q_i + q_{-i}) - c)\, q_i = \left(\frac{100}{(q_i + q_{-i})^{3/2}} - c \right) q_i.$$

Its best-response correspondence is characterized by

$$\left\{ q_i \geq 0 : \frac{\partial \pi_i(q_i, q_{-i})}{\partial q_i} = 0 \right\},$$

and thus given by

$$\mathrm{BR}_i(q_{-i}) = \left\{ q_i \geq 0 : \frac{100}{(q_1 + q_2)^{3/2}} - \frac{150 q_i}{(q_1 + q_2)^{5/2}} = c \right\}.$$

(ii) A symmetric Nash-equilibrium $q^* = (q_1^*, q_2^*)$ is such that $q_1^* = q_2^*$, whence with $Q^* = q_1^* + q_2^*$:

$$\frac{100}{(Q^*)^{3/2}} - \frac{150(Q^*/2)}{(Q^*)^{5/2}} = c.$$

As a result,

$$Q^* = \left(\frac{25}{c} \right)^{2/3},$$

and

$$q_i^* = Q^*/2.$$

Firm i's equilibrium profits are

$$\pi_i^* = \left(\frac{100}{(Q^*)^{3/2}} - c \right) \frac{Q^*}{2} = \frac{3}{2} (625\, c)^{1/3}.$$

(iii) Using the result in part (ii) we have that

$$\pi_i|_{c=10} \approx 27.63 < 34.81 \approx \pi_i|_{c=20},$$

and $\pi_i|_{c=0} = 0$, which indicates that the firms' equilibrium profits are increasing in c (at least in a certain range of costs). This means that a cost decrease in the industry has detrimental effects on the firms' equilibrium profits: the value of collective innovation is in fact negative. Examples of such industries are telecommunications and computers, where (quality-adjusted) costs have fallen dramatically but the firms' profits also declined. The intuition is that due to a cost increase the equilibrium price overadjusts, more than compensating for the decreased output.[5]

[5] This phenomenon was described in the context of taxation by Seade, J. (1985) "Equitable Cost Increases and the Shifting of Taxation: Equilibrium Responses and Markets in Oligopoly," Working Paper No. 260, Department of Economics, University of Warwick, Warwick, UK.

(iv) For $c \to 0+ = \check{c}$ the industry's equilibrium profits vanish. Hence, when the firms have zero marginal costs the industry is unable to earn any money. The reason is that the industry's aggregate output goes to infinity, which in turn drives the market price to zero. Collusion could help alleviate the firms' dilemma, but is in many cases illegal.

(v) [Bonus] Assume that $\Delta > 0$. Then, one can quickly see that there is no pure-strategy equilibrium in the firms' rent-seeking game. Assume that firm i's investment is $z_i \geq 0$. If the firm randomizes with density $f_i(z_i)$ over the support $[0, \Delta]$, its expected first-mover-advantage benefits are

$$\bar{B}_i = \int_0^\Delta f_i(z_i) \left(\Delta \int_0^{z_i} f_{-i}(z_{-i}) dz_{-i} - z_i \right) dz_i.$$

Firm $-i$ can make firm i indifferent between choosing any density by making sure that the bracketed expression is constant in z_i. The latter holds if and only if

$$\Delta f_{-i}(z_i) = 1.$$

Hence, a mixed-strategy equilibrium of the rent-seeking game is given by the investment densities $f_i(z_i) = 1/\Delta$ on the support $[0, \Delta]$ for $i \in \{1, 2\}$. One can show that this equilibrium is in fact unique (in the sense of Lebesgue).