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Optimal Depth of Discharge for Electric Batteries with Robust Capacity-Shrinkage Estimator

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***Abstract*—A user's benefit from the energy stored in a battery over its lifetime depends on the time-varying characteristics of the battery, which are in turn affected by the chosen usage behavior. Both the capacity shrinkage and the number of lifetime cycles are strongly impacted by the depth of discharge as a key decision variable. Available models of battery lifetime are rather complex and depend on many factors, such as temperature and other physical particularities, which complicate the user's decision problem. We propose a simple, relatively robust approach for determining an optimal robust depth of discharge, based on a cycle-discharge curve and an unknown exponential capacity-shrinkage curve. We characterize the optimal robust depth of discharge and describe the implied performance guarantees. A relative cost of robustness is obtained as the boundary of the uncertainty set varies. The method provides an example of parameter estimation based on minimizing the relative regret caused by the implied decisions. In the special case where the cycle-discharge curve is exponential as well, we find a closed-form solution.**

***Index Terms*—Battery aging management, depth of discharge, lifetime user benefit, relative robustness, robust optimization**

I. Introduction

How aggressively a battery is used on each recharging cycle has an important influence on how long the battery lasts. Conversely, using a significant portion of the available battery capacity drives the economic return from the storage device in the short run. At a constant capacity the user's lifetime payoff is proportional to the chosen depth of discharge (or "cycle depth"), as well as the number of available cycles, depending on the cycle-discharge curve. In reality, the capacity shrinks from cycle to cycle—also as a function of the applied cycle depth. Moreover, the user often does not have access to reliable information about the battery characteristics and its current condition. The latter tend to depend on physical boundary conditions and "aging factors" which are generally difficult to track and incorporate into useful battery-aging models [1]–[4]. As a simple decision tool, we propose a robust-optimization approach to determine a (relative) depth of discharge, together with a performance guarantee with respect to a family of (exponential) capacity-shrinkage curves, based on the information available to the user. In case the given cycle-discharge curve is also exponential (as, e.g., in [4] over significant discharge intervals),[1] our robust method yields a closed-form solution.

[1]A piecewise-exponential dependence is sufficient, corresponding to a piecewise-linear approximation of cycle-discharge curves in $(\ln n, \delta)$-space.

We evaluate the performance of a *candidate* parameter estimate by a "performance index" which tracks a worst-case "performance ratio." The performance ratio is thereby the user's lifetime payoff at the optimal depth of discharge implied by the candidate parameter, evaluated when a different *true* parameter value determines the actual battery characteristics, divided by the user's *ex-post optimal* lifetime payoff achieved when the missing parameter is perfectly known. Distribution-free robust decision-making based on maximizing the minimum payoff across realizations of uncertainty dates back to Wald's seminal work [5], which in turn relies on insights from zero-sum games by von Neumann and Morgenstern [6]. Minimizing the difference of ex-post optimal and achieved payoffs, termed minimax absolute regret, was proposed by Savage [7] and has found a large following. Our *relatively robust* decision-making approach, which can be viewed as following a minimax *relative regret* criterion is examined in [8]. In computer science, the general idea of a "competitive ratio" has been used for the comparative performance evaluation of algorithms (e.g., online vs. offline); see, e.g., [9]. Relatively robust decisions have found some applications in economics (see [10], [11]), and appear also useful in the context of robustly optimizing the depth of discharge so as to maximize the lifetime payoff from battery usage. The latter problem belongs to the realm of battery management [12], which has been investigated especially in the context of electric vehicles [13], [14], due to the substantial role the battery price plays in the cost of the entire vehicle.

II. Model

While our considerations are fairly general and should apply to any type of physical battery, we consider here the battery of an electric vehicle as a leading example. Such a battery stores electric energy for the ultimate purpose of propelling a car forward, thus converting stored energy into distance travelled at the rate ϕ (measured in kWh/km). Travelled distance provides value at the rate v to the user (measured in dollars/km). On the other hand, recharging a battery has a constant marginal cost c (measured in dollars/kWh). The main reason for varying the depth of discharge $\delta \in [0, 1]$ which we define here as the fraction of *current* battery capacity used in a single discharge-charge cycle is that the battery lifetime depends negatively on it.[2] Specifically, the *cycle-discharge curve* $n(\delta)$, specifying the number of cycles over the entire use life of a battery, is a decreasing and continuously differentiable function of $\delta \in [0,1]$. The capacity $K(i)$ (measured in kWh) is nonincreasing in the cycle index $i \in \{0,1,2,\ldots\}$, starting with a given *nominal capacity* $K(0) = K_0 > 0$, so

$$K(i) \leq K(i-1), \quad i \geq 1.$$

We first consider the case where $K(i) \equiv K_0$ as a reference, and then turn our attention to a more general case with exponential capacity shrinkage. To avoid trivialities we assume that $c < v/\phi$, so that using the car makes economic sense, as long as the battery is intact.

A. Base Case: Constant Capacity

Let $K(i) = K_0$ for all relevant cycle indices $i \geq 0$. The user's (undiscounted) *lifetime payoff* (measured in dollars) is

$$\pi_0(\delta) = \left(\frac{v\delta K_0}{\phi} - c(\delta K_0)\right) n(\delta) = \frac{K_0}{\phi}\,(v - c\phi)\,\delta n(\delta),$$

for all $\delta \in [0,1]$. Hence, finding the lifetime-optimal depth of discharge δ_0 amounts to solving a problem that does not actually depend on any parameters other than the shape of the cycle-discharge curve $n(\cdot)$, since

$$\delta_0 \in \arg\max_{\delta\in[0,1]} \pi_0(\delta) = \arg\max_{\delta\in[0,1]} \delta n(\delta). \tag{1}$$

A solution to the preceding optimization problem exists, as by the extreme-value theorem [15, p. 89] any continuous real-valued function attains its maximum on a compact set. If the maximizer is interior, i.e., in $\delta_0 \in (0,1)$, it needs to satisfy the first-order necessary optimality condition (for $\delta = \delta_0$):

$$n(\delta) + \delta n'(\delta) = 0.$$

The latter is equivalent to

$$\varepsilon(\delta) = 1, \tag{2}$$

where we refer to $\varepsilon(\delta) = -\delta n'(\delta)/n(\delta) \geq 0$ as the *cycle-discharge elasticity*. Since the maximand vanishes for $\delta = 0$, it is straightforward to see that the optimal depth of discharge δ_0 is an element of $(0,1]$. Moreover, if $\varepsilon(\cdot)$ (with $\varepsilon(0) = 0$ and $\varepsilon(1) > 1$) is nondecreasing, then the preceding optimality condition is also sufficient.

Example 1. For $n(\delta) = n_0 \exp(-\kappa\delta)$, where n_0 is the maximum number of charging cycles (as $\delta \to 0^+$), and where $\kappa > 0$ describes an exponential decay, one obtains the cycle-discharge elasticity $\varepsilon(\delta) = \kappa\delta$, which increases linearly in δ. The optimality condition (2), together with the imposed upper limit (using a standard Lagrangian constrained-optimization approach) implies $\delta_0 = \min\{1/\kappa, 1\}$ as the unique solution to the cycle-discharge problem (1) in the base case.

[2] The relative depth of discharge (DoD), as a fraction current capacity (instead of the nominal capacity K_0), has been used by [12]. With capacity shrinkage, a constant relative DoD implies a decreasing absolute DoD.

B. Use Case: Exponential Capacity Shrinkage

We now allow for a capacity decrease from cycle to cycle at the (nonnegative) exponential rate λ, so

$$K(i) = K_0 \exp(-\lambda i), \quad i \geq 0. \tag{3}$$

The precise value of $\lambda > 0$ may not be known. Its robust determination based on minimizing the relative regret for any admissible incorrect estimate is discussed below. Using the geometric-series formula, the user's corresponding lifetime payoff becomes

$$\begin{aligned}\pi_\lambda(\delta) &= \frac{K_0}{\phi}\,(v - c\phi)\,\delta \sum_{i=0}^{n(\delta)-1} \exp(-\lambda i) \\ &= \frac{K_0}{\phi}\,(v - c\phi)\,\delta\,\frac{1 - \exp(-\lambda n(\delta))}{1 - \exp(-\lambda)},\end{aligned}$$

where by l'Hôpital's rule it is $\lim_{\lambda\to 0^+} \pi_\lambda(\delta) = \pi_0(\delta)$, for all $\delta \in [0,1]$. The user's cycle-discharge optimization problem is therefore to find

$$\delta_\lambda \in \arg\max_{\delta\in[0,1]} \pi_\lambda(\delta) = \arg\max_{\delta\in[0,1]} \delta\big(1 - \exp(-\lambda n(\delta))\big). \tag{4}$$

As in the base case, the objective has a maximizer $\delta_\lambda \in (0,1]$, which in the interior $(0,1)$ would have to satisfy the first-order necessary optimality condition,

$$\varepsilon(\delta) = \frac{1 - \exp(-\lambda n(\delta))}{\lambda n(\delta)} \quad (< 1). \tag{5}$$

For $\lambda \to 0^+$ this condition specializes to that in the base case, where the cycle-discharge elasticity was required to be unity (cf. Eq. (2)). In the case where capacity decreases from cycle to cycle (with $\lambda > 0$), the cycle-discharge elasticity is below 1 at the optimum.

Example 2. In the setting of Ex. 1, the optimality condition (5) specializes to

$$\delta = \frac{1 - \exp(-\lambda n(\delta))}{\kappa\lambda n(\delta)}.$$

Given that $(1 - e^{-x})/x < 1$, for all $x = \lambda n(\delta) > 0$, the preceding equation has an (interior) solution by the Brouwer's fixed-point theorem [16, p. 118], as long as $\kappa \geq 1$. In that case, the solution is unique because the unit slope on the left-hand side strictly exceeds the slope on the right-hand side, $(1 - \exp(-\lambda n(\delta)))/(\lambda n(\delta)) - \exp(-\lambda n(\delta))$, which lies in the interval $(0, 1/3)$, for any $\lambda n(\delta) > 0$.

C. Comparative Statics

An interesting, and perhaps somewhat counterintuitive, insight is that the optimal solution δ_λ to the user's cycle-discharge problem in the general form (4) is increasing in λ, as long as the corresponding optimal cycle count is nontrivial.

Proposition 1. *Let $\delta_\lambda \in (0,1]$ be an optimal solution to the cycle-discharge problem (4). Then $\partial\delta_\lambda/\partial\lambda \geq 0$, as long as $n(\delta_\lambda) \geq 1$.*

Proof. To obtain the claim, it is sufficient to establish that the objective function $\pi_\lambda(\delta)$ is supermodular in (δ, λ); see, e.g., [17]. Indeed, by direct differentiation one obtains that

$$\mathrm{sgn}\left(\frac{\partial^2 \pi_\lambda(\delta)}{\partial\lambda\partial\delta}\right) = \mathrm{sgn}\big(n(\delta) - \delta n'(\delta)(n(\delta)-1)\big) > 0,$$

as long as $n(\delta) \geq 1$, which concludes our proof. □

The intuition for this result is that as the battery capacity decreases faster across cycles, it becomes a more perishable resource that is no longer worth preserving and thus should be exploited more aggressively, as each current capacity level will be succeeded by a significantly worse capacity level.

III. Robust Determination of Capacity Decay

Let us now reconsider the general cycle-discharge problem (4) when the capacity-shrinkage rate is only known to lie in a range $\Lambda = (\lambda_1, \lambda_2)$ with $0 \leq \lambda_1 < \lambda_2 \leq \infty$.[3] As a consequence of Prop. 1, all possible solutions δ_λ to the cycle-discharge problem (4) must lie in the set $\Delta = [\delta_{\lambda_1}, \delta_{\lambda_2}]$.[4]

A. Robustness Evaluation

By the solution monotonicity in Prop. 1 one can associate with any solution value in $\hat{\delta} \in \Delta$ a distinct candidate parameter value $\hat{\lambda}$, so $\delta_{\hat{\lambda}} = \hat{\delta}$. This in turn suggests the following robust approach where the performance of any potential estimate $\hat{\lambda} \in \Lambda$ is evaluated based on how well the implied action $\delta_{\hat{\lambda}}$ does in a scenario where $\lambda \in \Lambda$ realizes, which in turn would imply the ex-post optimal action δ_λ. Hence, we consider the *performance ratio*,

$$\varphi(\hat{\lambda}, \lambda) = \frac{\pi_\lambda(\delta_{\hat{\lambda}})}{\pi_\lambda(\delta_\lambda)} = \frac{\delta_{\hat{\lambda}}\big(1 - \exp(-\lambda n(\delta_{\hat{\lambda}}))\big)}{\delta_\lambda\big(1 - \exp(-\lambda n(\delta_\lambda))\big)} \in [0, 1], \quad (6)$$

for all $\lambda \in \Lambda$, which leads to the *performance index*,

$$\rho(\hat{\lambda}) = \inf_{\lambda \in \Lambda} \varphi(\hat{\lambda}, \lambda), \quad (7)$$

for all $\hat{\lambda} \in \Lambda$, as the worst-case performance ratio with respect to the ambiguity set Λ.[5]

Proposition 2. *The performance index can be written in the form*

$$\rho(\hat{\lambda}) = \min\left\{\varphi(\hat{\lambda}, \lambda_1), \varphi(\hat{\lambda}, \lambda_2)\right\}, \quad (8)$$

for all $\hat{\lambda} \in \Lambda$.

Proof. Fix $\hat{\lambda} \in \Lambda$. Then for $\lambda = \hat{\lambda}$ the performance ratio attains its maximum value, $\varphi(\hat{\lambda}, \hat{\lambda}) = 1$. We now show that $\varphi(\hat{\lambda}, \cdot)$ is quasiconcave, which implies the representation of the performance index in Eq. (8). For this, consider first

$$\frac{\partial\varphi(\hat{\lambda}, \lambda)}{\partial\lambda} \propto \frac{\delta_{\hat{\lambda}}}{\pi^*_\lambda}\left(n(\delta_{\hat{\lambda}})e^{-\lambda n(\delta_{\hat{\lambda}})} - \tfrac{\delta_\lambda}{\delta_{\hat{\lambda}}} n(\delta_\lambda)e^{-\lambda n(\delta_\lambda)}\varphi(\hat{\lambda}, \lambda)\right),$$

where $\pi^*_\lambda = \pi_\lambda(\delta_\lambda)$. As a direct consequence,

$$\mathrm{sgn}\left(\frac{\partial\varphi(\hat{\lambda}, \lambda)}{\partial\lambda}\right) = \mathrm{sgn}\left(\frac{\lambda n(\delta_{\hat{\lambda}})}{e^{\lambda n(\delta_{\hat{\lambda}})} - 1} - \frac{\lambda n(\delta_\lambda)}{e^{\lambda n(\delta_\lambda)} - 1}\right).$$

On the other hand, the function $x \mapsto x/(\exp(x) - 1)$ is decreasing, for all $x > 0$. Thus, by setting $x = \lambda n(\delta_\lambda)$ and $\hat{x} = \lambda n(\delta_{\hat{\lambda}})$, and realizing that $(\hat{\lambda} - \lambda)(\hat{x} - x) \leq 0$ by Prop. 1, one obtains that the slope of $\varphi(\hat{\lambda}, \cdot)$ has a single-crossing property,

$$(\hat{\lambda} - \lambda)\frac{\partial\varphi(\hat{\lambda}, \lambda)}{\partial\lambda} \geq 0, \quad \lambda \in \Lambda,$$

which in turn yields that $\varphi(\hat{\lambda}, \cdot)$ is quasiconcave. □

B. Robust Parameter-Estimation Problem

The robust parameter-estimation problem becomes

$$\hat{\lambda}^* \in \arg\max_{\hat{\lambda} \in \bar{\Lambda}} \rho(\hat{\lambda}), \quad (9)$$

where $\bar{\Lambda}$ denotes the closure of Λ. The *optimal performance index*, $\rho^* = \rho(\hat{\lambda}^*)$, provides a relative performance guarantee, in the sense that assuming $\hat{\lambda}^*$ as the true parameter leads to a maximum relative loss (with respect to an ex-post optimal payoff) that cannot exceed $1 - \rho^*$. For example, if $\rho^* = 70\%$, then the implied optimal battery discharge policy yields a lifetime payoff $\hat{\pi}^* = \pi_{\hat{\lambda}^*}(\delta_{\hat{\lambda}^*})$ that is within 30% of the ex-post optimal lifetime payoff $\pi^*_\lambda = \pi_\lambda(\delta_\lambda)$, for all possible parameter values $\lambda \in \Lambda$. The next result characterizes an optimal robust parameter estimate.

Proposition 3. *The parameter* $\hat{\lambda}^* \in \bar{\Lambda} = [\lambda_1, \lambda_2]$ *solves the robust parameter-estimation problem (9) if and only if*

$$\varphi(\hat{\lambda}^*, \lambda_1) = \varphi(\hat{\lambda}^*, \lambda_2). \quad (10)$$

Proof. Fix $\lambda \in \Lambda$. Consider first the slope of the performance ratio with respect to the candidate parameter $\hat{\lambda} \in \Lambda$,

$$\frac{\partial\varphi(\hat{\lambda}, \lambda)}{\partial\hat{\lambda}} \propto \frac{\hat{\delta}'}{\pi^*_\lambda}\left(1 - e^{-\lambda n(\delta_{\hat{\lambda}})} + \delta_{\hat{\lambda}}\lambda n'(\delta_{\hat{\lambda}})\right),$$

where $\hat{\delta}' = \partial\delta_{\hat{\lambda}}/\partial\hat{\lambda} \geq 0$ by Prop. 1, and where $\pi^*_\lambda = \pi_\lambda(\delta_\lambda)$ as before. Provided an interior solution $\delta_{\hat{\lambda}} \in (0, 1)$,[6] the optimality condition (5), evaluated at the candidate parameter $\hat{\lambda}$ (instead of the unknown true parameter λ) yields

$$-\hat{\lambda}\delta_{\hat{\lambda}} n'(\delta_{\hat{\lambda}}) = 1 - \exp(-\hat{\lambda} n(\delta_{\hat{\lambda}})).$$

Hence, we can conclude that

$$\mathrm{sgn}\left(\frac{\partial\varphi(\hat{\lambda}, \lambda)}{\partial\hat{\lambda}}\right) = \mathrm{sgn}\left(\frac{1 - e^{-\lambda n(\delta_{\hat{\lambda}})}}{\lambda n(\delta_{\hat{\lambda}})} - \frac{1 - e^{-\hat{\lambda} n(\delta_{\hat{\lambda}})}}{\hat{\lambda} n(\delta_{\hat{\lambda}})}\right),$$

as long as $\hat{\delta}' > 0$, and otherwise a slope of zero. But this implies that

$$(\hat{\lambda} - \lambda)\frac{\partial\varphi(\hat{\lambda}, \lambda)}{\partial\hat{\lambda}} \geq 0,$$

[3] See [18, Sec. 1.3] for details on the (affine) extension of the real numbers, $\bar{\mathbb{R}} = [-\infty, \infty]$.

[4] For $\lambda_2 = \infty$ the limiting value, $\delta_\infty = \lim_{\lambda\to\infty} \delta_\lambda$, exists because of the monotone-convergence theorem for sequences [15, p. 55] (given that the δ_λ-values are uniformly bounded by 1).

[5] Naturally, 1 minus the performance index corresponds to the (maximal) *relative regret*, $1 - \rho(\hat{\lambda}) = \sup_{\lambda\in\Lambda}\big(\pi_\lambda(\delta_\lambda) - \pi_\lambda(\delta_{\hat{\lambda}})\big)/\pi_\lambda(\delta_\lambda) \in [0, 1]$.

[6] For $\delta_{\hat{\lambda}} = 1$, it is $\hat{\delta}' = 0$, so $\varphi(\cdot, \lambda)$ becomes (locally) constant.

for all $\hat{\lambda} \in \Lambda$. As a result,

$$\frac{\partial \varphi(\hat{\lambda}, \lambda_1)}{\partial \hat{\lambda}} \leq 0 \leq \frac{\partial \varphi(\hat{\lambda}, \lambda_2)}{\partial \hat{\lambda}}; \quad (11)$$

that is, the boundary performance ratios exhibit countervailing monotonicities in the candidate parameter $\hat{\lambda} \in \Lambda$, with $\varphi(\cdot, \lambda_1)$ being nonincreasing and $\varphi(\cdot, \lambda_2)$ being nondecreasing. Their difference,

$$D(\hat{\lambda}) = \varphi(\hat{\lambda}, \lambda_2) - \varphi(\hat{\lambda}, \lambda_1), \quad (12)$$

must therefore be nondecreasing in $\hat{\lambda} \in \Lambda$. Note further that $D(\lambda_1) = \varphi(\lambda_1, \lambda_2) - 1 \leq 0$ and $D(\lambda_2) = 1 - \varphi(\lambda_2, \lambda_1) \geq 0$. Since $D(\cdot)$ is continuous, by the intermediate-value theorem [15, p. 93], there exists $\hat{\lambda}^* \in \Lambda$ such that $D(\hat{\lambda}^*) = 0$. By the representation of the performance index in Eq. (8) of Prop. 2, one obtains

$$\begin{aligned} \rho(\hat{\lambda}) &= \min\left\{0, D(\hat{\lambda})\right\} + \varphi(\hat{\lambda}, \lambda_1) \\ &= \min\left\{-D(\hat{\lambda}), 0\right\} + \varphi(\hat{\lambda}, \lambda_2), \end{aligned}$$

for all $\hat{\lambda} \in \Lambda$. But this means,

$$\rho(\hat{\lambda}) = \begin{cases} \varphi(\hat{\lambda}, \lambda_1), & \text{if } D(\hat{\lambda}) \geq 0, \\ \varphi(\hat{\lambda}, \lambda_2), & \text{if } D(\hat{\lambda}) < 0. \end{cases}$$

"$\Rightarrow$:" We can now establish the *sufficiency* of Eq. (10). By virtue of Eq. (11), the performance index $\rho(\cdot)$ is nondecreasing on $[\lambda_1, \hat{\lambda}^*]$ and nonincreasing on $[\hat{\lambda}^*, \lambda_2]$, which implies that $\hat{\lambda}^*$ solves the robust parameter-estimation problem (9).
"$\Leftarrow$:" If for a solution $\hat{\lambda}^*$ of the robust parameter-estimation problem (9) we have $D(\hat{\lambda}^*) > 0$, then by Eq. (8) it is $\rho^* = \rho(\hat{\lambda}^*) = \varphi(\hat{\lambda}^*, \lambda_1) < \varphi(\hat{\lambda}^*, \lambda_2)$. But by the countervailing monotonicities of the boundary performance ratios established earlier, there exists a $\hat{\lambda}^{**} \in (\lambda_1, \hat{\lambda}^*)$, so that $\varphi(\hat{\lambda}^{**}, \lambda_1) = \varphi(\hat{\lambda}^{**}, \lambda_2) > \rho^*$, which is a contradiction to ρ^* being the optimal performance index. Considering the possibility that $D(\hat{\lambda}^*) < 0$ leads to a similar impossibility. Hence, $D(\hat{\lambda}^*) = 0$ must hold, thus establishing the *necessity* of Eq. (10) for any solution $\hat{\lambda}^*$ to the robust parameter-estimation problem (9). This concludes our proof. □

Example 3. Consider a user without any prior knowledge about the capacity-shrinkage rate, corresponding to $\Lambda = \mathbb{R}_+$, so $\lambda_1 = 0$ and $\lambda_2 = \infty$. By Prop. 2 the performance index is

$$\rho(\hat{\lambda}) = \min\left\{\varphi(\hat{\lambda}, 0), \varphi(\hat{\lambda}, \infty)\right\} = \min\left\{\frac{\delta_{\hat{\lambda}} n(\delta_{\hat{\lambda}})}{\delta_0 n(\delta_0)}, \delta_{\hat{\lambda}}\right\}.$$

Hence, Eq. (10) in Prop. 3 implies that the optimal robust discharge rate is

$$\hat{\delta}^* = \delta_{\hat{\lambda}^*} = n^{-1}(\delta_0 n(\delta_0)),$$

taking into account that $n(\cdot)$ is invertible. In the setting of Exs. 1 and 2, this result specializes to

$$\hat{\delta}^* = \min\left\{\delta_0 - \frac{\ln \delta_0}{\kappa}, 1\right\} = \min\left\{\frac{1 + [\ln \kappa]_+}{\kappa}, 1\right\} \geq \delta_0,$$

where $\delta_0 = \min\{1/\kappa, 1\}$, as determined in Ex. 1. The optimal robust discharge rate $\hat{\delta}^*$ is decreasing in the cycle-discharge decay rate $\kappa > 1$ (while it is constant, $\hat{\delta}^* = 1$, for $\kappa \leq 1$).

IV. Cost of Robustness

To examine the consequences of shifts in the prior knowledge about the unknown capacity-shrinkage parameter λ, we assume that the bounds of Λ are strictly positive and finite, so $0 < \lambda_1 < \lambda_2 < \infty$. The following result summarizes the comparative statics.

Proposition 4. *(i) The optimal performance index ρ^* is nondecreasing in λ_1 and nonincreasing in λ_2. (ii) The optimal robust parameter $\hat{\lambda}^*$ is nondecreasing in the boundaries λ_1 and λ_2 of the parameter space $\Lambda = (\lambda_1, \lambda_2)$.*

Proof. (i) Fixing one of the two bounds of Λ, say λ_i, and varying λ_j (with $j = 3 - i$) to its new value λ_j' (without crossing λ_i) immediately implies the conclusion based on whether the new domain Λ' is a subset of Λ or the other way around, based on the fact that the worst-case performance ratio in Eq. (7), and thus the performance index, can only increase when the new optimization domain is a subset of the previous optimization domain. (ii) By Prop. 3 any optimal robust parameter $\hat{\lambda}^*$ satisfies Eq. (10). Differentiating this relation with respect to λ_i, for $i \in \{1, 2\}$, yields

$$D'(\hat{\lambda}^*) \frac{\partial \hat{\lambda}^*}{\partial \lambda_1} = \frac{\partial \varphi(\hat{\lambda}^*, \lambda_1)}{\partial \lambda} \quad (13)$$

and

$$-D'(\hat{\lambda}^*) \frac{\partial \hat{\lambda}^*}{\partial \lambda_2} = \frac{\partial \varphi(\hat{\lambda}^*, \lambda_2)}{\partial \lambda}, \quad (14)$$

where $D(\hat{\lambda})$ is the difference of the boundary performance ratios in Eq. (12). As shown in the proof of Prop. 3, it is $D'(\hat{\lambda}) \geq 0$ for all $\hat{\lambda} \in \Lambda$. Furthermore, by the quasiconcavity of $\varphi(\hat{\lambda}^*, \cdot)$, established in the proof of Prop. 2,

$$\frac{\partial \varphi(\hat{\lambda}^*, \lambda_2)}{\partial \lambda} \leq 0 \leq \frac{\partial \varphi(\hat{\lambda}^*, \lambda_1)}{\partial \lambda},$$

so that Eqs. (13) and (14) together imply [7]

$$\frac{\partial \hat{\lambda}^*}{\partial \lambda_i} \geq 0,$$

for $i \in \{1, 2\}$, thus proving our claim. □

Part (i) of Prop. 4 says that the optimal performance guarantee ρ^* improves whenever Λ shrinks. The gradient,

$$\frac{\partial \rho^*}{\partial \lambda_i} = \frac{\partial \varphi(\hat{\lambda}^*, \lambda_i)}{\partial \lambda_i}, \quad i \in \{1, 2\}, \quad (15)$$

represents the firm's (relative) "cost of robustness," measured as change in the performance index induced by a small deformation of the parameter space.[8] As might have been instinctively obvious from the outset: *additional parameter ambiguity lowers the performance guarantee.* Indeed, by quantifying this intuition Eq. (15) provides an indicator for the

[7] By the characterization of the optimal robust parameters in Prop. 3, naturally $D'(\hat{\lambda}^*) = 0$ implies that the slope of $\hat{\lambda}^*$ with respect to λ_i also vanishes (for $i \in \{1, 2\}$).

[8] The "price of robustness" was popularized by [19]. The specific measures employed may naturally vary across applications.

value of additional data, so as to inform decisions about conducting further costly battery-discharge experiments that could tighten the parameter space Λ. Part (ii) of Prop. 4 states that when the lower boundary λ_1 of Λ increases, for example, as a consequence of additional measurements, then the optimal robust parameter $\hat{\lambda}^*$ also increases (at least weakly), and so does (by Prop. 1) the optimal robust depth of discharge $\hat{\delta}^* = \delta_{\hat{\lambda}^*}$, with $\hat{\lambda}^*$ in Eq. (9). *Ceteris paribus*, an increase of the upper boundary λ_2 has qualitatively the same effect.

V. Numerical Example

Consider an electric vehicle (EV) whose (hypothetical lead-acid) battery has the nominal capacity $K_0 = 100$ kWh, which converts electric energy at the rate $\phi = 0.2$ kWh/km into distance travelled. The user's value of transportation is $v = 0.50$ \$/km, while the cost of electricity is given by $c = 0.25$ \$/kWh. The dollar-value of a full nominal charge becomes $\eta = (v - c\phi)K_0/\phi = \$225$. Thus, the user's lifetime payoff, as a function of the depth of discharge $\delta \in [0, 1]$, becomes

$$\pi_\lambda(\delta) = \eta\delta \frac{1 - \exp(-\lambda n(\delta))}{1 - \exp(-\lambda)},$$

where the capacity-shrinkage rate $\lambda \in \Lambda$ in Eq. (3) is unknown to the user. For the cycle-discharge curve $n(\cdot)$ the user identifies an exponential dependency $n(\delta) = n_0 \exp(-\kappa\delta)$ via least-squares regression, with $n_0 = \exp(8.9569) \approx 7761$ and $\kappa = 3.127997$, based on the measurements in Table I.[9]

TABLE I
Empirical Cycle-Discharge Curve[a]

Depth-of-Discharge (δ)	Cycle Counts	
	n	$\ln n$
0.05	15000	9.6158
0.1	7000	8.8537
0.2	3300	8.1017
0.3	2050	7.6256
0.4	1475	7.2964
0.5	1150	7.0475
0.6	950	6.8565
0.7	780	6.6593
0.8	675	6.5147
0.9	590	6.3801
1	500	6.2146

[a]Measurements for a lead-acid battery [20].

From the expression in Ex. 3 we can conclude that the optimal robust discharge rate is $\hat{\delta}^* = 0.6843$, leading to an optimal performance index of the same value: $\rho^* = 68.43\%$. The corresponding optimal robust parameter, determined so that $\delta_{\hat{\lambda}^*} = \hat{\delta}^*$, is $\hat{\lambda}^* = 0.001499$.

VI. Conclusion

For practical applications, it is useful to determine robust decision rules without having to rely on very complex models, for which the data and boundary conditions might be very difficult to determine. The present research, by focusing on a simple, imperfectly known relationship, namely the dependence of capacity shrinkage on the depth of discharge, examines a relatively robust approach which provides a performance guarantee. In a numerical example, we obtain a performance index of close to 70% for a robust parameter estimate without any prior knowledge about its range (other than that the capacity-shrinkage rate is positive). This means that no matter what the true parameter might be, the proposed optimal robust depth of discharge achieves a lifetime payoff within about 30% of what could be achieved with perfect ex-post knowledge about the parameter value. The optimization model here may therefore serve as a building block for larger system models that need to rely on less data and still remain useful in practice.

[9]This numerical example serves as a proof of concept. A similar exercise can be performed with appropriate data for other kinds of batteries (see, e.g., [4, p. 351]), such as lithium-ion which is the type usually deployed in EVs.

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