

Fair Allocation of an Exhaustible Resource

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Abstract—There is little agreement on the social objective that should guide the intergenerational allocation of an exhaustible resource. Thus, remaining agnostic about a broad class of possible societal goals, we propose assessing feasible consumption paths by a *fairness index* that measures the worst relative performance across generations. Each generation's performance ratio compares its attained welfare with the maximum welfare achievable for that generation in isolation. The resulting fairness criterion is robust with respect to a broad canonical class of social objectives (symmetric, increasing, and concave). The fair consumption path, which maximizes the fairness index subject to basic economic consistency, is characterized by three primitive requirements: implementability (a nonincreasing path), exhaustiveness (full use of the resource), and balancedness (equalized ratios across cohorts). Together, these conditions yield a unique allocation that is Pareto-efficient, Lorenz-undominated, and guarantees strictly positive consumption for all future generations. In the benchmark case of constant absolute risk aversion (CARA), the model admits a semi-closed-form solution, including an explicit value function and a unique fairness level. The proposed criterion provides a transparent standard, without the need for society to endorse any particular social welfare function, while guaranteeing even-handed treatment to all cohorts.

Keywords—Fairness, nonrenewable resources, optimal control, resource allocation, robust optimization

I. INTRODUCTION

A central difficulty in intertemporal resource allocation is that distinct social welfare functions justify different time profiles of consumption. Utilitarian and egalitarian (Rawlsian) objectives often point in opposite directions, and any Pareto-efficient allocation can be supported as the maximizer of a suitably chosen social objective. This variety makes it difficult to identify a single, defensible rule for sharing an exhaustible resource across generations (see, e.g., Rawls [1]; Dasgupta [2]).

Building on Lorenz's pathbreaking analysis of inequality and its characterization by majorization (Lorenz [3]; Hardy et al. [4]; Atkinson [5]), Goel et al. [6] propose a performance-based approach that evaluates an allocation by its worst relative welfare across a large canonical class of social objectives. These objectives are symmetric, increasing, and concave, encompassing the standard utilitarian and Rawlsian cases. Maximizing the worst relative performance over the feasible set, subject, where necessary, to context-specific implementability conditions, yields Pareto-efficient, Lorenz-undominated allocations without committing to any single welfare function.

We extend this idea to an intertemporal setting in which overlapping generations draw on a common, exhaustible resource. A feasible plan is a nonincreasing consumption path, reflecting the fact that postponement is costly under discounting. Each generation's *performance ratio* compares its achieved welfare with the maximum welfare achievable for that generation if treated in isolation. The *fairness index*, defined as the infimum of these performance ratios across all generations, measures the minimal relative welfare guarantee implied by a given consumption path. The fair allocation maximizes this index subject to nonwastefulness.

Three basic conditions jointly determine the fair path. *Implementability* demands that consumption be nonincreasing over time. *Exhaustiveness* requires full use of the resource in the limit. *Balancedness* equalizes, across all generations, the ratio of attained welfare to the generation-specific benchmark. If any cohort enjoyed a strictly higher relative welfare than the least favored, a small reallocation from the former to the latter would raise the minimum ratio, contradicting optimality.

Methodologically, our analysis extends a static fairness concept to a dynamic optimal-control framework. The value function from the single-generation depletion problem provides the benchmark for the intertemporal tradeoff between scarcity and discounting. The fairness criterion imposes a balancedness condition for intergenerational comparisons in terms of welfare achieved relative to the benchmark under exclusive treatment. This leads to a characterization of fair allocations, which are naturally Pareto-efficient, Lorenz-undominated, and relatively robust with respect to canonical social welfare functions.

The approach is illustrated under constant absolute risk aversion (CARA), where the classical optimal policy has a closed-form value function and a linear consumption decline to zero within a finite horizon. The fairness requirement reallocates consumption from early to later generations while preserving feasibility and monotonicity, resulting in an optimal fairness index: a performance ratio—common across cohorts.

The remainder of the paper is organized as follows. Section II formulates the resource-exploitation problem and the generation-specific benchmark. Section III derives the classical optimal consumption path and its value function. Section IV introduces the fairness index and establishes balancedness. Section V characterizes the fair allocation in the constant-risk-aversion case. Section VI concludes with implications and extensions.

II. EXPLOITATION OF A NONRENEWABLE RESOURCE

Consider a continuum of overlapping generations, indexed by birth time $s \geq 0$, that jointly consume a nonrenewable resource. Given the amount $x_s > 0$ available at time $t = s$, and a social discount rate $r > 0$, an admissible (bounded, measurable) collective consumption path $c(t)$, for $t \in [s, \infty)$, yields the generation- s lifetime benefit (as seen from the “present,” at $t = 0$),¹

$$P_s(c|x_s) = \int_s^\infty e^{-rt} U(c(t)) \mathbf{1}_{x(t)>0} dt, \quad (1)$$

where $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuously differentiable, strictly increasing, and strictly concave, normalized by $U(0) = 0$. The resource stock evolves as

$$\dot{x}(t) = -c(t) \cdot \mathbf{1}_{x(t)>0}, \quad x(s) = x_s, \quad t \geq s, \quad (2)$$

so that $x(t)$ denotes the available quantity at time t . Once exhausted, no further utility can be derived. This standard specification follows the nonrenewable resource models of Hotelling [7] and Solow [8].

Equation (1) measures the attainable welfare of generation s as the discounted utility flow derived from the joint resource use, conditional on the stock remaining positive. The law of motion (2) captures the essential intertemporal tradeoff: consumption today irreversibly diminishes the productive base of tomorrow. The economy therefore moves along a one-dimensional frontier between present utility and future capacity, with the social discount rate r mediating this exchange.

Remark 1 (Stationarity). For transparency, we assume that at every time $t \in \mathbb{R}_+$ the birth rate equals the average death rate λ (a constant hazard rate), yielding a stationary population of normalized size 1.² Because expected lifetimes equal $1/\lambda$, the social discount rate must include λ , implying $r = r_0 + \lambda$.

Remark 2 (Social Discounting). The interpretation of the social discount rate has long been debated in intertemporal welfare analysis (see, e.g., [9]–[12]). Arrow et al. [13] emphasize a complementary view: social discounting should preserve inclusive wealth (i.e., the aggregate of reproducible, human, and natural capital), so that future generations inherit undiminished productive capacity. Within our framework, r is thus interpreted as an ethically adjusted rate balancing present consumption against intergenerational equity.

This formulation provides a minimal yet complete environment for intertemporal fairness. Each feasible path $c(\cdot)$ traces a trajectory of collective welfare across generations, constrained only by the simple but irreversible law of depletion (2). The planner’s problem is to select among these trajectories one that respects feasibility while embodying an equitable sharing of opportunity over time.

¹The reference time is immaterial; identical results obtain under any consistent normalization.

²Exponential population (de-)growth at rate $\gamma < r_0 + \lambda$ can be absorbed into the discount rate, so $r = r_0 + \lambda + \gamma > 0$.

III. OPTIMAL GENERATION- s CONSUMPTION

An admissible consumption path $c(\cdot) = c_s(\cdot|x_s)$ is an *optimal generation- s consumption path*, if it solves the optimal control problem (OCP)

$$P_s(c|x_s) \longrightarrow \max, \quad \text{s.t. (1)–(2), } c(t) \in [0, \bar{c}], \quad t \geq s, \quad (3)$$

with $\bar{c} > 0$ being a given upper consumption bound.³

A. *Special Case: $s = 0$*

Each generation faces essentially the same consumption problem, whose solution depends on the quantity of the resource available at birth. We therefore consider the initial generation, setting $s = 0$. To ensure nonnegativity of the resource in (2), we convert problem (3) into a finite-horizon OCP with free end time T :

$$J(T, x, u|x_0) = \int_0^T e^{-rt} u(t) dt \longrightarrow \max, \quad (4)$$

subject to

$$\dot{x} = -g(u), \quad x(0) = x_0, \quad x(T) = 0, \quad T \text{ free}, \quad (5)$$

and

$$u \in [0, \bar{u}], \quad (6)$$

where u is a measurable control, with upper bound $\bar{u} = U(\bar{c})$, and $g = U^{-1}$ is therefore strictly increasing and strictly convex (by the assumptions on U), with $g(0) = 0$.

The problem admits a particularly simple structure. Because utility is discounted and the resource is exhaustible, it is optimal to consume at a declining rate until the resource is fully depleted at some finite time T , after which consumption is zero. The optimal policy can therefore be characterized by a depletion phase followed by inactivity, with the switching time T determined endogenously.

Proposition 1. *For any initial resource stock $x_0 > 0$, there exists an optimal generation-0 consumption path with finite exhaustion time $T > 0$, and it is unique almost everywhere. It is given by*

$$c_0(t|x_0) = g(u^*(t)), \quad t \geq 0,$$

where

$$u^*(t) = h(e^{-rt}/\psi) \mathbf{1}_{t \leq T}, \quad t \geq 0,$$

with $h = (g')^{-1}$ defined on the range of g' and the constants $\psi, T > 0$ determined by Eqs. (11) and (12).

Proof. Existence. The control set $[0, \bar{u}]$ is compact and the velocity set $\{-g(u) : u \in [0, \bar{u}]\}$ is compact and convex. The dynamics in (5) satisfy the Carathéodory conditions, and the running payoff is measurable and bounded on finite intervals. Hence, by standard existence results for free-terminal-time optimal control problems (see, e.g., Filippov [14], Vinter [15, Ch. 8]), there exists an optimal triple (T, x^*, u^*) .

Characterization and uniqueness. Under the standing assumption that $\bar{c}$ is sufficiently large, the upper control bound

³To avoid inessential complications, $\bar{c}$ is assumed large enough; indeed, by the strict concavity of U , the optimal consumption rate is naturally bounded.

does not bind along the optimal path. By the Pontryagin maximum principle (Pontryagin et al. [16]; Weber [17]), an optimal pair (x^*, u^*) admits an adjoint $\psi(\cdot)$ and u^* maximizes the Hamiltonian

$$H(t, u, \psi) = e^{-rt}u - \psi g(u) \quad (7)$$

pointwise (a.e.). The adjoint equation $-\dot{\psi} = \partial_x H = 0$ implies that ψ is constant. Economically, ψ represents the shadow value of the remaining resource stock and governs the tradeoff between current utility and future scarcity.

Since g is strictly convex and $\psi > 0$ along a depletion arc, the Hamiltonian is strictly concave in $u \in [0, \bar{u}]$.⁴ Therefore, for almost every t , the maximizer

$$u^*(t) \in \arg \max_{u \in [0, \bar{u}]} \{e^{-rt}u - \psi g(u)\} \quad (8)$$

is unique. For $\psi > 0$, the first-order condition yields

$$e^{-rt} - \psi g'(u) = 0,$$

so that $u = h(e^{-rt}/\psi)$ whenever the interior solution lies in $[0, \bar{u}]$, where $h = (g')^{-1}$ is strictly increasing on the range of g' . Therefore the optimal control is

$$u^*(t) = h(e^{-rt}/\psi) \mathbf{1}_{t \leq T}, \quad t \geq 0, \quad (9)$$

and the corresponding optimal resource trajectory is

$$x^*(t) = x_0 - \int_0^t g(u^*(\theta)) d\theta, \quad t \geq 0, \quad (10)$$

so the optimal consumption path is $c_0(t|x_0) = g(u^*(t))$ on $[0, T]$ and 0 thereafter.

The free-terminal-time transversality condition gives $H(T, u^*(T), \psi) = 0$, which yields

$$(e^{-rT}/\psi) h(e^{-rT}/\psi) = g(h(e^{-rT}/\psi)). \quad (11)$$

The state constraint $x(T) = 0$ implies the resource-exhaustion condition

$$\int_0^T g(h(e^{-rt}/\psi)) dt = x_0. \quad (12)$$

These two equations determine (ψ, T) uniquely.

On $\{t : x^*(t) = 0\}$ admissibility enforces $u^*(t) = 0$ (a.e.). Hence, the optimal control is unique up to a null set, and so is the induced consumption path. $\square$

Since e^{-rt} is strictly decreasing and h is increasing, it follows that u^* is decreasing over time. Thus, the largest resource consumption occurs at the beginning:

$$c_0(0|x_0) = \sup_{t \geq 0} c_0(t|x_0).$$

In other words, the optimal policy places greater weight on early consumption, reflecting the combined effects of discounting and resource scarcity.

⁴Indeed, if $\psi = 0$, then $H(t, u, \psi) = e^{-rt}u$, so the unique maximizer is $u^*(t) = \bar{u}$ for a.e. $t \in [0, T]$, which implies $H(T, u^*(T), \psi) = e^{-rT}\bar{u} > 0$, contradicting the free-terminal-time transversality condition $H(T, u^*(T), \psi) = 0$. Thus, ψ must be strictly positive at the optimum.

B. General Case: $s \geq 0$

It is straightforward to generalize our solution to the optimal consumption problem (3) for the first generation ($s = 0$), to future generations $s \geq 0$ starting with the resource stock x_s .

Proposition 2. *The (unique) optimal generation- s consumption path is $c_s(t|x_s) = c_0(t-s|x_s)$ for all $t \geq s \geq 0$.*

Proof. Fix $s \geq 0$ and an initial stock $x_s > 0$. For any admissible $c(\cdot)$ on $[s, \infty)$, define the time shift

$$\tilde{c}(\tau) = c(s + \tau), \quad \tilde{x}(\tau) = x(s + \tau), \quad \tau \geq 0.$$

Then $\tilde{x}$ satisfies the same law of motion as in Eq. (5),

$$\dot{\tilde{x}}(\tau) = -\tilde{c}(\tau) \mathbf{1}_{\tilde{x}(\tau) > 0}, \quad \tilde{x}(0) = x_s,$$

and, by the change of variables $\tau = t - s$,

$$P_s(c|x_s) = e^{-rs} \int_0^\infty e^{-r\tau} U(\tilde{c}(\tau)) \mathbf{1}_{\tilde{x}(\tau) > 0} d\tau.$$

Hence, the generation- s problem is identical to the generation-0 problem with initial stock x_s , up to the positive multiplicative constant e^{-rs} , which does not change the solution to the variational problem. By uniqueness of the optimal solution for $s = 0$, the optimal tail control is $\tilde{c}(\tau) = c_0(\tau|x_s)$ (a.e.), whence $c_s(t|x_s) = c_0(t-s|x_s)$ for $t \geq s$. Uniqueness for generation s follows from uniqueness at $s = 0$. $\square$

C. Value Function

The value function $V_s(x) = P_s(c_s(\cdot|x)|x)$ of the optimal consumption problem (3) for generation s is multiplicatively related to that of the initial generation,

$$V_s(x) = e^{-rs} V(x), \quad s, x \geq 0, \quad (13)$$

where $V(x) = J(T^*, x^*, u^*|x)$ satisfies the Bellman equation

$$rV(x) = \max_{u \in [0, \bar{u}]} \{u - V'(x)g(u)\}, \quad x > 0, \quad (14)$$

with boundary condition $V(0) = 0$. The first-order condition for the maximization in Eq. (14) is

$$1 - V'(x)g'(u) = 0, \quad x > 0.$$

On the compact control set $[0, \bar{u}]$, this gives the optimal (utility-)feedback law⁵

$$u = \mu(x) \triangleq \begin{cases} \min\{h(1/V'(x)), \bar{u}\}, & x > 0, \\ 0, & x = 0, \end{cases} \quad (15)$$

where $h = (g')^{-1}$ and $g = U^{-1}$.

Proposition 3. *The value function $V : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing and strictly concave. The optimal feedback law $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing and concave whenever the map $u \mapsto g'(u)/(g(u)g''(u))$ is nonincreasing on $(0, \bar{u}]$.*

Proof. Properties of V . Monotonicity is immediate: if $x' > x \geq 0$, any admissible policy for x remains feasible for x' by inserting an initial idle interval, and strict monotonicity of U

⁵The corresponding consumption-feedback is $c = g(\mu(x))$.

implies $V(x') > V(x)$. At the maximizer $\mu(x)$, the Bellman equation (14) reads

$$rV(x) = \mu(x) - V'(x)g(\mu(x)), \quad x > 0.$$

By the envelope theorem,

$$rV'(x) = -V''(x)g(\mu(x)), \quad x > 0. \quad (16)$$

On any optimal depletion arc $\mu(x) > 0$, hence $g(\mu(x)) > 0$. Since $r > 0$ and $V'(x) > 0$, Eq. (16) yields $V''(x) < 0$. Thus, V is strictly concave on $\mathbb{R}_{++}$, with $V(0) = 0$.

Properties of μ . For interior points, the first-order condition is

$$1 - V'(x)g'(\mu(x)) = 0, \quad x > 0. \quad (17)$$

Differentiating Eq. (17) gives

$$0 = V''(x)g'(\mu(x)) + V'(x)g''(\mu(x))\mu'(x),$$

so that

$$\mu'(x) = -\frac{V''(x)g'(\mu(x))}{V'(x)g''(\mu(x))} \geq 0, \quad (18)$$

because $V'' < 0 < g', g''$. Hence, μ is increasing wherever interior, and nondecreasing on $[0, \bar{u}]$ when the constraint binds.

Combining Eqs. (16) and (18) eliminates V'' :

$$\mu'(x) = r \frac{g'(\mu(x))}{g(\mu(x))g''(\mu(x))}, \quad x > 0. \quad (19)$$

If $u \mapsto g'(u)/(g(u)g''(u))$ is nonincreasing on $(0, \bar{u}]$, and since μ is nondecreasing, the right-hand side of (19) is non-increasing in x . Therefore $\mu''(x) \leq 0$ and μ is concave. This condition holds for the CARA benchmark used below. $\square$

Remark 3. Equation (19) shows that the comparative statics of μ depend solely on the curvature of $g = U^{-1}$. With $c = g(\mu(x))$, the right-hand side equals

$$-\frac{rU'(c)}{cU''(c)} = \frac{r}{\rho(c)}, \quad c > 0,$$

where $\rho(c) = -cU''(c)/U'(c)$ denotes the local measure of relative risk aversion. This expression is constant for constant relative risk aversion (CRRA) and decreasing for constant absolute risk aversion (CARA). More generally, μ remains concave for all utility functions exhibiting increasing relative risk aversion, since $1/\rho(c)$ is then nonincreasing in c . Thus, monotonicity and concavity of the feedback law follow directly from the curvature of U , without recourse to higher-order derivatives of V .

IV. FAIR CONSUMPTION

Future generations are more expensive to serve than the present generation because delaying consumption (or foregoing it altogether) entails an opportunity cost. This implies that generations are cost-ordered, just as in the example of Goel et al. [6], where farmers are located at different distances

from a water source with limited capacity, inducing higher evaporation losses to serve those farther away.⁶

A. Implementability and Resource Balance

Generations being necessarily cost-ordered leads to the *implementability condition*:

$$\hat{t} > t \Rightarrow c(\hat{t}) \leq c(t), \quad (20)$$

for all $t, \hat{t} \in \mathbb{R}_+$, meaning that any admissible consumption path $c \in L^\infty(\mathbb{R}_+)$ must be nonincreasing. If an implementable consumption path is used to maximize the lifetime benefit of a particular generation $s \geq 0$, then the best a planner can do for that generation—while maintaining implementability—is to keep consumption constant at the level $\bar{c}_s = g(\mu(\bar{x}_s))$, where $\bar{x}_s$ is the resource level at which the optimal generation- s consumption path $c_s(\cdot|\bar{x}_s)$ begins at time $t = s$.⁷ The maximum feasible stock $\bar{x}_s$ for generation s is determined (implicitly) by the *focused generation- s resource allocation*:

$$\bar{x}_s = x_0 - s \cdot c_0(0|\bar{x}_s), \quad s \geq 0. \quad (21)$$

It is important to note that $\bar{x}_s$, for $s \geq 0$, is *not* a feasible resource trajectory in itself, as it presumes a consumption path that preserves the resource solely to maximize the lifetime benefit of generation s , without regard to other generations—apart from satisfying the implementability condition (20).

B. Fairness Index

Consider any implementable consumption path c in $L^\infty(\mathbb{R}_+)$, which by Eq. (2) induces the resource trajectory⁸

$$x(t) = \left[x_0 - \int_0^t c(\theta) d\theta \right]_+, \quad t \geq 0. \quad (22)$$

The relative benefit achievement of c for a given generation s is measured by the *performance ratio*,

$$\varphi_s(c) = \frac{P_s(c|x_s)}{P_s^*} \in [0, 1], \quad (23)$$

where $P_s^* = V_s(\bar{x}_s) = e^{-rs}V(\bar{x}_s)$ (cf. Eq. (13)) is the optimal lifetime benefit of generation s when provided with the maximum amount $\bar{x}_s$ (cf. Eq. (21)) through maximal generosity of all other generations, and where $x_s = x(s)$ is the resource stock left for generation s under c (cf. Eq. (22)).

The overall achievement of the consumption path c is then summarized by the *fairness index*,

$$\phi(c) = \inf_{s \geq 0} \varphi_s(c) \in [0, 1], \quad (24)$$

⁶All else equal, it is always better to consume a marginal unit $\delta > 0$ of the resource today rather than tomorrow, since $U(c(t) + \delta) > e^{-rt}U(c(t) + \delta)$, for all $t > 0$, as a consequence of social discounting. The capital generated by the immediate use of δ can accrue interest and contribute to overall economic growth, which otherwise would have to be delayed.

⁷Since by Eq. (2) the resource trajectory is decreasing, the monotonicity of the optimal feedback $\mu(\cdot)$ (cf. Prop. 3) implies that the optimal generation- s consumption path is decreasing.

⁸For any $\xi \in \mathbb{R}$, it is $[\xi]_+ = \max\{0, \xi\}$.

which captures the minimum relative benefit attained across generations—that is, the welfare level of the least-favored generation under c .

Remark 4 (Lorenz Dominance). To clarify the normative power of the fairness index, consider a finite planning horizon $[0, \tau]$, where $\tau > 0$ is chosen large enough that the residual tail of the discounted utility flow beyond τ is negligible, that is, $\int_{\tau}^{\infty} e^{-rt} U(\bar{c}) dt \leq \varepsilon$, for some arbitrarily small $\varepsilon > 0$. Partition $[0, \tau]$ into N equal intervals of length $\Delta t = \tau/N$ and approximate the generation-0 lifetime benefit in Eq. (1) by its Riemann sum,

$$P_0(c) \approx \sum_{k=1}^N e^{-rt_k} U(c(t_k)) \Delta t,$$

where $t_k = k\Delta t$. Each term can be interpreted as the discounted welfare of the generation indexed by k . Because future generations consume later, they are systematically poorer in discounted utility terms.

We may thus associate with c the finite welfare vector

$$w^{(N, \tau)}(c) = \Delta t (e^{-rt_1} U(c(t_1)), \dots, e^{-rt_N} U(c(t_N))) \in \mathbb{R}_+^N.$$

Plotting the cumulative welfare of the k poorest components of $w^{(N, \tau)}(c)$ against k (with linear interpolation) yields a Lorenz curve for intergenerational inequality. Allocation $\hat{c}$ is said to *Lorenz-dominate* allocation c if the Lorenz curve of $w^{(N, \tau)}(\hat{c})$ lies nowhere below and somewhere strictly above that of $w^{(N, \tau)}(c)$. By the classical result of Hardy et al. [4] (cf. also [18]), this is equivalent to the ordering induced by all *canonical* social welfare functions, i.e., all functions on $\mathbb{R}_+^N$ that are *increasing*, *symmetric*, and *concave*. Thus, intergenerational fairness can be identified with the Lorenz criterion under finer and finer discretizations of time, and the fairness index in Eq. (24) is obtained as the continuous-time limit of the discrete fairness ratios (cf. [6, Prop. 3]):

$$\phi(c) = \inf_{s \geq 0} \frac{P_s(c|x_s)}{P_s^*}. \quad (25)$$

Equivalently, the right-hand side is the limit of the worst relative welfare over all canonical welfare aggregations of the finite vectors $w^{(N_m, \tau_m)}(c)$ along any refining sequence with $\tau_m \rightarrow \infty$ and $\tau_m/N_m \rightarrow 0$ as $m \rightarrow \infty$.⁹ This provides the Lorenz-welfare interpretation of intertemporal fairness.

C. Fair Consumption Problem

An admissible and implementable consumption path $\hat{c}(t)$, $t \geq 0$, is called *fair* if it solves the fair-consumption problem

$$\phi(c) \longrightarrow \max, \quad \text{s.t. (2), (20), (24)}. \quad (26)$$

The corresponding optimal value $\hat{\phi} = \phi(\hat{c})$ provides a lifetime-benefit guarantee for all generations, in the sense that

$$P_s(\hat{c}|\hat{x}_s) \geq \hat{\phi} P_s^*, \quad s \geq 0, \quad (27)$$

⁹For each integer $m \geq 1$, one considers the discrete class of increasing, symmetric, and concave welfare functions on $\mathbb{R}_+^{N_m}$. The associated discrete fairness ratios converge pointwise in c to the right-hand side of Eq. (25).

where $\hat{x}_s = x(s)$, obtained from Eq. (22) with $c = \hat{c}$ and $t = s$, denotes the (by construction feasible) fair resource trajectory.

D. Balancedness

The following balancedness condition provides the key to solving the fair-consumption problem (26), because it implies a law of motion up to a constant; that constant is then determined by full exhaustion of the initial resource stock.¹⁰

Proposition 4 (Balancedness). *If $\hat{c}$ is fair, then*

$$\varphi_s(\hat{c}) = \hat{\phi}, \quad s \geq 0, \quad (28)$$

where $\hat{\phi} = \phi(\hat{c}) \in (0, 1]$ is the optimal fairness index.

Proof. Let $\hat{c}$ be a fair consumption path and $\hat{\phi} = \phi(\hat{c})$ the associated optimal fairness index. We first note that $s \mapsto \varphi_s(\hat{c})$ is continuous on $\mathbb{R}_+$. Indeed,

$$x_s = \left[x_0 - \int_0^s \hat{c}(\theta) d\theta \right]_+$$

is continuous in s , hence

$$P_s(\hat{c}|x_s) = \int_s^\infty e^{-rt} U(\hat{c}(t)) \mathbf{1}_{x(t) > 0} dt$$

depends continuously on s by dominated convergence (boundedness of $U(\hat{c})$ and exponential discount). Moreover, $P_s^* = e^{-rs} V(\bar{x}_s)$ is continuous in s because $\bar{x}_s$ is continuous by implicit-function arguments applied to (21) and V is continuous. Therefore $\varphi_s(\hat{c}) = P_s(\hat{c}|x_s)/P_s^*$ is continuous.

Suppose, for contradiction, that there exists $\hat{s} \in \mathbb{R}_+$ with $\varphi_{\hat{s}}(\hat{c}) > \hat{\phi}$. By continuity there exist $\eta > 0$ and an interval

$$I = [\hat{s} - \varepsilon, \hat{s} + \varepsilon] \cap \mathbb{R}_+$$

such that

$$\varphi_s(\hat{c}) \geq \hat{\phi} + 3\eta, \quad s \in I.$$

At the same time, since $\hat{\phi} = \inf_{s \geq 0} \varphi_s(\hat{c})$, the near-minimizer set

$$\mathcal{S}_\eta = \{s \geq 0 : \varphi_s(\hat{c}) \leq \hat{\phi} + \eta\}$$

is nonempty. By continuity, every $s \in \mathcal{S}_\eta$ has a neighborhood on which $\varphi(\hat{c}) \leq \hat{\phi} + 2\eta$.

Now perform a Pigou–Dalton transfer across generations that preserves implementability: remove a sufficiently small total amount $\delta > 0$ of consumption from the interval I and redistribute the same amount over a sequence of later disjoint subintervals chosen so that every neighborhood intersecting $\mathcal{S}_\eta$ receives positive mass, with the weights summing to δ and decreasing fast enough to preserve the nonincreasing property of the path.¹¹ Denote the perturbed path by $\tilde{c}$.

¹⁰Intuitively, a fair path leaves no generation systematically better treated than the worst-off one, since any such slack could be reallocated to raise the minimum performance ratio.

¹¹This is the standard balancing construction for max-min fair allocations with cost-ordered agents; cf. Goel et al. [6]. Since $\hat{c}$ is nonincreasing, the perturbation can be chosen so that total resource use is unchanged and implementability is preserved.

Because the perturbation is positive on every neighborhood intersecting $\mathcal{S}_\eta$, continuity of $P_s(\cdot|x_s)$ implies that, for sufficiently small δ , all performance ratios on $\mathcal{S}_\eta$ increase above $\hat{\phi} + \eta$. On the other hand, the loss on I can be made arbitrarily small, so the performance ratios on I remain above $\hat{\phi} + 2\eta$. Outside $\mathcal{S}_\eta \cup I$, all performance ratios were already above $\hat{\phi} + \eta$ and are affected only negligibly. Hence, for sufficiently small δ , we obtain

$$\inf_{s \geq 0} \varphi_s(\tilde{c}) > \hat{\phi},$$

contradicting the optimality of $\hat{c}$. Therefore

$$\varphi_s(\hat{c}) \equiv \hat{\phi}, \quad s \geq 0.$$

To establish that $\hat{\phi} > 0$, consider the exponential path $c_\beta(t) = x_0 \beta e^{-\beta t}$, $t \geq 0$, which is admissible and implementable with full resource use. For all sufficiently small $\beta > 0$, one can show that $\phi(c_\beta) > 0$. Since $\hat{c}$ maximizes the fairness index ϕ , it follows that

$$\hat{\phi} \geq \phi(c_\beta) > 0.$$

Finally, $\hat{\phi} \leq 1$ by Eq. (23). Thus, $\hat{\phi} \in (0, 1]$. $\square$

E. Fair Consumption Path

By balancedness (cf. Prop. 4), there exists $\kappa \in (0, 1]$ with

$$P_s(c|x_s) = \kappa P_s^* = \kappa e^{-rs} V(\bar{x}_s), \quad s \geq 0,$$

whenever c is fair, so each generation attains the same fraction of its benchmark; differentiating and using Eq. (1) yields

$$-e^{-rs} U(c(s)) = \kappa e^{-rs} \left(V'(\bar{x}_s) \frac{d\bar{x}_s}{ds} - rV(\bar{x}_s) \right), \quad s \geq 0.$$

Thus, balancedness of a fair allocation amounts to

$$\frac{U(c(s))}{\kappa} = rV(\bar{x}_s) - V'(\bar{x}_s) \frac{d\bar{x}_s}{ds}, \quad s \geq 0, \quad (29)$$

where the missing constant $\kappa \in (0, 1]$ is determined by the *exhaustiveness condition*,¹²

$$x_0 = \int_0^\infty c(t) dt, \quad (30)$$

meaning that no part of the resource goes unconsumed. *The (unique) joint solution $(\hat{c}, \hat{\kappa})$ of Eqs. (29) and (30) determines the fair consumption path $\hat{c}$ and the optimal fairness index $\hat{\phi} = \phi(\hat{c}) = \hat{\kappa}$. Meanwhile, the term $\bar{x}_s$ evolves according to the focused-resource condition (21), which, by implicit differentiation, gives*

$$\frac{d\bar{x}_s}{ds} = -\frac{c_0(0|\bar{x}_s)}{1 + s \frac{d}{dx} \Big|_{x=\bar{x}_s} c_0(0|x)} < 0, \quad (31)$$

since $c_0(0|x) = g(\mu(x))$ is nondecreasing in $x \geq 0$, as $\mu' \geq 0$ by Eq. (18). Although $\bar{x}_s$ is not a feasible resource trajectory, its monotonic decline with s describes the *opportunity tradeoff* implicit in fair planning: as the fairness horizon extends, the maximum stock hypothetically assignable to later generations decreases, reflecting the growing cumulative opportunity cost of deferral rather than an actual depletion process.

¹²Exhaustiveness follows from asking for $\lim_{t \rightarrow \infty} x(t) = 0$ in Eq. (22).

V. EXAMPLE: CARA ECONOMY

Consider an economy with a representative agent who has a constant-absolute-risk-aversion (CARA) utility function $U(c) = 1 - e^{-\alpha c}$, a widely used specification in continuous-time consumption-savings models (cf. Merton [19]), where $\alpha > 0$ is the coefficient of absolute risk aversion. Then

$$g(u) = -\frac{\ln(1-u)}{\alpha}, \quad u \in [0, 1),$$

with a (positive, increasing) derivative,

$$g'(u) = \frac{1}{(1-u)\alpha}, \quad u \in [0, 1),$$

which possesses a (nonnegative) inverse,

$$h(z) = 1 - \frac{1}{\alpha z}, \quad z \geq 1/\alpha.$$

A. Optimal Consumption

By Eq. (9) together with the endpoint conditions (11)–(12), the optimal generation-0 utility path is

$$u^*(t) = h(e^{-rt}/\psi) = \left[1 - \frac{e^{rt}\psi}{\alpha} \right]_+ \in [0, 1).$$

Hence, the optimal consumption path becomes

$$g(u^*(t)) = \left(\frac{1}{\alpha} \right) \left[\ln \left(\frac{\alpha}{\psi} \right) - rt \right], \quad t \geq 0.$$

Full resource use implies

$$x_0 = \int_0^T g(u^*(t)) dt = \frac{1}{\alpha} \left(T \ln \frac{\alpha}{\psi} - \frac{rT^2}{2} \right). \quad (32)$$

The free-time transversality condition $(H(T, u^*(T), \psi) = 0$; cf. Prop. 1) yields

$$\left(\frac{e^{-rT}}{\psi} \right) \left(1 - \frac{e^{rT}\psi}{\alpha} \right) = \left(\frac{1}{\alpha} \right) \left[\ln \left(\frac{\alpha}{\psi} \right) - rT \right],$$

which simplifies to

$$\left(\frac{\alpha}{\psi} \right) e^{-rT} - 1 = \ln \left(\frac{\alpha}{\psi} \right) - rT. \quad (33)$$

From Eq. (32) we obtain

$$\frac{\alpha}{\psi} = \exp \left[\frac{\alpha x_0}{T} + \frac{rT}{2} \right]. \quad (34)$$

Substituting this into Eq. (33) yields

$$\exp \left[\frac{\alpha x_0}{T} - \frac{rT}{2} \right] = 1 + \frac{\alpha x_0}{T} - \frac{rT}{2},$$

equivalent to $\alpha x_0/T = rT/2$, so that the optimal exploitation horizon is

$$T = \sqrt{\frac{2\alpha x_0}{r}}.$$

Substitution into Eq. (34) gives the adjoint variable,

$$\psi = \alpha e^{-rT} = \alpha \exp[-\sqrt{2r\alpha x_0}],$$

implying the optimal utility path

$$u^*(t) = 1 - e^{-r[T-t]_+} \in [0, 1), \quad t \geq 0.$$

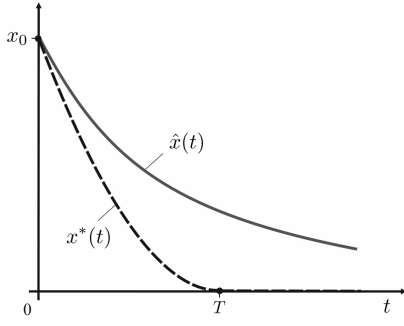


Fig. 1. Trajectory $x^*(t)$ under the optimal generation-0 path (exhausting the resource in finite time T) and $\hat{x}(t)$ under the fair path (never exhausting the resource), for $t \geq 0$ $[(x_0, \alpha, r) = (100, 0.5, 3\%)]$.

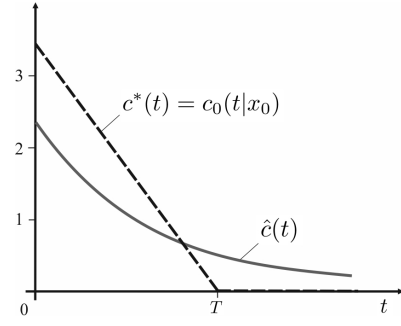


Fig. 2. Optimal generation-0 consumption path $c^*(t)$ (dropping to zero in finite time T) and strictly positive fair consumption path $\hat{c}(t)$, for $t \geq 0$ $[(x_0, \alpha, r) = (100, 0.5, 3\%)]$.

Taking into account that $r/\alpha = 2x_0/T^2$, the optimal resource trajectory in Eq. (10) becomes

$$x^*(t) = x_0 - \int_0^t g(u^*(\theta)) d\theta = x_0 \left[1 - \frac{t}{T} \right]_+^2, \quad t \geq 0.$$

Hence, the optimal consumption path (see Fig. 2)

$$c^*(t) = c_0(t|x_0) = g(u^*(t)) = \sqrt{\frac{2x_0 r}{\alpha}} \left[1 - \frac{t}{T} \right]_+, \quad t \geq 0, \quad (35)$$

drops linearly to zero on $[0, T]$. The value function is

$$V(x) = \int_0^\infty e^{-rt} u^*(t) dt = \frac{1}{r} - \left(\frac{1}{r} + T \right) e^{-rT}.$$

Substituting $T = \sqrt{2\alpha x/r}$ yields

$$V(x) = \frac{1}{r} \left(1 - \frac{1 + \sqrt{2r\alpha x}}{e^{\sqrt{2r\alpha x}}} \right) \in [0, 1/r), \quad (36)$$

with derivative

$$V'(x) = \alpha e^{-\sqrt{2r\alpha x}} \in (0, \alpha], \quad x \geq 0. \quad (37)$$

Hence, the optimal feedback law in Eq. (15) is

$$\mu(x) = 1 - e^{-\sqrt{2r\alpha x}}, \quad x \geq 0.$$

The trajectories $x^*(t)$ and $\hat{x}(t)$ are compared in Fig. 1. While $x^*(t)$ hits zero at the finite horizon T , the fair trajectory remains strictly positive for all t , reflecting the balanced use of the resource across generations.

B. Fair Consumption

For $t \geq s$, let $\bar{x}_s$ denote the resource trajectory implicitly defined by Eq. (21), which describes the tail problem starting at $s \geq 0$ with value $e^{-rs} V(\bar{x}_s)$. Since by Eq. (35)

$$c_0(0|\bar{x}_s) = \sqrt{\frac{2\bar{x}_s r}{\alpha}},$$

Eq. (21) yields

$$\bar{x}_s = x_0 - \left[\sqrt{\left(\frac{r}{\alpha} \right)^2 s^2 + 2 \left(\frac{r}{\alpha} \right) x_0} - \left(\frac{r}{\alpha} \right) s \right] s, \quad (38)$$

for all $s \geq 0$, so that $\bar{x}_0 = x_0$ and $\bar{x}_s \downarrow 0$ as $s \rightarrow \infty$. Combining Eq. (38) with Eqs. (36)–(37) gives

$$V(\bar{x}_s) = \frac{1}{r} \left(1 - \left(1 + \sqrt{2r\alpha \bar{x}_s} \right) e^{-\sqrt{2r\alpha \bar{x}_s}} \right),$$

with gradient

$$V'(\bar{x}_s) = \alpha e^{-\sqrt{2r\alpha \bar{x}_s}},$$

for all $s \geq 0$. By Prop. 4, the fair utility trajectory $\hat{u}(s)$ satisfies the balancedness condition (29) for some $\kappa \in (0, 1]$. Setting $\bar{y}_s = \sqrt{2r\alpha \bar{x}_s}$, this becomes

$$\frac{\hat{u}(s)}{\kappa} = 1 - \left(1 + \frac{(rs)\bar{y}_s}{\bar{y}_s + (rs)} \right) e^{-\bar{y}_s}, \quad s \geq 0.$$

If $\hat{c}$ denotes the fair consumption path (see Fig. 2), then the fairness level κ follows from the isoperimetric constraint

$$\begin{aligned} x_0 &= \int_0^\infty \hat{c}(s) ds = \int_0^\infty g(\hat{u}(s)) ds = C_\kappa \\ &= -\frac{1}{\alpha} \int_0^\infty \ln \left(1 - \kappa + \kappa \left(1 + \frac{(rs)\bar{y}_s}{\bar{y}_s + (rs)} \right) e^{-\bar{y}_s} \right) ds. \end{aligned}$$

The right-hand side C_κ increases strictly in $\kappa \in [0, 1]$ with $C_0 = 0$ and $C_1 > x_0$, ensuring a unique $\hat{\kappa} \in (0, 1)$ satisfying $C_{\hat{\kappa}} = x_0$. Fig. 3 shows the contrast between the performance ratio under the classical optimum, which collapses at T , and the fair trajectory, which maintains a constant $\varphi_s(\hat{c}) = \hat{\phi}$ for all generations.

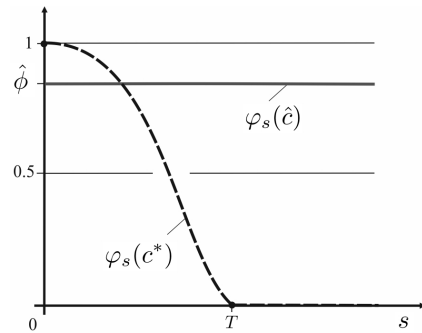


Fig. 3. Performance ratio $\varphi_s(c^*)$ under the optimal generation-0 path c^* (dropping to its vanishing fairness index in finite time T) and $\varphi_s(\hat{c})$ under the fair path $\hat{c}$ (constant at $\hat{\phi} \approx 0.84$), for $s \geq 0$ $[(x_0, \alpha, r) = (100, 0.5, 3\%)]$.

VI. CONCLUSION

The analysis has produced, as an evaluation criterion for consumption paths, a fairness index which is agnostic about the mode of social-welfare aggregation across a broad class of canonical (i.e., symmetric, increasing, and concave) social welfare functions. The canonical class, which includes—for example—both utilitarian and egalitarian social welfare functions, corresponds to the comparison domain that characterizes finite-dimensional Lorenz dominance. By continuous approximation we thus obtain a fair allocation rule that is relatively robust (cf. [20], [21]) in the sense that its worst-case relative performance weakly exceeds that of any other canonical welfare criterion.

Maximization of the fairness index, subject to economic consistency, is equivalent to three fundamental conditions—*implementability* (a nonincreasing path), *exhaustiveness* (full use of the resource), and *balancedness* (equalized performance ratios across cohorts)—which, when taken together, determine a unique fair allocation of an exhaustible resource over time. Implementability responds to the opportunity cost of delaying consumption. Exhaustiveness ensures that, at least in the limit, no resource goes to waste. Finally, balancedness compares the welfare delivered to each generation with the welfare that could be provided to that generation if treated in isolation. Equalization of these ratios expresses an intergenerational equity principle that does not depend on any particular social welfare function but is robust across the canonical class.

The CARA benchmark illustrates the construction. Indeed, a classical optimal policy pursuing a utilitarian objective tends to produce a finite depletion horizon. Its value function, together with the aforementioned conditions, can be used to obtain the fair consumption path and the associated maximum fairness index $\hat{\phi} \in (0, 1]$. By comparison to its classical counterpart, the fair consumption trajectory reallocates consumption from early to later cohorts while preserving full resource use and monotonicity. Beyond the benchmark example, which is solved in semi-closed-form, the proposed method applies to any strictly concave utility function and any positive social discount rate, consistent with the view, well established in public decision-making, that discounting reflects prudence as well as ethical concern for future claims.

A key feature of fair intertemporal resource allocation is that, quite unlike utilitarian consumption, a positive amount of the resource is left for every generation. This property reflects the notion of a *going concern*: the resource base is perpetually preserved at a diminishing but strictly positive level, ensuring that each generation enjoys a share of the common endowment. At the same time, the fair allocation achieves the *optimal fairness index* by construction. The allocation is relatively robust in the sense that (by taking discretized trajectories to a continuous-time limit) it guarantees a fraction $\hat{\phi}$ of the optimum across all canonical social welfare functions. In our numerical example, the fair consumption path yields over 80% of the maximal welfare that any admissible social objective could attain with its own optimal path.

It should be noted that the present analysis abstracts from production, technological change, and uncertainty in stock dynamics, and takes the social discount rate as exogenous. Incorporating stochastic resource growth, learning about future opportunities, or incomplete information about preferences would further enrich the setting and complicate the analysis. Moreover, allowing for renewable resources or environmental externalities could enable assessment of policies linking resource use with investment in natural capital.

Fair intertemporal allocation means choosing a feasible path that treats each generation by equalizing attainable performance guarantees and maximizing the fairness index. When society cannot agree on a social objective, it can still adopt a rule accountable to all reasonable societal goals, granting each generation a nontrivial claim on the common resource, subject to the unavoidable depletion logic that present consumption irreversibly limits the opportunities of future generations.

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