Constructing Efficient Equilibria in Games Played Through Agents

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Abstract

For the game of complete information with multiple principals and multiple common agents discussed by Prat and Rustichini (2003), we construct a general set of equilibrium transfers which implement any efficient outcome as a weakly truthful equilibrium, and the subset of such equilibria that are Pareto optimal for the principals. We provide conditions under which the general set completely characterizes the set of all weakly truthful equilibria implementing a given efficient outcome. In addition, we show that as long as payoffs are concave any efficient outcome can be implemented with principals using affine strategies.

JEL-Classification: C72, D62, L14, L24.

Keywords: Common Agency, Multi-Principal Multi-Agent Contracting, Weakly Truthful Equilibria.

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1 Introduction

Recently, Prat and Rustichini (2003) [P&R] analyzed a class of games of complete information in which N agents noncooperatively implement an outcome (i.e., a collection of actions) after having received offers of outcome-contingent transfer schedules from M principals. Based on Bernheim and Whinston's (1986) truthful Nash equilibrium for games of common agency with one principal and N agents, P&R introduce weakly truthful equilibrium (WTE) as a Nash-equilibrium refinement for their class of *games played through agents*. They focus on the implementation of an *efficient outcome* (that maximizes the sum of all principals' and agents' payoffs). P&R provide a characterization of WTEs and prove that any WTE must be efficient. Moreover, they show that *any* efficient outcome can be implemented as a WTE when agents have convex action sets and all parties have bounded, concave, and continuous payoff functions (P&R, Theorem 8). Their existence proof, based on a generalization of Farkas' Lemma (Aubin and Ekeland, 1984, p. 144), is nonconstructive. Hardly any insight is gained about *how* weakly truthful equilibria can actually be implemented, i.e., which transfers to specify in the equilibrium contracts. As Weber and Xiong (2004) [W&X] demonstrate, it is precisely the latter question of equilibrium implementation which is of great importance in practical applications such as the coordination of supply chains. Indeed, as off-equilibrium payoffs supporting the implementation of an efficient outcome vary, in-equilibrium payoffs to principals and agents are reallocated. We provide several direct algorithms for implementing any efficient outcome as a WTE in a game played through agents, under the assumption that all principals' and agents' payoff functions are concave and continuous.[1]

Given any efficient outcome of a game played through agents, this paper pursues the following *three main objectives*, corresponding to the underlying practical contract design problem: first, to construct a general set of (and, whenever possible, the set of *all*) WTEs implementing the efficient outcome and to provide a simple representation of these WTEs; second, to characterize the subset of these WTEs that yield Pareto-optimal in-equilibrium transfers for the principals; third, given any attainable Pareto-optimal in-equilibrium transfer, to provide a WTE that implements the efficient outcome and results in the prescribed in-equilibrium

[1]While these properties require assumptions on the domains of the payoff functions, our results are valid when the actions sets are finite, or more generally are any subsets of these domains.

transfers. The paper is organized as follows. Section 2 describes several applications of the games played through agents. Section 3 reviews the setting of games played through agents and recalls a simple equilibrium characterization (Proposition 1) by P&R and W&X. This characterization reduces the problem of finding a WTE that implements a given efficient outcome to a problem of constructing separating "excess transfers." In Section 4 we provide an inductive algorithm (Theorem 1) to obtain separating transfers which are affine for each principal except one. In the generic case where the payoff functions are differentiable at the efficient outcome, we show how to implement the outcome with excess transfers that are affine for all principals (Proposition 2). We use the first algorithm to systematically find additional "maximal" excess transfers through outcome-contingent convex combination (relation (21)). Using a levelling algorithm, it is possible to further extend the set of equilibrium excess transfers such that "minimal" excess transfers appear as limits of the algorithm (Proposition 3). We are thus able to describe a general set of (and, whenever these minimal excess transfers are unique, the set of *all*) equilibrium excess transfers (Theorem 2) in terms of an extremal basis which can be directly computed from the minimal excess transfers. In Section 5, we describe the subset of these transfers which are Pareto optimal for the principals and the set of attainable best in-equilibrium transfers for each principal (Theorem 3). We also show how to implement any of these in-equilibrium transfers as a WTE. Section 6 discusses and summarizes our results.

2 Applications of Games Played Through Agents

We now provide several applications of games played through agents to illustrate the significance of a representation of equilibrium transfer profiles implementing an efficient outcome (see also Bernheim and Whinston (1986) and P&R). Before proceeding with examples, we note that the contracting game is played in two stages. In the first stage each principal noncooperatively proposes transfer schedules to all agents, while in the second stage agents implement an outcome by individually selecting their actions. Investigating equilibria that yield Pareto-optimal payoffs for the principals is motivated by the possibility that principals, who move first, may coordinate at the beginning of the game. Principal-Pareto perfection is therefore a natural refinement for equilibria of games played through agents.

Supply Chains. By definition, supply chains are "coordinated" if the outcome maximizes the sum of the payoffs of all firms involved (Cachon, 2003). Much of the literature on coordinated supply chains is focused on a two-echelon single-agent single-principal context, which already involves nonlinear contracts ("quantity-dependent pricing"), and contracts with discount across products and orders ("generalized tying"). In a multi-supplier context, there can also be provisions relative to actions for other suppliers ("exclusive dealing"). When demand is random, as studied in Bernstein and Federgruen (2005), contracts bear on expected values. Some contracts involve multiple components of actions, for example when suppliers must choose both capacities and quantities. In this case, buyers may be induced to enter "royalty schemes," such as pay-back and revenue-sharing contracts. Principals can either be at the lower or at the higher echelon of the supply chain, depending on the allocation of the bargaining power. The model addresses both situations, by a simple transformation of the framework, as described in W&X (Section 2.5).

Labor Economics. When several firms grant the right to make certain decisions to common intermediaries, they behave as principals influencing agents. For example, insurance companies use common insurance brokers to sell their policies.[2]

Lobbying. In order to influence public decisions, interest groups engage in transactions (e.g., by financing electoral campaigns) with decision makers (Dixit, Grossman, and Helpman, 1997). As pointed out by P&R, most decisions are now made collegially, which makes it necessary to consider a multi-agent setting.

Multi-Object Auctions. When bidders submit their offers to an auctioneer, they behave as principals influencing an agent, whose decision is to allocate the auctioned good (Bernheim and Whinston, 1986). In the context of multi-object auctions with complementary goods, bidders are sometimes allowed to condition the payment of their bids to a given auctioneer on the result of another auction, in which case a multi-agent setting is required for the analysis

[2]In 2004, such contracts were the focus of an investigation in New York on the count of possible bid rigging, when several of the nation's largest insurers admitted to having payed kickbacks ('contingent commissions') to obtain business from insurance brokers (Source: CFO Magazine, December 2004).

of the game. A particular instance of such auctions (with one principal) are take-over bids, where the potential buyer conditions his purchase of the stocks (from stockholders, who are the auctioneers and agents of this game) on his obtaining a minimal percentage of the outstanding float of the stock, without which he does not gain the desired control over the targeted firm.

3 Games Played Through Agents

Let $\mathcal{M} = \{1, \ldots, M\}$ be the set of all principals and $\mathcal{N} = \{1, \ldots, N\}$ be the set of all agents. Each agent n can implement an *action* $x_n \in \mathcal{X}_n$ where $\mathcal{X}_n$ is a compact, convex subset of $\mathbb{R}_+^{ML}$, for some $L \geq 1$, and $x_n = (x_n^1, \ldots, x_n^M)$. The component $x_n^m \in \mathbb{R}_+^L$ of agent n's action can be thought as a transfer of L goods and services ("actions") between agent n ("he") and principal m ("she"). A game played through agents consists of two periods. In the first period, each principal $m \in \mathcal{M}$ proposes an outcome-contingent transfer $t_n^m \in C(\mathcal{X}_n, \mathbb{R}_+)$ to agent n.[3] In the second period, the transfer schedule $t = [t_n^m]$ is announced publicly and each agent n implements his most preferred action to obtain a respective net payoff of

$$U_n(x_n; t) = \Gamma_n(x_n) + \sum_{m \in \mathcal{M}} t_n^m(x_n), \tag{1}$$

where $\Gamma_n \in C(\mathcal{X}_n, \mathbb{R})$ is agent n's gross payoff function. Provided the *outcome* $x = (x_1, \ldots, x_N) \in \mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ is implemented by the agents, principal m's net payoff is given by

$$V^m(x; t^m) = \Pi^m(x) - \sum_{n \in \mathcal{N}} t_n^m(x_n), \tag{2}$$

where $\Pi^m \in C(\mathcal{X}, \mathbb{R})$ is principal m's gross payoff and $t^m = (t_1^m, \ldots, t_N^m)$ is her transfer. Principal m's *transfer* t^m *is weakly truthful* relative to an outcome $\hat{x} \in \mathcal{X}$ if

$$V^m(\hat{x}; t^m) = \max_{x \in \mathcal{X}} V^m(x; t^m). \tag{3}$$

A subgame-perfect *pure-strategy (Nash) equilibrium* of the two-period game

$$\mathcal{G} = \big\{\{\mathcal{M}, \mathcal{N}\}, \{V^m(\cdot), U_n(\cdot)\}, \{C(\mathcal{X}, \mathbb{R}_+^N), \mathcal{X}_n\}\big\}$$

[3]Given topological spaces $\mathcal{R}$ and $\mathcal{S}$, we denote by $C(\mathcal{R}, \mathcal{S})$ the set of continuous functions $f : \mathcal{R} \to \mathcal{S}$.

is a pair $(\hat{t}, \hat{x}) \in C(\mathcal{X}, \mathbb{R}^{MN}) \times \mathcal{X}$ such that (i) for every $n \in \mathcal{N}$, and given any $t \in C(\mathcal{X}, \mathbb{R}^{MN})$,

$$\hat{x}_n(t) \in \arg\max_{x_n \in \mathcal{X}_n} U_n(x_n, \hat{x}_{-n}; t), \tag{4}$$

and (ii) for every $m \in \mathcal{M}$, and given $\hat{t}^{-m} \in C(\mathcal{X}, \mathbb{R}_+^{N-1})$,

$$\hat{t}^m \in \arg\max_{t^m \in C(\mathcal{X}, \mathbb{R}_+^N)} V^m(\hat{x}(t^m, \hat{t}^{-m}); t^m). \tag{5}$$

A *pure-strategy equilibrium* $(\hat{t}, \hat{x})$ *is weakly truthful* if every principal m's transfer is weakly truthful with respect to the equilibrium outcome $\hat{x}$. As P&R (Proposition 3) note, *any WTE is efficient* in the sense that the associated outcome $\hat{x}$ maximizes joint surplus,

$$W(x) = \sum_{m \in \mathcal{M}} V^m(x; t^m) + \sum_{n \in \mathcal{N}} U_n(x; t) = \sum_{m \in \mathcal{M}} \Pi^m(x) + \sum_{n \in \mathcal{N}} \Gamma_n(x_n).$$

For any given efficient outcome $\hat{x} \in \arg\max_{x \in \mathcal{X}} W(x)$ let $F^m(x) = \Pi^m(x) - \Pi^m(\hat{x})$ denote principal m's *excess payoff* and let $G_n(x_n) = \Gamma_n(\hat{x}_n) - \Gamma_n(x_n)$ be agent n's *excess cost* relative to their respective payoffs at $\hat{x}$. Based on P&R's results, W&X characterize WTEs of the game $\mathcal{G}$ in the following compact form, which they term the *reduced contract design problem.*

Proposition 1 (Reduced Contract Design Problem) *The pair $(\hat{t}, \hat{x})$ is a WTE of the game $\mathcal{G}$ if and only if for all $(m, n) \in \mathcal{M} \times \mathcal{N}$,*

$$F^m - \sum_{n \in \mathcal{N}} \Delta_n^m \leq 0 \leq G_n - \sum_{m \in \mathcal{M}} \Delta_n^m, \tag{R}$$

where $\Delta_n^m(x_n) = \hat{t}_n^m(x_n) - \hat{t}_n^m(\hat{x}_n)$ is principal m's excess transfer to agent n contingent on the feasible outcome $x_n \in \mathcal{X}_n$.

In fact, any excess transfer $\Delta = [\Delta_n^m]$ that solves (R) for a given efficient outcome $\hat{x} \in \mathcal{X}$ (which we also refer to as reduced (equilibrium) transfer) can be mapped to an admissible equilibrium transfer $t \in C(\mathcal{X}, \mathbb{R}_+^{MN})$ by setting $\hat{t}_n^m(x_n) = \hat{\Delta}_n^m(x_n) + \alpha_n^m$. The nonnegative constants α_n^m ("in-equilibrium transfers") correspond to the equilibrium transfers contingent on the implemented efficient equilibrium outcome $\hat{x}$, i.e., $[\alpha_n^m] = \hat{t}(\hat{x})$. For instance, by using for any fixed $n \in \mathcal{N}$ the recursive construction suggested by P&R (Lemma 2), one obtains

$$\alpha_n^m = -\min_{x_n \in \mathcal{X}_n} \left\{ G_n(x_n) - \sum_{i=1}^{m-1} \hat{\Delta}_n^i(x_n) - \sum_{i=m+1}^{M} \Delta_n^i(x_n) \right\} \geq 0 \tag{6}$$

and

$$\hat{\Delta}_n^m(x_n) = -\min\left\{\alpha_n^m, -\Delta_n^m(x_n)\right\}, \tag{7}$$

for all $m = 1, \ldots, M$. More generally, if $\hat{\Delta}$ is an "admissible" equilibrium excess transfer (e.g., obtained through (6)–(7)), then the in-equilibrium transfers α_n^m can be obtained through[4]

$$\alpha_n^m = -\min_{x_n \in \mathcal{X}_n} \left\{ \hat{\Delta}_n^m(x_n) + \left(G_n(x_n) - \sum_{i \in \mathcal{M}} \hat{\Delta}_n^i(x_n) \right) \right\} \geq 0 \tag{8}$$

for all $(m, n) \in \mathcal{M} \times \mathcal{N}$. The continuous equilibrium excess transfer $\hat{\Delta} = [\hat{\Delta}_n^m]$ is thereby *admissible*, if it solves the reduced contract design problem (R) *and* it is such that

$$\hat{t}_n^m = \hat{\Delta}_n^m + \alpha_n^m \geq 0 \tag{9}$$

for all $(m, n) \in \mathcal{M} \times \mathcal{N}$. The latter inequality ensures that the equilibrium transfer is nonnegative and is thus an element of $C(\mathcal{X}_n, \mathbb{R}_+)$, as required at the outset.

The difficulty with the recursive construction (6)–(7) is twofold. First, the resulting in-equilibrium transfer $\alpha = [\alpha_n^m]$ is potentially not Pareto optimal for the principals, in the sense that it may be possible to strictly lower in-equilibrium transfers to agents for some principals while making no other principal worse off. Second, the set of attainable in-equilibrium transfers using the recursive construction is in general strictly contained in the set $\mathcal{A}(\hat{x})$ of all in-equilibrium transfers that can be implemented as a WTE $(\hat{t}, \hat{x})$. Our direct equilibrium construction bypasses these shortcomings by providing a precise representation of *all* attainable WTEs of $\mathcal{G}$. This is accomplished as follows. To find all WTEs that implement an efficient outcome $\hat{x} \in \mathcal{X}$, it is by Proposition 1 necessary to find all excess transfers Δ whose elements Δ_n^m satisfy the $M + N$ inequalities (R). We thus provide a complete set of solutions to this "reduced contract design problem" (Section 4). In particular, we show that any solution to (R) can be represented as an outcome-contingent convex combination of extremal basis functions, which implies a simple representation of all WTEs of $\mathcal{G}$ including an explicit representation of the set of attainable in-equilibrium transfers. In Section 5 we then provide a simple mapping from any solution Δ of (R) to any admissible equilibrium

[4] The reasoning behind expression (8) is that in equilibrium each principal m is minimizing her expenditure on transfers by paying only the amount needed to compensate agent n for the payoff difference, had he chosen his otherwise best action without principal m. The worst "punishment" principal m can inflict on agent n is thereby limited to paying him zero, since the transfer t_n^m is by assumption an element of $C(\mathcal{X}_n, \mathbb{R}_+)$.

transfer $\hat{t} = \hat{\Delta} + \alpha$ with a Pareto-optimal in-equilibrium transfer α. Again one finds that any equilibrium transfer in the subset of Pareto-optimal equilibrium transfers can be represented as a convex combination of extremal basis functions.

4 Solving the Reduced Contract Design Problem

We now construct a complete solution to the reduced contract design problem (R) given an efficient outcome $\hat{x}$. Any efficient outcome is thereby a maximizer of the joint surplus W on the compact set $\mathcal{X}$ of the agents' feasible actions. Since W is continuous, by the Weierstrass theorem (Berge 1963, p. 69) there exists at least one efficient outcome in $\mathcal{X}$. Fixing an efficient outcome $\hat{x}$ we look for solutions to the set of $M + N$ inequalities (R). For this it is useful to note that by efficiency of $\hat{x}$ any excess welfare $W(x) - W(\hat{x}) = \sum_{m \in \mathcal{M}} F^m(x) - \sum_{n \in \mathcal{N}} G_n(x_n)$ is nonpositive, i.e.,

$$\sum_{m \in \mathcal{M}} F^m(x) \leq \sum_{n \in \mathcal{N}} G_n(x_n) \tag{10}$$

for all $x \in \mathcal{X}$. Since our construction makes repeated use of the separating hyperplane theorem (Berge, 1963, p. 163) it is necessary to assume that all gross payoff functions are concave (for a characterization of existence without such an assumption, see footnote 11).

Assumption 1 (Payoff Concavity) *Principal m's gross payoff Π^m and agent n's gross payoff Γ_n are concave and bounded for all $(m, n) \in \mathcal{M} \times \mathcal{N}$.*

Naturally Assumption 1 implies that the excess measures F^m and $-G_n$ in Proposition 1 are concave. A solution to (R) can now be obtained by induction on m. Since $F^1 \leq \sum_{n \in \mathcal{N}} G_n - \sum_{m \geq 2} F^m$, F^1 is concave, $\sum_{n \in \mathcal{N}} G_n - \sum_{m \geq 2} F^m$ is convex, and both functions vanish at $\hat{x}$, the separating hyperplane theorem implies the existence of vectors $\delta_n^1 \in \mathbb{R}^L$ for $n \in \mathcal{N}$ such that

$$F^1(x) \leq \sum_{n \in \mathcal{N}} \langle \delta_n^1, x_n - \hat{x}_n \rangle \leq \sum_{n \in \mathcal{N}} G_n(x_n) - \sum_{m \geq 2} F^m(x). \tag{11}$$

We set $\Delta_n^1(x_n) = \langle \delta_n^1, x_n - \hat{x}_n \rangle$ for $n \in \mathcal{N}$. In the second step of the induction we observe that from (11),

$$F^2 \leq \sum_{n \in \mathcal{N}} (G_n - \Delta_n^1) - \sum_{m \geq 3} F^m.$$

Moreover, F^2 is concave, whereas $\sum_{n\in\mathcal{N}}(G_n - \Delta_n^1) - \sum_{m\geq 3} F^m$ is convex (the functions Δ_n^1 are affine for all $n \in \mathcal{N}$), and both functions vanish at $\hat{x}$. Another application of the separating hyperplane theorem, then, implies the existence of vectors $\delta_n^2 \in \mathbb{R}^L$ for $n \in \mathcal{N}$ such that

$$F^2(x) \leq \sum_{n\in\mathcal{N}} \langle \delta_n^2, x_n - \hat{x}_n \rangle \leq \sum_{n\in\mathcal{N}} (G_n - \Delta_n^1)(x_n) - \sum_{m\geq 3} F^m(x). \tag{12}$$

We set $\Delta_n^2(x_n) = \langle \delta_n^2, x_n - \hat{x}_n \rangle$, where the brackets denote the usual scalar product. Let us now describe a generic iteration of the induction. Suppose that for any $\mu \in \{1, \ldots, M-2\}$ we have affine functions Δ_n^m for $m \in \{1, \ldots, \mu - 1\}$ and $n \in \mathcal{N}$ such that

$$F^\mu \leq \sum_{n\in\mathcal{N}} \left(G_n - \sum_{m\leq \mu-1} \Delta_n^m \right) - \sum_{m\geq \mu+1} F^m.$$

By assumption, F^μ is concave and $\sum_{n\in\mathcal{N}} \left(G_n - \sum_{m\leq\mu-1} \Delta_n^m \right) - \sum_{m\geq\mu+1} F^m$ is convex. Both functions vanish at $\hat{x}$. The separating hyperplane theorem guarantees the existence of vectors $\delta_n^\mu \in \mathbb{R}^L$ for $n \in \mathcal{N}$ such that

$$F^\mu(x) \leq \sum_{n\in\mathcal{N}} \langle \delta_n^\mu, x_n - \hat{x}_n \rangle \leq \sum_{n\in\mathcal{N}} \left(G_n - \sum_{m\leq\mu-1} \Delta_n^m \right)(x_n) - \sum_{m\geq\mu+1} F^m(x).$$

This defines functions $\Delta_n^\mu(x_n) = \langle \delta_n^\mu, x_n - \hat{x}_n \rangle$ which are affine and vanish at $\hat{x}$, such that

$$F^\mu \leq \sum_{n\in\mathcal{N}} \Delta_n^\mu \leq \sum_{n\in\mathcal{N}} \left(G_n - \sum_{m\leq\mu-1} \Delta_n^m \right) - \sum_{m\geq\mu+1} F^m. \tag{13}$$

As a result,

$$F^{\mu+1} \leq \sum_{n\in\mathcal{N}} \left(G_n - \sum_{m\leq\mu} \Delta_n^m \right) - \sum_{m\geq\mu+2} F^m,$$

i.e., the induction hypothesis holds for $\mu + 1$. We continue the induction until $\mu = M - 1$ and obtain functions $\Delta_n^{M-1}(x_n) = \langle \delta_n^{M-1}, x_n - \hat{x}_n \rangle$ for $n \in \mathcal{N}$ that satisfy

$$F^{M-1} \leq \sum_{n\in\mathcal{N}} \langle \delta_n^{M-1}, x_n - \hat{x}_n \rangle \leq \sum_{n\in\mathcal{N}} \left(G_n - \sum_{m\leq M-2} \Delta_n^m \right) - F^M. \tag{14}$$

The algorithm terminates by setting

$$\Delta_n^M = G_n - \sum_{m\leq M-1} \Delta_n^m \tag{15}$$

for all $n \in \mathcal{N}$. We have therefore found MN functions Δ_n^m that all vanish at $\hat{x}$. From relations (13)–(15), one obtains that

$$F^m \leq \sum_{n\in\mathcal{N}} \Delta_n^m$$

for all $m \in \mathcal{M}$. Moreover, equation (15) implies that

$$\sum_{m \in \mathcal{M}} \Delta_n^m = G_n$$

for all $n \in \mathcal{N}$. Consequently the excess transfer $\Delta = [\Delta_n^m]$ solves (R) and implements the efficient outcome $\hat{x}$, as intended. We have thus provided a constructive proof of the following result.

Theorem 1 (Existence of a WTE of $\mathcal{G}$) *Under Assumption 1 there exists a weakly truthful equilibrium $(\hat{t}, \hat{x})$ of the game $\mathcal{G}$.*

This construction is useful when the payoff functions are not differentiable at $\hat{x}$.[5] On the other hand, when the principals' payoffs are differentiable at the efficient outcome, it is possible to provide an explicit representation of an affine WTE.

Assumption 2 (Principal Payoff Regularity) *Principal m's gross payoff Π^m is differentiable*[6] *at the efficient outcome $\hat{x} \in \mathcal{X}$ for all $m \in \mathcal{M}$.*

Since by the Rademacher theorem (Magaril-Il'yaev and Tikhomirov, 2003, p. 160) payoff concavity (i.e., Assumption 1) already implies differentiability of Π^m and Γ_n almost everywhere for all $(m, n) \in \mathcal{M} \times \mathcal{N}$, Assumption 2 is typically not a strong additional requirement. In case it is not satisfied, the following result can still be applied in a weaker form using subdifferentials.[7]

Proposition 2 (Affine WTE of $\mathcal{G}$) *Under Assumptions 1–2 any efficient outcome $\hat{x} \in \mathcal{X}$ can be implemented as a weakly truthful equilibrium $(\hat{t}, \hat{x})$ of the game $\mathcal{G}$, where $\hat{t} = [\hat{t}_n^m]$ with $\hat{t}_n^m(x_n) = \hat{\Delta}_n^m(x_n) + \alpha_n^m$, where $\hat{\Delta}_n^m$ and α_n^m are given by the recursion (6)–(7) as a function of $\Delta = [\Delta_n^m]$, and*

$$\Delta_n^m(x_n) = \langle \frac{\partial F^m(\hat{x})}{\partial x_n}, x_n - \hat{x}_n \rangle \tag{16}$$

[5]This may for example occur if, at least for some m, the excess payoff F^m is the pointwise minimum of concave functions, two of which intersect at $\hat{x}$.

[6]If the efficient outcome under consideration is a noninterior point of the feasible set $\mathcal{X}$, then differentiability is to be interpreted with respect to any differentiable path of points in $\mathcal{X}$ leading to $\hat{x} \in \partial\mathcal{X}$. We implicitly assume that $\mathcal{X}$ is indeed path-connected in a neighborhood of $\hat{x}$.

[7]In that case one can assert that there exists an appropriate element of the subdifferential of F^m at $\hat{x}$ with respect to x_n, which can be used in (16) instead of the regular directional derivative.

for all $(m,n) \in \mathcal{M} \times \mathcal{N}$*.*

Proof. By Assumption 2, the directional derivative $\partial F^m(\hat{x})/\partial x_n$ is well defined for all $(m,n) \in \mathcal{M} \times \mathcal{N}$. Moreover, the choice of $\delta_n^m = \partial F^m(\hat{x})/\partial x_n \in \mathbb{R}^L$ leads to excess transfers $\Delta_n^m(x_n) = \langle \delta_n^m, x_n - \hat{x}_n \rangle$ that satisfy

$$F^m(x) \leq \sum_{n \in \mathcal{N}} \Delta_n^m(x_n) \tag{17}$$

for all $m \in \mathcal{M}$ and all $x \in \mathcal{X}$. In fact, these excess transfers Δ_n^m are the *only* affine excess transfers vanishing at $\hat{x}$ to satisfy this inequality, for the lower epigraph of F^m is clearly supported at $\hat{x}$ by the hyperplane defined by the expression on the right-hand side of (17). Since $\sum_{m \in \mathcal{M}} F^m \leq \sum_{n \in \mathcal{N}} G_n$ by (10), the supporting hyperplane of $\sum_{m \in \mathcal{M}} F^m$, which is precisely $\sum_{(m,n) \in \mathcal{M} \times \mathcal{N}} \Delta_n^m$, lies below $\sum_{n \in \mathcal{N}} G_n$. That is,

$$\sum_{n \in \mathcal{N}} \left(\sum_{m \in \mathcal{M}} \Delta_n^m(x_n) \right) \leq \sum_{n \in \mathcal{N}} G_n(x_n) \tag{18}$$

for all $x \in \mathcal{X}$. Setting $x_j = \hat{x}_j$ for all $j \neq n$, (18) implies that $\sum_{m \in \mathcal{M}} \Delta_n^m(x_n) \leq G_n(x_n)$ for all $x_n \in \mathcal{X}_n$, since all the other terms vanish. This concludes our proof. ■

The affine solution (16) to the reduced contract design problem (R) is particularly simple and thus seems intuitively appealing. Nevertheless, since in-equilibrium transfers obtained from affine solutions of (R) are generally not Pareto-optimal, we have an interest to find all other solutions to the reduced contract design problem. These additional solutions are either "below" the affine solution ("infra-affine solutions") or "above" the affine solution ("ultra-affine solutions"); "mixed" solutions which are partially above and partially below the affine solution can of course be obtained by convex combination.[8]

Infra-Affine Excess Transfers. Reducing excess transfers relative to the affine solution can be accomplished by the following iterative process which starting from any excess transfer that solves (R) yields a unique infra-affine lower limit $\underline{\Delta}$ that also solves (R). The construction

[8]Given two solution excess transfers Δ and $\check{\Delta}$, any convex combination $\lambda\Delta + (1-\lambda)\check{\Delta}$ for $\lambda \in (0,1)$ also constitutes a solution excess transfer.

provided here is somewhat related to the levelling algorithm by Diliberto and Straus (1951), and follows Strulovici and Weber (2006). In the limit it provides a best approximation of any function of two variables by a sum of two functions of one variable. From a given excess transfer $\Delta = [\Delta_n^m]$ it is possible to obtain a new modified transfer $\tilde{\Delta} = [\tilde{\Delta}_n^m]$ with admissible out-of-equilibrium transfers below Δ as follows. For each n, let $x_{-n} = (x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_N)$, $\mathcal{X}_{-n} = \times_{j \neq n} \mathcal{X}_j$, and set

$$\tilde{\Delta}_n^m(x_n) = \max_{x_{-n} \in \mathcal{X}_{-n}} \left\{ F^m(x) - \sum_{j \neq n} \Delta_j^m(x_j) \right\} \tag{19}$$

for any single $(m, n) \in \mathcal{M} \times \mathcal{N}$. Then replace Δ_n^m in Δ by $\tilde{\Delta}_n^m$ and repeat (19) with the new Δ and another index $(m, n) \in \mathcal{M} \times \mathcal{N}$. From the definition of $\tilde{\Delta}_n^m$ the following result is immediate.

Lemma 1 (Excess-Transfer Decrement) *The modified excess transfer $[\tilde{\Delta}_n^m]$ is such that (i) $\tilde{\Delta}_n^m \leq \Delta_n^m$, and (ii) $\tilde{\Delta}_n^m + \sum_{j \neq n} \Delta_j^m \geq F^m$.*

Proof. (i) Since Δ solves (R), we obtain from (19) that

$$\tilde{\Delta}_n^m(x_n) \geq F^m(x) - \sum_{j \neq n} \Delta_j^m(x_j) \tag{20}$$

for all $x \in \mathcal{X}$. Moreover, since

$$\Delta_n^m(x_n) \geq F^m(x) - \sum_{j \neq n} \Delta_j^m(x_j)$$

for all x, taking the maximum on the right-hand side with respect to x_{-n}, shows that $\Delta_n^m(x_n) \geq \tilde{\Delta}_n^m(x_n)$ for all x_n. (ii) This assertion follows directly from (20). ■

We denote by $\tilde{\Delta}$ the matrix obtained after successive application of (19) for each $(m, n) \in \mathcal{M} \times \mathcal{N}$. It is clear that starting from any infra-affine solution $\Delta = [\Delta_n^m]$ the new excess transfer $\tilde{\Delta} = [\tilde{\Delta}_n^m]$ remains an infra-affine solution to (R). We also note that the monotonicity in part (i) of Lemma 1 is pointwise. Replacing Δ_n^m by $\tilde{\Delta}_n^m$, we can repeat this procedure with another index $j \neq n$ which leads to the following sequential algorithm:

1. Set $m = n = 1$.

2. Compute $\tilde{\Delta}_n^m$.

3. Replace Δ_n^m by the function found in 2.

4. If $n < N$, increase n by one, or else if $m < M$ set $n = 1$ and increase m by 1, otherwise terminate.

5. Go back to step 2 with the new values of m and n.

Iterating the sequential algorithm k times for $k = 1, 2, \ldots$ yields a sequence of excess transfer matrices $\sigma(\Delta) = \{\tilde{\Delta}_{(k)}(\Delta)\}_{k=0}^{\infty}$ with $\tilde{\Delta}_0 = \Delta$, where the starting matrix Δ satisfies the system of inequalities (R). The following result asserts that the limit $\Delta_\infty(\Delta) = \lim_{k\to\infty} \Delta_{(k)}(\Delta)$ is well defined and constitutes an admissible (i.e., continuous) equilibrium excess transfer.

Proposition 3 (Lower Excess-Transfer Bound) *For any admissible equilibrium excess transfer Δ the limit $\Delta_\infty(\Delta)$ of the sequence $\sigma(\Delta)$ exists, is continuous on $\mathcal{X}$, and solves the reduced contract design problem (R).*

Proof. The proof follows from Theorem 1 in Strulovici and Weber (2006) and is therefore omitted. ∎

In general, the limit $\Delta_\infty(\Delta)$ is not unique: it depends on the starting point Δ. In order to simplify the analysis for the description of the WTE implementing an efficient outcome, we restrict our attention to the limit $\underline{\Delta} = \Delta_\infty(\Delta_{\text{aff}})$ where Δ_{aff} is the matrix of affine excess transfers obtained by Proposition 2. Under some conditions on the payoff functions given in Assumption 3, the lower limit $\Delta_\infty(\Delta)$ is in fact independent of Δ, in which case the restriction is vacuous. For example, $\Delta_\infty(\Delta)$ is clearly independent of Δ when the payoff functions $\{F^m\}_{m\in\mathcal{M}}$ are additive in $(x_1, \ldots, x_N)$. However, the following proposition shows that independence holds under more general conditions.

Assumption 3 (Payoff Specification) *Each principal m's excess payoff function $F^m : \mathcal{X} \to \mathbb{R}$ is of the form*

$$F^m(x_1, \ldots, x_N) = f^m\left(g_1^m(x_1), \ldots, g_N^m(x_N)\right),$$

where f^m is submodular[9] *and vanishes at $(0,\ldots,0)$, and the function $g_n^m : \mathcal{X}_n \to \mathbb{R}_+$ vanishes at $\hat{x}_n$ for all $n \in \mathcal{N}$.*

Proposition 4 *Suppose that Assumption 3 holds. Then, $\Delta_\infty(\Delta)$ is independent of Δ.*

Proof. Using the levelling algorithm, the unique "additive upper envelope" of each F^m is uniquely determined as

$$\sum_n \underline{\Delta}_n^m = \sum_n f^m(0,\ldots,0,g_n^m(x_n),0,\ldots,0).$$

Letting $\underline{\Delta} = [\underline{\Delta}_n^m]$, $\underline{\Delta}$ is clearly below any limit $\Delta_\infty(\Delta)$. Since $\Delta_\infty(\Delta)$ is minimal, it must equal $\underline{\Delta}$. ■

Examples. Suppose that the set of outcomes is $\mathcal{X} = [0,\bar{x}] \times [0,\bar{y}]$ for some positive constants $\bar{x},\bar{y}$, that the efficient outcome is $(\hat{x},\hat{y}) = (0,0)$, and that $F^m(x,y) = \ln(1 + k_1^m x + k_2^m y)$, with $k_i > 0$. Then the additive upper envelope of the payoff function F^m vanishing at $(0,0)$ is $\underline{\Delta}_1^m(x) = \ln(1+k_1^m x)$ and $\underline{\Delta}_2^m(y) = \ln(1+k_2^m y)$. If we consider instead $F^m(x,y) = -(x^{k_1^m} + y^{k_2^m})^2$ with k_1^m, k_2^m positive, then the additive upper envelope is $\underline{\Delta}_1^m(x) = -x^{2k_1^m}$ and $\underline{\Delta}_2^m(y) = -y^{2k_2^m}$. Last consider the constant elasticity of substitution function $F^m(x,y) = [k_1^m x^\rho + k_2^m y^\rho]^{\frac{1}{\rho}}$, with k_1^m, k_2^m positive and $\rho > 1$. In this case, the additive upper envelope is $\underline{\Delta}_1^m(x) = (k_1^m)^{\frac{1}{\rho}} x$ and $\underline{\Delta}_2^m(y) = (k_2^m)^{\frac{1}{\rho}} y$. □

From now on we consider the lower excess-transfer bound $\underline{\Delta}$ generated by the affine excess transfer of Proposition 2. By Proposition 3, $\underline{\Delta}$ is infra-affine. We thus obtain a general class of WTEs implementing the efficient outcome. When Assumption 3 holds, this set entirely characterizes the set of *all* WTEs implementing the efficient outcome.[10] Note also that, by construction, $\underline{\Delta}$ solves the reduced contract design problem (R). When the limit obtained by the above procedure independent of Δ, it is the smallest additive function lying above φ.

[9] A function h of k variables $(x_1,\ldots x_k) \in \mathbb{R}^{qk}$ where q is a positive integer is *submodular* if and only if for any $(x,y) \in \mathbb{R}^{2qk}$, $h(\min\{x,y\}) + h(\max\{x,y\}) \leq h(x) + h(y)$, where the minimum and maximum are taken componentwise.

[10] In principle, it is also possible to consider the distinct lower limits in order to describe all WTEs, but this task is likely to involve a continuum of lower limit functions, depending on the form of the payoff functions.

For that reason, we call this limit the *additive upper envelope* of φ.[11]

Ultra-Affine Excess Transfers. The solution constructed in the proof of Theorem 1 was affine for all principals except the last. By relabelling principals such that the role of the last is assigned to principal $i \in \mathcal{M}$ we obtain in general M different solutions which all lie above the affine solution described in Proposition 2. An outcome-contingent convex combination of these solutions is given by

$$\Delta_n^m(x_n;\theta_n^m) = \langle \frac{\partial F^m(\hat{x})}{\partial x_n}, x_n - \hat{x}_n \rangle + \theta_n^m(x_n) \left(G_n(x_n) - \sum_{i \in \mathcal{M}} \langle \frac{\partial F^i(\hat{x})}{\partial x_n}, x_n - \hat{x}_n \rangle \right) \tag{21}$$

for all $(m,n) \in \mathcal{M} \times \mathcal{N}$, whereby $\theta_n^m \in C(\mathcal{X}_n, [0,1])$ such that $\sum_{m \in \mathcal{M}} \theta_n^m = 1$. It is clear that each $\Delta_n^m(\cdot;\theta_n^m)$ is ultra-affine, as a convex combination of excess transfers which are ultra-affine by construction. Replacing the affine excess transfers of the form (16) in (21) by the infra-affine lower limit $\underline{\Delta}_n^m$ we obtain a tight *upper bound* for solutions to the reduced contract design problem,

$$\bar{\Delta}_n^m(x_n;\theta_n^m) = \underline{\Delta}_n^m(x_n) + \theta_n^m(x_n) \left(G_n(x_n) - \sum_{i \in \mathcal{M}} \underline{\Delta}_n^i(x_n) \right) \tag{22}$$

for all $x_n \in \mathcal{X}_n$.

Proposition 5 (Upper Excess-Transfer Bound) *The excess transfer $[\bar{\Delta}_n^m(\cdot;\theta_n^m)]$ solves the reduced contract design problem (R) for any matrix of outcome-contingent weights $\theta_n^m \in C(\mathcal{X}_n, [0,1])$ with $\sum_{m \in \mathcal{M}} \theta_n^m = 1$.*

Proof. We prove more generally that given *any* solution $\Delta = [\Delta_n^m]$ of (R) the excess transfer $\tilde{\Delta} = [\tilde{\Delta}_n^m]$ with

$$\tilde{\Delta}_n^m = \Delta_n^m + \theta_n^m \left(G_n - \sum_{i \in \mathcal{M}} \Delta_n^i \right) \tag{23}$$

[11] The additive upper envelope $\underline{\Delta}$ is very much related to the existence of a WTE in the absence of concavity (Assumption 1), provided the payoff functions are at least continuously differentiable. Indeed, it is clear from our construction of the lower limit (which did not depend on payoff concavity) that a WTE of $\mathcal{G}$ exists if $\underline{\Delta} = \Delta_\infty(\Delta)$ solves (R), where the starting matrix $\Delta = [\Delta_n^m]$ can be any excess transfer matrix that satisfies the first inequality in (R). For instance, one could take $\Delta_n^m(x_n) = \max_{x_{-n} \in \mathcal{X}_{-n}} \{F(x_n, x_{-n})\}$.

solves (R) provided that the weights $\theta_n^m \in C(\mathcal{X}_n, [0,1])$ satisfy $\sum_{m\in\mathcal{M}} \theta_n^m = 1$. Our assertion then follows immediately for $\Delta = \underline{\Delta}$. Since $G_n \geq \sum_{i\in\mathcal{M}} \Delta_n^m$ by (R), $\tilde{\Delta}_n^m \geq \Delta_n^m$. Thus,

$$\sum_{n\in\mathcal{N}} \tilde{\Delta}_n^m \geq \sum_{n\in\mathcal{N}} \Delta_n^m \geq F^m,$$

for all $m \in \mathcal{M}$, since Δ satisfies (R). Moreover, for any $n \in \mathcal{N}$ the identity

$$\sum_{m\in\mathcal{M}} \tilde{\Delta}_n^m = G_n,$$

holds for any admissible θ, so that the excess transfer $\tilde{\Delta}$ solves the reduced contract design problem (R). ■

The lower and upper bounds for solutions to the reduced contract design problem (R) can now be used to construct an *extremal basis* $\mathcal{B}$ containing $M+1$ transfers as follows. Let

$$\mathcal{B} = \{[B_n^{m,0}], \ldots, [B_n^{m,M}]\}, \tag{24}$$

where

$$B_n^{m,\mu}(\cdot) = \begin{cases} \bar{\Delta}_n^m(\cdot;1), & \text{if } m = \mu, \\ \bar{\Delta}_n^m(\cdot;0), & \text{otherwise,} \end{cases} \tag{25}$$

for all $(m,\mu,n) \in \mathcal{M} \times \bar{\mathcal{M}} \times \mathcal{N}$ with the abbreviation $\bar{\mathcal{M}} = \{0\} \cup \mathcal{M}$. The central result of this section is convex combination of elements of $\mathcal{B}$ is a solution to (R), and that the converse is true under Assumption 3. For each n, let $\hat{\mathcal{X}}_n = \{x_n \in \mathcal{X}_n : \sum_{m\in\mathcal{M}} \underline{\Delta}_n^m(x_n) < G_n(x_n)\}$.

Definition 1 (Admissible Weights) *A matrix $\theta = [\theta_n^m]$ of functions defines admissible weights, denoted $\theta \in \mathcal{W}$, if its entries satisfy*

(i) for each (m,n) in $\mathcal{M} \times \mathcal{N}$, $\theta_n^m : \mathcal{X}_n \to [0,1]$;

(ii) for all n, $\sum_{m\in\mathcal{M}} \theta_n^m = 1$;

(iii) for each (m,n) in $\mathcal{M} \times \mathcal{N}$, θ_n^m is continuous on $\hat{\mathcal{X}}_n$.

Theorem 2 (Representation of Reduced Equilibrium Contracts) *Suppose that Assumption 1 holds and define* $\operatorname{co}\mathcal{B}$ *by*

$$\operatorname{co}\mathcal{B} = \left\{ [\Delta_n^m] : \Delta_n^m = \sum_{\mu\in\bar{\mathcal{M}}} \vartheta_n^\mu B_n^{m,\mu}, [\vartheta_n^\mu] \in \mathcal{W} \right\}. \tag{26}$$

Then: (i) any element of $\operatorname{co}\mathcal{B}$ *solves the reduced contract design problem (R) (ii) when the lower limit* $\Delta_\infty(\Delta)$ *is independent of* Δ*, the set of solutions to the reduced contract design problem (R) is exactly* $\operatorname{co}\mathcal{B}$*.*

Proof. We start by proving (ii): Let $[\Delta_n^m]$ be a solution to the reduced contract design problem (R). We need to show that $[\Delta_n^m] \in \operatorname{co}\mathcal{B}$. Indeed, for any fixed $n \in \mathcal{N}$ and $x_n \in \mathcal{X}_n$, let $\beta = (\Delta_n^1(x_n), \ldots, \Delta_n^M(x_n))$ and $\beta^\mu = (B_n^{1,\mu}(x_n), \ldots, B_n^{M,\mu}(x_n))$ for all $\mu \in \bar{\mathcal{M}}$. These $M+2$ vectors all belong to $\mathbb{R}^M$. Since $\beta^0 \leq \beta$ and $\langle \beta, (1, \ldots, 1)\rangle = \sum_{m\in\mathcal{M}} \Delta_n^m(x_n) \leq G_n(x_n)$, the vector β belongs to the M-simplex with vertices $\beta^0, \ldots, \beta^M$. As a result, there exist $M+1$ nonnegative numbers $\vartheta_n^0(x_n), \ldots, \vartheta_n^M(x_n)$ with $\sum_{\mu\in\bar{\mathcal{M}}} \vartheta_n^\mu(x_n) = 1$, such that $\beta = \sum_{\mu\in\bar{\mathcal{M}}} \vartheta_n^\mu(x_n)\beta^\mu$. If $x_n \in \hat{\mathcal{X}}_n$, then the simplex is nondegenerate in a neighborhood of x_n, which implies uniqueness and continuity of the coefficient functions $\vartheta_n^0, \ldots, \vartheta_n^M$ in the representation (26) on that neighborhood.[12] We now prove (i): Consider any element $\Delta \in \operatorname{co}\mathcal{B}$. Using (24)–(26),

$$\Delta_n^m(x_n) = \sum_{\mu\in\bar{\mathcal{M}}} \vartheta_n^\mu(x_n) B_n^{m,\mu}(x_n) = \vartheta_n^m(x_n)\bar{\Delta}_n^m(x_n;1) + \bar{\Delta}(x_n;0)(1-\vartheta_n^m(x_n))$$

on $\mathcal{X}_n$, and thus

$$\Delta_n^m = \vartheta_n^m\left(G_n - \sum_{i\neq m} \underline{\Delta}_n^i\right) + (1-\vartheta_n^m)\underline{\Delta}_n^m = \underline{\Delta}_n^m + \vartheta_n^m\left(G_n - \sum_{i\in\mathcal{M}} \underline{\Delta}_n^i\right) \tag{27}$$

for all $(m,n) \in \mathcal{M}\times\mathcal{N}$, where $\sum_{m\in\mathcal{M}} \vartheta_n^m \leq 1$. Since by Proposition 3 the transfer $\underline{\Delta}$ solves the reduced contract design problem, (27) implies that $\Delta_n^m \geq \underline{\Delta}_n^m$, whence

$$\sum_{n\in\mathcal{N}} \Delta_n^m \geq \sum_{n\in\mathcal{N}} \underline{\Delta}_n^m \geq F^m$$

for all $n \in \mathcal{N}$. In addition,

$$\sum_{m\in\mathcal{M}} \Delta_n^m = (1-\vartheta_n^0)G_n + \vartheta_n^0 \sum_{i\in\mathcal{M}} \underline{\Delta}_n^i = G_n - \vartheta_n^0\left(G_n - \sum_{i\in\mathcal{M}} \underline{\Delta}_n^i\right) \leq G_n,$$

where $\vartheta_n^0 = 1 - \sum_{m\in\mathcal{M}} \vartheta_n^m$ takes on values in $[0,1]$. Thus, $\Delta = [\Delta_n^m]$ indeed solves the reduced contract design problem (R). The continuity of Δ_n^m comes from the continuity of the weights on $\hat{\mathcal{X}}_n$, and the fact that all convex combination yield the same point when the

[12]If $F = \sum_{m\in\mathcal{M}} F^m$ is strictly concave in a neighborhood of the efficient outcome $\hat{x}$, it is easily shown that the simplex is everywhere nondegenerate.

simplex is degenerate (i.e., $x_n \in \mathcal{X}_n \setminus \hat{\mathcal{X}}_n$). ■

Theorem 2 provides, when Assumption 3 holds, a simple characterization of all solutions to the reduced contract design problem (R). We denote $\mathcal{R}'$ the subset of excess transfers solving (R) which have the representation (26). Which particular equilibrium excess transfer Δ to choose depends on the desired allocation of surplus at the implemented efficient equilibrium outcome $\hat{x}$. The next section constructs Pareto optimal WTEs based on the obtained reduced transfers.

5 Constructing Weakly Truthful Equilibria

From Theorem 2, any element $\Delta = [\Delta_n^m]$ of $\mathcal{R}'$ can be represented as

$$\Delta_n^m = \underline{\Delta}_n^m + \vartheta_n^m \left(G_n - \sum_{i \in \mathcal{M}} \underline{\Delta}_n^i \right)$$

where $[\vartheta_n^m]$ defines admissible weights (over $\bar{\mathcal{M}}$ instead of $\mathcal{M}$). From (8), the in-equilibrium transfer α_n^m for any solution to $\mathcal{R}'$ is given by

$$\alpha_n^m = - \min_{x_n \in \mathcal{X}_n} \left\{ \underline{\Delta}_n^m + (\vartheta_n^m + \vartheta_n^0) \left(G_n - \sum_{i \in \mathcal{M}} \underline{\Delta}_n^i \right) \right\}, \tag{28}$$

where $\vartheta_n^0 = 1 - \sum_{i \in \mathcal{M}} \vartheta_n^i$ as in the previous section. For any given agent n, each principal m prefers the smallest possible in-equilibrium transfer α_n^m. Thus, by reducing ϑ_n^0 it is possible to simultaneously reduce in-equilibrium payments for all principals, which thus leads to (weak) Pareto-improvements. To find Pareto-optimal WTEs of $\mathcal{R}'$, we can thus restrict our attention to the frontier $\mathcal{F} = \{[\Delta_n^m] \in \operatorname{co} \mathcal{B} : [\vartheta_n^0] = 0\}$. In general, however, $\mathcal{F}$ contains elements of $\mathcal{R}'$ which are not Pareto optimal. Nevertheless, using the following procedure it is possible to filter out all Pareto-optimal allocations. We now construct the set of WTEs in $\mathcal{R}'$ which are Pareto optimal. For this, we first introduce

$$L_n^m(\Delta) = \min_{x_n \in \mathcal{X}_n} \Delta_n^m(x_n) \tag{29}$$

as the largest lower bound for the excess transfer from principal m to agent n. For $[\Delta_n^m] \in \mathcal{F}$ fixed, we observe from (8) that $\alpha_n^m = -\min_{x_n \in \mathcal{X}_n} \Delta_n^m(x_n)$. For each $(m,n) \in \mathcal{M} \times \mathcal{N}$,

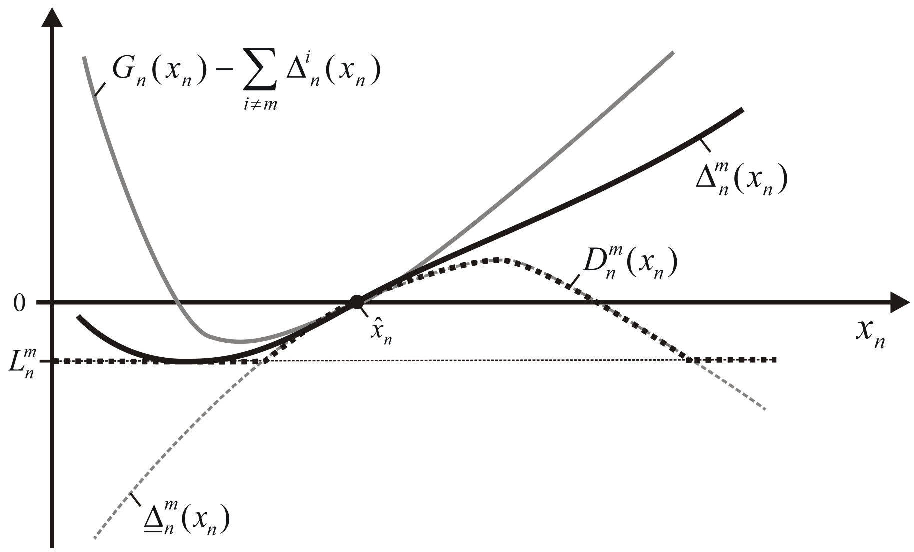

Figure 1: Construction of an Additive Upper Envelope of F^m as in (29) and (30).

define $D = [D^m_n]$ with

$$D^m_n(x_n) = \max\{\underline{\Delta}^m_n(x_n), L^m_n(\Delta)\}. \tag{30}$$

By construction, $D^m_n \leq \Delta^m_n$ but $L^m_n(D) = L^m_n(\Delta)$, which is also illustrated in Figure 1. A decrease in any D^m_n causes an increase in α^m_n, making principal m strictly worse off (when changes are limited to transactions with agent n). However, $[D^m_n]$ does not necessarily lie on the frontier $\mathcal{F}$. Therefore, it might be possible to Pareto-improve on $[D^m_n]$. For given $(m, n) \in \mathcal{M} \times \mathcal{N}$, let $\mathcal{X}^m_n = \{x_n \in \mathcal{X}_n : D^m_n = L^m_n(D)\}$,

$$d^m_n(D) = \min_{x_n \in \mathcal{X}^m_n} \left\{ G_n(x_n) - \sum_{\mathcal{M}} D^m_n(x_n) \right\} \geq 0. \tag{31}$$

and denote by $\underline{x}^m_n$ a point that reaches this minimum. If $d^m_n(D) = 0$, L^m_n cannot be increased without making another principal worse off, since at $\underline{x}^m_n$, $\sum_{m \in \mathcal{M}} D^m_n = G_n$ and $D^m_n = L^m_n$. If, however, $d^m_n(D) > 0$, then it is possible to increase L^m_n by setting

$$\tilde{D}^m_n = \max\{L^m_n(D) + d^m_n(D), \underline{\Delta}^m_n\}. \tag{32}$$

Performing this step for $m = 1$, we replace D^1_n by $\tilde{D}^1_n$ and repeat this step for $m = 2, \ldots, M$. This sequential procedure, illustrated in Figure 2 yields new transfers $\tilde{D}^m_n$ for fixed n, and in-equilibrium transfers that are Pareto optimal in $\mathcal{R}'$ when attention is restricted to agent n.

Lemma 2 (n-Pareto Optimality) *If for some excess transfers* $(\hat{D}^m_n)_{m \in \mathcal{M}}$

$$L^m_n(\hat{D}) \geq L^m_n(\tilde{D}) \quad \forall m \in \mathcal{M},$$

then

$$L_n^m(\hat{D}) = L_n^m(\tilde{D}) \quad \forall m \in \mathcal{M}.$$

Proof. We first observe that each step weakly increases the transfers, so that d_n^i remains at zero for $i < m$ when the m^{th} step occurs. By construction, $d_n^m(\tilde{D}) = 0$ for all m, so that $L_n^m(\tilde{D})$ cannot be increased without making another principal worse off. ■

In order to stay in $\mathcal{F}$, we conclude the procedure by letting $\tilde{D}_n^M = G_n - \sum_{m \neq M} \tilde{D}_n^m$. Note that this last transformation does not affect the in-equilibrium transfers since $\tilde{D}_n^m$ was already n-Pareto optimal, and increasing $\tilde{D}_n^M$ is a weak n-Pareto improvement. We can perform the procedure for all $n \in \mathcal{N}$ and obtain a new excess transfer matrix $[\tilde{D}_n^m]$. Denote by $\tilde{T} : \mathcal{F} \to \mathcal{F}$ the operator that maps $\Delta = [\Delta_n^m]$ to $\tilde{D} = [\tilde{D}_n^m]$. To state our result in its more general form, we introduce an equivalence class of excess transfer matrices.

Definition 2 (Excess-Transfer Equivalence) *Two excess transfer matrices $[\Delta_n^m]$ and $[\tilde{\Delta}_n^m]$ in $\mathcal{F}$ are equivalent, denoted $[\Delta_n^m] \equiv [\tilde{\Delta}_n^m]$, if $\alpha_n^m(\Delta) = \alpha_n^m(\tilde{\Delta})$ for all $(m, n) \in \mathcal{M} \times \mathcal{N}$.*

Note that an equivalence class is entirely characterized by the $M \times N$-dimensional matrix $\alpha = [\alpha_n^m]$ of in-equilibrium transfers. Let

$$\mathcal{A}_n = \{(\alpha_n^1(\Delta), \ldots, \alpha_n^M(\Delta)) : \Delta \in \mathcal{R}'\}$$

be the set of in-equilibrium transfers to agent n that are reachable from $\mathcal{R}'$ and $\mathcal{A} = \times_1^N \mathcal{A}_n$ the cartesian product of these sets.

Definition 3 *An equivalence class α of excess transfers is agent-wise Pareto optimal (in $\mathcal{R}'$), denoted $\alpha \in \mathcal{L}$, if for each n the vector $\alpha_n = (\alpha_n^1, \ldots, \alpha_n^M)$ belongs to the lower boundary $\partial_- \mathcal{A}_n \subset \mathbb{R}_+^{ML}$ of $\mathcal{A}_n$ defined by*

$$\partial_- \mathcal{A}_n = \{\alpha_n \in \mathcal{A}_n : \nexists \tilde{\alpha}_n \in \mathcal{A}_n, \tilde{\alpha}_n < \alpha_n\}.$$

Denoting $T : \mathcal{F} \to \mathcal{A}$ the operator that maps $\Delta = [\Delta_n^m]$ to $[\alpha_n^m(\tilde{T}(\Delta))]$, we have the following result.

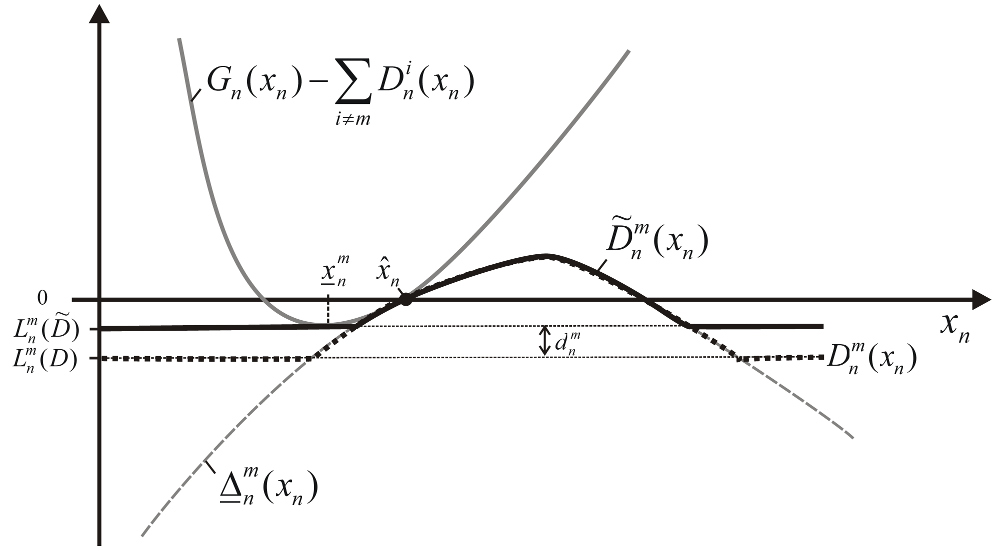

Figure 2: Construction of an Additive Upper Envelope of F^m as in (31) and (32).

Theorem 3 (Pareto-Optimal In-Equilibrium Transfers) *The image $T(\mathcal{F})$ is exactly the set of agent-wise Pareto-optimal in-equilibrium transfers.*

Proof. From Lemma 2, the image of T is clearly included in $\mathcal{L}$. Now take any in-equilibrium transfer matrix α in $\mathcal{L}$ and let Δ be an excess transfer matrix that implements it. By construction, $\tilde{T}(\Delta)$ is a weak Pareto improvement of Δ. Since α is agent-wise Pareto optimal, $T(\Delta)$ is less than α componentwise, and thus $T(\Delta) = \alpha$. ■

If each principal m only cares about the sum of her in-equilibrium transfers, $\alpha_1^m + \cdots + \alpha_N^m$, agent-wise Pareto optimality does *not* imply Pareto optimality. Once the lower boundaries $\partial_- \mathcal{A}_n$ of each $\mathcal{A}_n$ have been constructed, the set of Pareto-optimal excess transfer matrices corresponds to the lower boundary of the sum $\partial_- \mathcal{A}_1 + \cdots + \partial_- \mathcal{A}_N \in \mathbb{R}_+^{ML}$. We now determine the "absolute" lower boundaries in $\mathcal{R}'$ for each principal. This is the best scenario for a given principal, where only her profit is taken into account to implement the efficient outcome.

Proposition 6 (Transfer Lower Bound) *For each $n \in \mathcal{N}$, the lowest in-equilibrium transfer that can be obtained by principal m is*

$$\underline{\alpha}_n^m = - \min_{x_n \in \mathcal{X}_n} \left\{ G_n(x_n) - \sum_{i \neq m} \underline{\Delta}_n^i(x_n) \right\}.$$

Proof. $G_n(x_n) - \sum_{i \neq m} \underline{\Delta}_n^i(x_n)$ is the highest excess transfer $\bar{\Delta}_n^m$ that can be attained by principal m, and thus yields the lowest α_n^m. Moreover, $\alpha_n^m = -\min\left\{\bar{\Delta}_n^m\right\}$ since, in that case, the sum of the excess transfers equals G_n. ■

We can also derive an upper bound for $\partial_- \mathcal{A}_n$, implying a limit on the worst case for principal m in an n-Pareto-optimal transfer.

Proposition 7 (Transfer Upper Bound) *For each $n \in \mathcal{N}$, $\partial_- \mathcal{A}_n$ is bounded from above by $\bar{\alpha}_n = (\bar{\alpha}_n^1, \ldots, \bar{\alpha}_n^M)$, where*

$$\bar{\alpha}_n^m = -\min_{x_n \in \mathcal{X}_n} \underline{\Delta}_n^m(x_n).$$

Proof. For any excess transfer Δ in $\mathcal{F}$,

$$\alpha_n^m(\Delta) = -\min_{x_n \in \mathcal{X}_n} \Delta_n^m(x_n) \leq -\min_{x_n \in \mathcal{X}_n} \underline{\Delta}_n^m(x_n) = \bar{\alpha}_n^m.$$

Since $\tilde{T}(\Delta) \geq \Delta$,

$$\alpha_n^m\left(\tilde{T}(\Delta)\right) \leq \alpha_n^m(\Delta).$$

Therefore, $T(\Delta) \leq \bar{\alpha}_n$ for all $\Delta \in \mathcal{F}$. By Theorem 3, the set of agent-wise Pareto-optimal transfers precisely is the image of T, which concludes our proof. ■

In the next result, we construct an excess transfer matrix Δ that attains a given in-equilibrium transfer α in $T(\mathcal{F})$. Since the construction can be done agent-wise, we fix $n \in \mathcal{N}$ and construct $(\Delta_n^m)_{m \in \mathcal{M}}$. Thus suppose that $\alpha_n = (\alpha_n^m)_{m \in \mathcal{M}}$ is a given vector in the boundary $\partial_- \mathcal{A}_n$. Let

$$\Delta_n^m(x_n) = \max\{\underline{\Delta}_n^m(x_n), -\alpha_n^m\} \tag{33}$$

for all $x_n \in \mathcal{X}_n$. By construction, we have

$$-\min_{x_n \in \mathcal{X}_n} \Delta_n^m(x_n) \geq \alpha_n^m. \tag{34}$$

Moreover,

$$\underline{\Delta}_n^m \leq \Delta_n^m \tag{35}$$

and Δ_n^m is the smallest excess transfer that satisfies (34) and (35). Since by assumption α_n is attainable, this implies that $\sum_{m\in\mathcal{M}} \Delta_n^m \leq G_n$. To ensure that Δ_n^m belongs to $\mathcal{F}$, we redefine Δ_n^M as

$$\Delta_n^M = G_n - \sum_{m\neq M} \Delta_n^m. \tag{36}$$

Moreover, n-Pareto optimality of α_n implies that in fact equality obtains in (34). Therefore, $(\Delta_n^m)_{m\in\mathcal{M}}$ implements α_n. From the construction, all the excess transfers are clearly continuous, which together with (R) implies their admissibility. We have thus shown the following result.

Theorem 4 (Pareto-Optimal Equilibrium Implementation) *Suppose that α_n belongs to the set $\partial_-\mathcal{A}_n$ of n-Pareto optimal in-equilibrium transfers. Then, the vector of excess transfers $(\Delta_n^m)_{m\in\mathcal{M}}$ defined by (33) for $m < M$ and by (36) for $m = M$ results in a WTE with in-equilibrium transfer α_n.*

Theorem 4 provides a practical implementation of Pareto-optimal equilibrium contracts for the principals.

6 Discussion

We have shown that it is possible to construct weakly truthful equilibria for any game played through agents, as long as all principals' and agents' payoffs are concave. The constructions can be used in a variety of practical settings with complete information, such as the coordination of multi-principal multi-agent supply chains. Note that no payoff externalities between agents are permitted in the model we discussed. The reason is that – as Segal (1999) shows – efficient outcomes may not be implementable when agents' payoffs depend on each others' actions, at least when principals' transfers are action-contingent, i.e., depend on the agent's action who receives the transfer. The situation also becomes more delicate when information about the contracts is asymmetric, as in bilateral contracting,[13] or when renegotiation is allowed, as in Matthews (1995). We have seen that any efficient outcome in games played

[13] Segal and Whinston (2003) examine a setting where a single principal writes bilateral contracts with N different agents without announcing the contracts publicly.

through agents can be implemented with all principals using affine transfers. Nevertheless, to change the allocation of surplus within the principal-agent system for a given efficient outcome, it may be desirable to choose particular equilibrium excess transfer matrices. Using essentially convex combinations and a levelling algorithm, we have provided a (under general assumptions, complete) class of admissible equilibrium excess transfers. Our analysis of the relation between desired in-equilibrium surplus allocations α and off-equilibrium excess transfers Δ allows us to characterize and implement WTEs with Pareto-optimal in-equilibrium transfers. We have thus obtained a global solution to the following practical problem: given an efficient outcome, the principals determine the set of WTEs of $\mathcal{G}$. After agreeing on a point in the set of attainable Pareto-optimal in-equilibrium transfers (e.g., using a bargaining procedure, not discussed here) they implement the efficient outcome as a WTE. Bernheim and Whinston (1986) prove an equivalence between their truthful Nash equilibria and the concept of "coalition-proof Nash equilibrium" (see also Bernheim et al., 1997). This latter concept contains the idea that even if principals were allowed to collude, the corresponding Nash equilibria would still be outcomes of the games. Strengthening the concept of Nash equilibrium to coalition-proof ones is therefore important for the predictive power of this theory, whenever collusion among principals is possible or imperfectly monitored. While coalition-proofness is lost when using the more flexible concept of weakly truthful equilibrium, our characterization of Pareto optimal WTE may allow to retrieve coalition-proof Nash equilibria. Indeed, coalition-proofness is equivalent to the in-equilibrium transfers being Pareto optimal when the game is limited to any subset of principals, and the remaining principals and all agents take their equilibrium actions.

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